Module MAU23302: Euclidean and Non-Euclidean Geometry
Hilary Term 2020
Part II, Section 1
Stereographic Projection

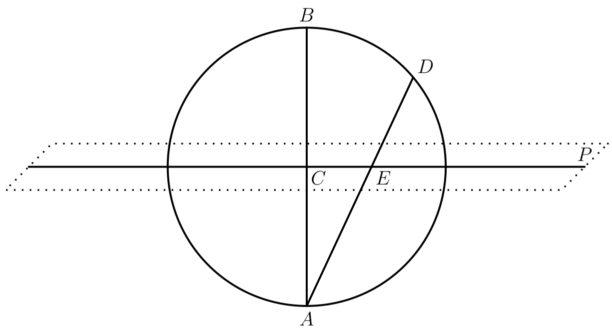
D. R. Wilkins

Contents

1 Stereographic Projection

1.1 The Basic Equations of Stereographic Projection

Let a sphere in three-dimensional spaces be given, let C be the centre of that sphere, let AB be a diameter of that sphere with endpoints A and B, and let P be the plane through the centre of the sphere that is perpendicular to the diameter AB. Given a point D of the sphere distinct from the point A, the image of D under *stereographic projection* from the point A is defined to be the point E at which the line passing through the points A and D intersects the plane P.



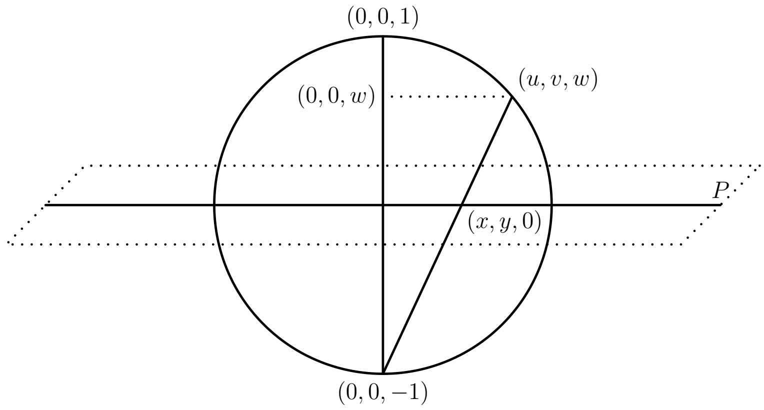
Proposition 1.1 *Let S^2 be the unit sphere in $\mathbb{R}^3$, consisting of those points (u, v, w) of $\mathbb{R}^3$ that satisfy the equation $u^2 + v^2 + w^2 = 1$, and let P be the plane consisting of those points (u, v, w) of $\mathbb{R}^3$ for which $w = 0$. Then, for each point (u, v, w) of S^2 distinct from the point $(0, 0, -1)$, the straight line passing through the points (u, v, w) and $(0, 0, -1)$ intersects the plane P at the point $(x, y, 0)$ at which*

$$x = \frac{u}{w+1} \quad \textit{and} \quad y = \frac{v}{w+1}.$$

Proof Let $A = (0, 0, -1)$, $D = (u, v, w)$ and $E = (x, y, 0)$. Then the displacements of the points D and E from the point A are represented by the vectors $(u, v, w + 1)$ and $(x, y, 1)$ respectively. These vectors are parallel because the points A, D and E are collinear. Consequently

$$\frac{x}{u} = \frac{y}{v} = \frac{1}{w+1}.$$

The result follows. ∎



Corollary 1.2 *Let S be a sphere with centre C, let A and B be the endpoints of a diameter of that sphere, let P be the plane through the centre C of the sphere that is the perpendicular bisector of the diameter AB, let D be a point on the sphere distinct from the point A, and let E be the point where the infinite straight line passing through the points A and D intersects the plane P. Then*

$$|AD|\,|AE| = 2|AC|^2.$$

Proof We may assume, without loss of generality, that $|AC|$ is the unit of length and that the sphere S is the unit sphere centred on the origin of Cartesian coordinates, so that $C = (0, 0, 0)$ and $|AC| = 1$. We may also assume that the Cartesian coordinates of the points A and B are $(0, 0, -1)$ and $(0, 0, 1)$ respectively. Let the points D and E have Cartesian coordinates (u, v, w) and $(x, y, 0)$. Then $w \neq -1$ and $u^2 + v^2 + w^2 = 1$. Also

$$x = \frac{u}{w+1} \quad \text{and} \quad y = \frac{v}{w+1}$$

(see Proposition 1.1). It follows that $|AD| = (w+1)|AE|$. Moreover $|AE|$ is the length of the line segment joining the points $(0, 0, -1)$ and $(x, y, 0)$, and therefore

$$|AE|^2 = x^2 + y^2 + 1.$$

It follows that

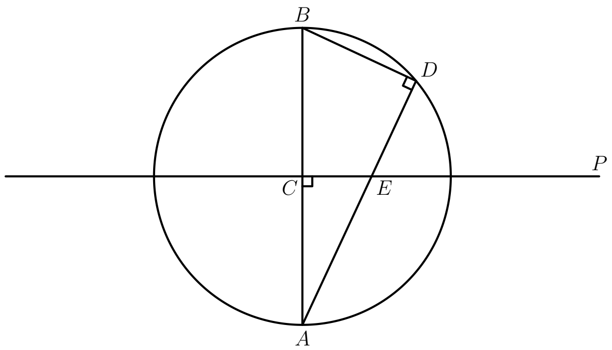
$$\begin{aligned} |AD|\,|AE| &= (w+1)|AE|^2 = (w+1)(x^2 + y^2 + 1) \\ &= \frac{u^2 + v^2 + (w+1)^2}{w+1} = \frac{u^2 + v^2 + w^2 + 2w + 1}{w+1}. \end{aligned}$$

But $u^2 + v^2 + w^2 = 1$. It follows that

$$|AD|\,|AE| = 2 = 2\,|AC|^2,$$

as required. ∎

Alternative Proof The angle ADB, being the angle in a semicircle, is a right angle (Euclid, *Elements*, III, 31). The angle ACE is also a right angle. Thus the triangles ADB and ACE are similar (or equiangular), and consequently corresponding sides of those triangles are proportional (Euclid, *Elements*, VI, 4).



Accordingly AD is to AC as AB is to AE, and thus

$$\frac{|AD|}{|AC|} = \frac{|AB|}{|AE|}.$$

Cross-multiplying, it follows that

$$|AD|\,|AE| = |AB|\,|AC| = 2\,|AC|^2,$$

as required. ∎

Definition Let (u, v, w) be a point on the unit sphere distinct from the point $(0, 0, -1)$, where $u^2 + v^2 + w^2 = 1$, and let (x, y) be a point of the plane $\mathbb{R}^2$. We say that the point (x, y) is the *image* of the point (u, v, w) under *stereographic projection* from the point $(0, 0, -1)$ if

$$x = \frac{u}{w+1} \quad \text{and} \quad y = \frac{v}{w+1}.$$

Proposition 1.3 *Each point (x, y) of $\mathbb{R}^2$ is the image, under stereographic projection from the point $(0, 0, -1)$, of the point (u, v, w) of the unit sphere for which*

$$u = \frac{2x}{1+x^2+y^2}, \quad v = \frac{2y}{1+x^2+y^2} \quad \textit{and} \quad w = \frac{1-x^2-y^2}{1+x^2+y^2}.$$

This point (u, v, w) is distinct from the point $(0, 0, -1)$.

Proof Given a point (x, y) of $\mathbb{R}^2$, the straight line passing through the points $(0, 0, -1)$ and $(x, y, 0)$ is not tangent to the unit sphere, and therefore intersects the unit sphere at some point distinct from $(0, 0, -1)$. It follows that every point of $\mathbb{R}^2$ is the image, under stereographic projection from $(0, 0, -1)$, of some point of the unit sphere distinct from the point $(0, 0, -1)$.

Let (x, y) be the image, under stereographical projection from the point $(0, 0, -1)$, of a point (u, v, w), where $u^2 + v^2 + w^2 = 1$ and $w \neq -1$. Then

$$x = \frac{u}{w+1}, \quad y = \frac{v}{w+1}.$$

It follows that

$$x^2 + y^2 = \frac{u^2 + v^2}{(w+1)^2} = \frac{1 - w^2}{(w+1)^2} = \frac{1 - w}{w+1}.$$

It follows that

$$w(x^2 + y^2) + x^2 + y^2 = 1 - w,$$

and therefore

$$w = \frac{1 - x^2 - y^2}{1 + x^2 + y^2}.$$

But then

$$1 + w = 1 + \frac{1 - x^2 - y^2}{1 + x^2 + y^2} = \frac{2}{1 + x^2 + y^2},$$

and therefore

$$\begin{aligned} u &= (1 + w)x = \frac{2x}{1 + x^2 + y^2}, \\ v &= (1 + w)y = \frac{2y}{1 + x^2 + y^2}. \end{aligned}$$

Conversely if

$$u = \frac{2x}{1 + x^2 + y^2}, \quad v = \frac{2y}{1 + x^2 + y^2} \quad \text{and} \quad w = \frac{1 - x^2 - y^2}{1 + x^2 + y^2}.$$

then

$$u^2 + v^2 + w^2 = \frac{4(x^2 + y^2) + (1 - x^2 - y^2)^2}{(1 + x^2 + y^2)^2} = 1,$$

because

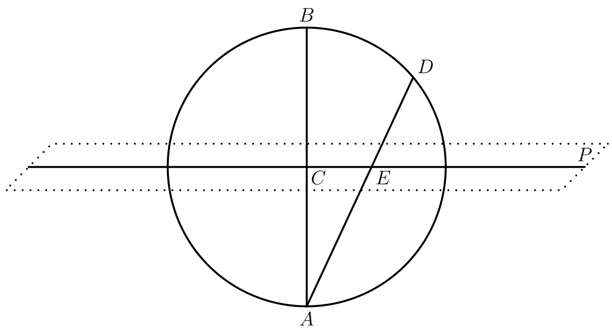
$$\begin{aligned} &4(x^2 + y^2) + (1 - x^2 - y^2)^2 \\ &\qquad = 4(x^2 + y^2) + 1 - 2(x^2 + y^2) + (x^2 + y^2)^2 \\ &\qquad = 1 + 2(x^2 + y^2) + (x^2 + y^2)^2 \\ &\qquad = (1 + x^2 + y^2)^2. \end{aligned}$$

Also $w > -1$ and

$$x = \frac{u}{w+1} \quad \text{and} \quad y = \frac{v}{w+1}.$$

The result follows. $\blacksquare$

Alternative Proof Let $A = (0,0,-1)$, $B = (0,0,1)$, $C = (0,0,0)$ and $E = (x,y,0)$. Then E is the image, under stereographic projection from A, of the unique point D distinct from A at which the line passing through A and E intersects the unit sphere. Let $D = (u,v,w)$.



Now the displacement vectors $\overrightarrow{AD}$ and $\overrightarrow{AE}$ representing the displacements of the points D and E respectively from the point A point in the same direction. Moreover $|AD|\,|AE| = 2|AC|^2$ (Corollary 1.2). It follows that

$$\overrightarrow{AD} = \frac{|AD|}{|AE|}\,\overrightarrow{AE} = \frac{|AD|\,|AE|}{|AE|^2}\,\overrightarrow{AE} = \frac{2|AC|^2}{|AE|^2}\,\overrightarrow{AE}.$$

Now $\overrightarrow{AD} = (u,v,w+1)$ and $\overrightarrow{AE} = (x,y,1)$. Also $|AC| = 1$ and $|AE|^2 = 1 + x^2 + y^2$. It follows that

$$(u,v,w+1) = \frac{2}{1+x^2+y^2}\,(x,y,1),$$

and thus

$$u = \frac{2x}{1+x^2+y^2}, \quad v = \frac{2y}{1+x^2+y^2}$$

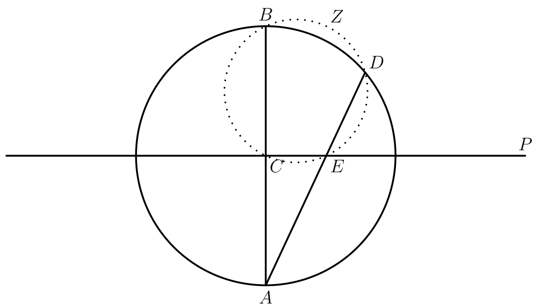
and

$$w = \frac{2}{1+x^2+y^2} - 1 = \frac{1-x^2-y^2}{1+x^2+y^2},$$

as required. $\blacksquare$

Proposition 1.4 *Let S be a sphere with centre C, let A and B be the end-points of a diameter of that sphere, let P be the plane through the centre C of the sphere that is the perpendicular bisector of the diameter AB, let D be a point on the sphere distinct from the point A, and let E be the point where the infinite straight line passing through the points A and D intersects the plane P (so that E is the image of D under stereographic projection from the point A). Then the points C, B, D and E lie on a circle.*

Proof We show that the point D lies on the circle that passes through the points C, B and E. Now we can assume, without loss of generality, that the sphere is the unit sphere centred on the origin of coordinates, that $A = (0, 0, -1)$ and $B = (0, 0, 1)$. Let $E = (x, y, 0)$, and let Z be the circle through the points C, B and E.



The centre of the circle Z lies on the perpendicular bisector of the line segment CE. This perpendicular bisector consists of those points of three-dimensional space whose Cartesian coordinates are of the form $(\frac{1}{2}x, \frac{1}{2}y, w)$ for some real number w. The centre of the circle also lies on the perpendicular bisector of the line segment CB, where $C = (0, 0, 0)$ and $B = (0, 0, 1)$. It follows that $w = \frac{1}{2}$, and thus the centre of the circle Z is located at the point $(\frac{1}{2}x, \frac{1}{2}y, \frac{1}{2})$. The radius of the circle Z is the distance from the origin $(0, 0, 0)$ to the centre of the circle. The square of the radius of the circle Z is therefore equal to $\frac{1}{4}(x^2 + y^2 + 1)$, and thus the circle Z itself consists of those points in the plane of this circle whose Cartesian coordinates (u, v, w) satisfy the equation

$$(u - \tfrac{1}{2}x)^2 + (v - \tfrac{1}{2}y)^2 + (w - \tfrac{1}{2})^2 = \tfrac{1}{4}(x^2 + y^2 + 1).$$

Expanding out and cancelling terms, this equation reduces to the equation

$$u^2 + v^2 + w^2 - xu - yv - w = 0.$$

Now let (u, v, w) be the Cartesian coordinates of the point D. Then

$$u = \frac{2x}{1 + x^2 + y^2}, \quad v = \frac{2y}{1 + x^2 + y^2} \quad \text{and} \quad w = \frac{1 - x^2 - y^2}{1 + x^2 + y^2}$$

(see Proposition 1.3). It follows that

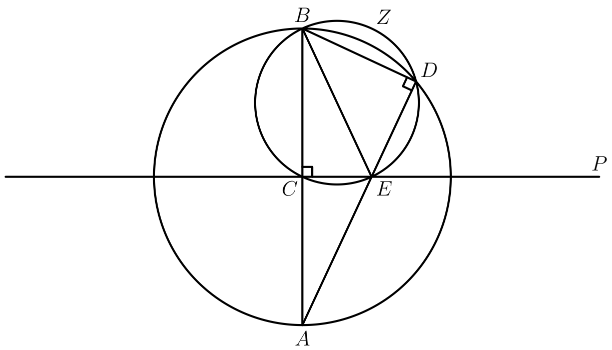
$$xu + yv + w = \frac{2x^2 + 2y^2 + 1 - x^2 - y^2}{1 + x^2 + y^2} = 1.$$

Also $u^2 + v^2 + w^2 = 1$, because the point D lies on the unit sphere. It follows that the Cartesian coordnates (u, v, w) of the point D satisfy the equation

$$u^2 + v^2 + w^2 - xu - yv - w = 0.$$

The point D also lies on the plane of the circle Z. It follows that the point D lies on the circle Z. The result follows. ∎

Alternative Proof The configuration is as depicted in the figure below. In particular the angle BDE is a right angle, because it is the angle in a semicircle (Euclid, *Elements*, III, 31) and the angle BCD is also a right angle. It follows (as an immediate corollary of the results stated in Euclid, *Elements*,



III, 31) that the region bounded by the straight line BE and the circular arc BDE is a semicircle. Similarly the region bounded by the straight line BE and the circular arc BCE is a semicircle, and thus $BCED$ is a circle. The result follows. ∎

1.2 Small Circles on the Unit Sphere

If a plane in three-dimensional space contains points lying inside some given sphere then the intersection of the plane and the given sphere takes the form of a circle on that sphere. Every circle on a sphere is the intersection of that sphere with some plane in three-dimensional space. If the centre of the sphere lies on that plane then the circle is said to be a *great circle* on the sphere. On the other hand, if the centre of the sphere does not lie on the plane then the circle is said to be a *small circle* on the sphere.

Proposition 1.5 *Let S be the unit sphere in three-dimensional Euclidean space, consisting of those points whose Cartesian coordinates (u, v, w) satisfy the equation $u^2 + v^2 + w^2 = 1$, and let Q be a plane in that space consisting of those points of space whose Cartesian coordinates (u, v, w) satisfy an equation of the form*

$$pu + qv + rw + s = 0,$$

where p, q, r and s are real constants and p, q and r are not all equal to zero. Then the plane Q intersects the sphere S along a circle if and only if

$$s^2 < p^2 + q^2 + r^2.$$

Proof The vector with components (p, q, r) is orthogonal to the plane Q, and therefore the perpendicular dropped from the origin of Cartesian coordinates to the given plane meets that plane at a point whose Cartesian coordinates are of the form (kp, kq, kr) for some real number k. That intersection point lies on the plane Q, and therefore

$$k(p^2 + q^2 + r^2) + s = 0.$$

It follows that the point with Cartesian coordinates

$$\left(\frac{-sp}{p^2 + q^2 + r^2}, \frac{-sq}{p^2 + q^2 + r^2}, \frac{-sr}{p^2 + q^2 + r^2} \right)$$

is the point on the plane Q that lies closest to the origin.

Now the plane Q intersects the unit sphere S in a circle if and only if the point on the plane Q closest to the origin lies inside the unit sphere. This is the case if and only if

$$\left(\frac{-sp}{p^2 + q^2 + r^2} \right)^2 + \left(\frac{-sq}{p^2 + q^2 + r^2} \right)^2 + \left(\frac{-sr}{p^2 + q^2 + r^2} \right)^2 < 1,$$

and the latter inequality holds if and only if

$$s^2 < p^2 + q^2 + r^2.$$

The result follows. ∎

Corollary 1.6 *Let S be the unit sphere consisting of those points of three-dimensional Euclidean space whose Cartesian coordinates (u, v, w) with respect to a chosen Cartesian coordinate system satisfy the equation $u^2 + v^2 + w^2 = 1$, and let Z be a small circle on the unit sphere S. Then there exist real constants p, q and r, where $p^2 + q^2 + r^2 > 1$, such that the circle Z consists of those points of the unit sphere S whose Cartesian coordinates (u, v, w) satisfy the equations*

$$u^2 + v^2 + w^2 = 1 \quad \text{and} \quad pu + qv + rw = 1.$$

Proof Let Q be the plane that contains the circle Z. Then the plane Q does not contain the centre of the sphere, because the circle Z is a small circle. But the centre of the unit sphere is the origin $(0,0,0)$ of the chosen Cartesian coordinate system. Now the points of the plane Q are those points of space whose Cartesian coordinates (u,v,w) satisfy an equation of the form

$$p'u + q'v + r'w + s' = 0,$$

where p', q', r' and s' are real constants and p', q' and r' are not all zero. Moreover $s' \neq 0$, because the origin $(0,0,0)$ does not lie in the plane Q. Let

$$p = -\frac{p'}{s'}, \quad q = -\frac{q'}{s'}, \quad r = -\frac{r'}{s'}.$$

Then the points of the plane Q are those points whose Cartesian coordinates satisfy the equation

$$pu + qv + rw = 1.$$

Moreover it follows from Proposition 1.5 that $p^2 + q^2 + r^2 > 1$. The result follows. ∎

1.3 Images of Circles under Stereographic Projection

We consider the images of circles on the unit sphere under stereographic projection. This sphere is the sphere of unit radius centred on the origin of Cartesian coordinates, and consists of those points of three-dimensional space whose Cartesian coordinates (u,v,w) satisfy the equation

$$u^2 + v^2 + w^2 = 1.$$

Let some plane in three-dimensional space be given. Then the given plane consists of those points of three-dimensional Euclidean space whose Cartesian coordinates (u,v,w) satisfy an equation of the form

$$pu + qv + rw + s = 0,$$

where p, q, r and s are real constants and p, q and r are not all equal to zero. Given real constants p', q', r' and s', where p', q' and r' are not all zero, the plane consisting of those points that satisfy the equation

$$p'u + q'v + r'w + s' = 0$$

coincides with the given plane if and only if p', q', r' and s' are respectively proportional to p, q, r and s, in which case there exists some non-zero real number k such that $p' = kp$, $q' = kq$, $r' = kr$ and $s' = ks$.

Proposition 1.7 *Let p, q, r and s be real constants, where p, q and r are not all equal to zero and $s^2 < p^2 + q^2 + r^2$, and let $P_{p,q,r,s}$ be the plane in three-dimensional space consisting of those points whose Cartesian coordinates (u, v, w) satisfy the equation*

$$pu + qv + rw + s = 0.$$

A point (x, y) of $\mathbb{R}^2$ belongs to the image, under stereographic projection from the point $(0, 0, -1)$, of the circle on the unit sphere along which that sphere interects the plane $P_{p,q,r,s}$ if and only if

$$(r - s)(x^2 + y^2) = 2px + 2qy + r + s.$$

Proof Given a point (x, y) of $\mathbb{R}^2$ there is a unique point (u, v, w) of the unit sphere in $\mathbb{R}^3$ distinct from $(0, 0, -1)$ that maps to the point (x, y) under stereographic projection from the point $(0, 0, -1)$. Moreover

$$u = \frac{2x}{1 + x^2 + y^2}, \quad v = \frac{2y}{1 + x^2 + y^2} \quad \text{and} \quad w = \frac{1 - x^2 - y^2}{1 + x^2 + y^2}.$$

It follows that (x, y) is the image, under stereographic projection from the point $(0, 0, -1)$ of a point on the circle along which the plane $P_{p,q,r,s}$ intersects the unit sphere if and only if

$$\frac{2px + 2qy + r - r(x^2 + y^2)}{1 + x^2 + y^2} + s = 0.$$

This equation is satisfied if and only if

$$2px + 2qy + r - r(x^2 + y^2) + s + s(x^2 + y^2) = 0.$$

Thus (x, y) is on the image, under stereographic projection, of the specified circle if and only if

$$(r - s)(x^2 + y^2) = 2px + 2qy + r + s.$$

The result follows. ∎

Corollary 1.8 *Circles on the unit sphere in three-dimensional Euclidean space correspond, under stereographic projection, to lines and circles in the Euclidean plane. A circle on the unit sphere is projected to a line in the Euclidean plane, under stereographic projection from the point $(0, 0, -1)$, if and only if that circle on the unit sphere passes through the point $(0, 0, -1)$.*

Proof If $r \neq s$ then the equation

$$(r - s)(x^2 + y^2) = 2px + 2qy + r + s$$

is the equation of a circle in the Euclidean plane. If $r = s$ then the above equation reduces to

$$2px + 2qy + r + s = 0,$$

and the latter equation is the equation of a line in the Euclidean plane. The circle along which the plane $pu + qv + rw + s = 0$ intersects the unit sphere passes through the point $(0, 0, -1)$ if and only if $r = s$. The result follows. ∎

Corollary 1.9 *Let (p, q) be a point of the Euclidean plane $\mathbb{R}^2$, and let R be a positive real number. Then the circle of radius R centred on the point (p, q) is the image, under stereographic projection from the point $(0, 0, -1)$, of the circle on the unit sphere in which that sphere intersects the plane consisting of those points (u, v, w) of $\mathbb{R}^3$ that satisfy the equation*

$$pu + qv + \tfrac{1}{2}(R^2 - p^2 - q^2 + 1)(w + 1) = 1.$$

Proof The points (x, y) lying on the circle of radius R about the point (p, q) in $\mathbb{R}^2$ are those points of $\mathbb{R}^2$ that satisfy the equation

$$(x - p)^2 + (y - q)^2 = R^2.$$

This equation is satisfied by (x, y) if and only if

$$x^2 + y^2 = 2px + 2qy + R^2 - p^2 - q^2.$$

This equation is of the form $(r - s)(x^2 + y^2) = 2px + 2qy + r + s$ provided that $r = s + 1$ and $R^2 - p^2 - q^2 = 2s + 1$, in which case

$$R^2 = p^2 + q^2 + r^2 - s^2,$$

and therefore $s^2 < p^2 + q^2 + r^2$. Applying the result of Proposition 1.7, we conclude that the circle of radius R about the point (p, q) is the image, under stereographic projection from $(0, 0, -1)$, of the circle on the unit sphere along which that unit sphere intersects the plane consisting of those points (u, v, w) that satisfy the equation

$$pu + qv + \tfrac{1}{2}(R^2 - p^2 - q^2 + 1)(w + 1) = 1.$$

The result follows. ∎

1.4 Pole and Polar

Let S be the unit sphere in three-dimensional space centred on the origin of a Cartesian coordinate system (u, v, w), so that the sphere S consists of those points of space whose Cartesian coordinates (u, v, w) satisfy the equation $u^2 + v^2 + w^2 = 1$, and let Q be a plane in three-dimensional space that does not contain the origin of a Cartesian coordinate system (u, v, w). Then there exist real constants p, q and r such that the points of space that lie on the plane Q are those whose Cartesian coordinates satisfy the equation $pu + qv + rw = 1$. Let F be the point of three-dimensional Euclidean space whose Cartesian coordinates are (p, q, r). The point F is then said to be the *pole* of the plane Q, and the plane Q is said to be the *polar plane* (or *polar*) of the point F with respect to the unit sphere S. The terminology of *pole* and *polar* can be extended to define poles of planes and polar planes of points with respect to spheres of arbitrary radius and centre, as in the definitions that follow.

Definition Let S be a sphere in three-dimensional Euclidean space, and let Q be a plane in that space that does not contain the centre of the sphere S. The sphere S consists of those points (x, y, z) of space for which

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = R_0^2,$$

where R_0 is the radius of the sphere and (a, b, c) is the centre of the sphere. Real numbers p, q and r may then be determined so that the plane Q consists of those points (x, y, z) of space that satisfy the equation

$$(p - a)(x - a) + (q - b)(y - b) + (r - c)(z - c) = R_0^2.$$

The *pole* of the plane Q with respect to the sphere S is then the point of space whose Cartesian coordinates are (p, q, r).

Definition Let S be a sphere in three-dimensional Euclidean space, and let F be a point in that space that is distinct from centre of the sphere S. The sphere S consists of those points (x, y, z) of space for which

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = R_0^2,$$

where R_0 is the radius of the sphere and (a, b, c) is the centre of the sphere. Let the point F have Cartesian coordinates (p, q, r). The *polar plane* (or *polar*) of the point F with respect to the sphere S consists of those points (x, y, z) of space for which

$$(p - a)(x - a) + (q - b)(y - b) + (r - c)(z - c) = R_0^2.$$

Lemma 1.10 *Let S be a sphere in three-dimensional Euclidean space, let Q be a plane that does not contain the centre of the sphere S, and let F be a point of space that is distinct from the centre of the sphere S. Then the point F is the pole of plane Q with respect to the sphere S if and only if the plane Q is the polar plane of the point F with respect to the sphere S.*

Proof This result follows directly from the relevant definitions. ∎

Proposition 1.11 *Let S be a sphere in three-dimensional Euclidean space, and let Q be a plane that intersects the sphere S along a circle Z but does not pass through the centre of the sphere. Let F be the pole of the plane Q with respect to the sphere S. Then the line DF joining any point D of the circle Z to the point F is contained in the tangent plane to the sphere S at the point D.*

Proof We may choose the unit of length and the origin of the Cartesian coordinate system (u, v, w) so that the sphere S has unit radius and is centred on the origin of the Cartesian coordinate system. The centre C of the sphere S then has Cartesian coordinates $(0, 0, 0)$. Let F be the pole of the plane Q. Then $F = (p, q, r)$, where $p^2 + q^2 + r^2 > 1$, and the plane Q consists of those points of space whose Cartesian coordinates (u, v, w) satisfy the equation $pu + qv + rw = 1$.

Let D be a point that lies on the circle Z along which the plane Q intersects the sphere S, and let $D = (u, v, w)$, where $u^2 + v^2 + w^2 = 1$ and $pu + qv + rw = 1$. Then

$$\begin{aligned} |DF|^2 &= (u - p)^2 + (v - q)^2 + (w - r)^2 \\ &= u^2 + v^2 + w^2 - 2(pu + qv + rw) + p^2 + q^2 + r^2 \\ &= p^2 + q^2 + r^2 - 1. \end{aligned}$$

Also $|CD| = 1$ and $|CF|^2 = p^2 + q^2 + r^2$. It follows that $|CF|^2 = |CD|^2 + |DF|^2$, and thus the angle of the triangle CDF at the vertex D is a right angle (see for example Euclid, *Elements*, I, 48). Now a line passing through the point D is tangent to the unit sphere S at D if and only if it is makes a right angle with the line DC joining D to the centre C of the unit sphere. We conclude therefore that the line DF is indeed tangent to the sphere S at D. ∎

Corollary 1.12 *Let S be a sphere in three-dimensional Euclidean space, and let Q be a plane that intersects the sphere S in a circle Z but does not contain the centre C of the sphere, and let F be the pole of the plane Q with respect to the sphere S. Let D be a point lying on the circle Z. Then $|DF|^2 = |CF|^2 - R_0^2$, where R_0 is the radius of the sphere S.*

Proof Let D be a point lying on the circle Z. Then the angle CDF is a right angle, because the line DF lies in the tangent plane to the sphere S at the point D and is therefore perpendicular to the line CD joining the centre C of the sphere to the point D. It follows from Pythagoras's Theorem (Euclid, *Elements*, I, 47) that $|CF|^2 = |CD|^2 + |DF|^2 = R_0^2 + |DF|^2$. The result follows. ∎

Definition Given a circle in three-dimensional Euclidean space, the *axis* of the circle is the line, perpendicular to the plane containing the given circle, that passes through the centre of that circle.

The axis of a circle in three-dimensional Euclidean space consists of those points of Euclidean space that are equidistant from all points of the circle. (This result follows easily using the definition of the centre of a circle together with Pythagoras's Theorem.)

Lemma 1.13 *Let S be a sphere in three-dimensional Euclidean space, and let Q be a plane that intersects the sphere S in a circle Z but does not contain the centre C of the sphere, and let F be the pole of the plane Q with respect to the sphere S. Then the axis of the circle Z is the line that passes through the points C and F.*

Proof All points of the circle Z are equidistant from the centre C of the sphere S. There are also equidistant from the pole F of the plane Q, because $|DF|^2 = |CF|^2 - R_0^2$ for all points D of the circle Z, where R_0 is the radius of the sphere S (see Corollary 1.12). Therefore the points C and F must lie on the axis of the circle Z, and therefore determine the axis of this circle. The result follows. ∎

1.5 Stereographic Projection of Vertical Circles

Let S be a sphere in three-dimensional space, let A and B be the endpoints of a diameter AB of that sphere, let C be the centre of the sphere S, and let P be the plane containing C that is perpendicular to the diameter AB. In studying stereographic projection of the sphere S from the point A onto the plane P it is convenient to think of the point A as being located at the bottom of the sphere S. The point B will then be located at the top of the sphere. We may regard any line or plane parallel to the plane P as being *horizontal* and any line parallel to the diameter AB as being *vertical*. We may regard a plane Q as being *vertical* if the diameter AB joining the bottom to the top of the sphere is parallel to the plane Q. We regard a circle as being *horizontal* if it is contained in a horizontal plane, and as being *vertical* if it is contained in a vertical plane.

If we introduce a Cartesian coordinate system (u, v, w) so as to ensure that the sphere S has unit radius and is centred on the origin of those Cartesian coordinate system, and if we take the point A to be that point whose Cartesian coordinates are $(0, 0, -1)$, then the u-axis and the v-axis are horizontal and the w-axis is vertical. One can regard the plane P through the centre of the sphere that is perpendicular to the diameter AB as being the *equatorial plane*. This plane P is the unique horizontal plane passing through the centre C of the sphere.

If a point F with Cartesian coordinates $(p, q, 0)$ lies on the plane P and is distinct from the centre of the sphere S then its polar plane Q with respect to the sphere S consists of those points of space whose Cartesian coordinates (u, v, w) satisfy the equation $pu+qv = 1$. The polar plane Q is then a vertical plane, and the small circle Z along which the polar plane cuts the sphere S is a vertical circle.

Proposition 1.14 *Let S be the unit sphere in three-dimensional Euclidean space, let A and B be endpoints of a diameter of that sphere, and let P be the plane passing through the centre C of the sphere S that is perpendicular to the diameter AB. Let F be a point on the plane P located outside the sphere S, let Q be the polar plane of F with respect to the sphere S, and let Z be the vertical circle along which the plane Q intersects the sphere S. Also let W be the circle, contained in the plane P and centred on the point F, that passes through those points where the circle Z intersects the plane P. Then stereographic projection from the point A maps the circle Z onto the circle W.*

Proof Without loss of generality, we can take S to be the unit sphere centred on the origin of the Cartesian coordinate system (u, v, w), and also assign coordinates so that $A = (0, 0, -1)$, $B = (0, 0, 1)$, $C = (0, 0, 0)$. The plane P then consists of those points of space whose Cartesian coordinates (u, v, w) satisfy the equation $w = 0$. It follows that $F = (p, q, 0)$, where p and q are real numbers satisfying $p^2 + q^2 > 1$. The plane Q then consists of those points (u, v, w) of space for which $pu + qv = 1$, and the circle Z, being the intersection of the sphere S and the plane Q consists of those points (u, v, w) of space for which $u^2 + v^2 + w^2 = 1$ and $pu + qv = 1$.

Now the distance of all points of the circle Z from the point F is equal to $\sqrt{p^2 + q^2 - 1}$ (see Corollary 1.12). Accordingly the radius R of the circle W is determined by the equation $R^2 = p^2+q^2-1$. It follows from Corollary 1.9 that the circle W is the image, under stereographic projection from the point A, of the circle Z at which the polar plane Q of the point F intersects the sphere S, as required. $\blacksquare$

Proposition 1.15 *Let S be a sphere, let A and B be endpoints of a diameter of that sphere, and let P be the plane perpendicular to the diameter AB that passes through the centre C of the sphere S. Let F_1 and F_2 be points of the plane P that lie outside the sphere S, let Q_1 and Q_2 be the polar planes of F_1 and F_2 respectively with respect to the sphere S, and let Z_1 and Z_2 be the vertical circles along which the polar planes Q_1 and Q_2 intersect the sphere S. Let W_1 and W_2 be the circles in the plane P that are the images of the circles Z_1 and Z_2 under stereographic projection from the point A. Suppose that the circles Z_1 and Z_2 intersect at a point D of the sphere, and let E be the image of the point D under stereographic projection from the point A. Then the angle between the tangent lines to the circles Z_1 and Z_2 at the point D is equal to the corresponding angle between the tangent lines to the circles W_1 and W_2 at the point E.*

Proof It follows from Proposition 1.14 that stereographic projection from the point A maps the circles Z_1 and Z_2 onto circles W_1 and W_2 with centres F_1 and F_2 respectively.

Now the angle between the tangent lines to the circles W_1 and W_2 at the point E is equal to the angle F_1EF_2 between the lines joining the point E to the centres F_1 and F_2 of those circles, because the tangent lines are perpendicular to the lines joining E to the centres of the circles W_1 and W_2.

We next show that the lines DF_1 and DF_2 are perpendicular to the tangent lines of the circles Z_1 and Z_2 at the point D. Let G_1 be the centre of the circle Z_1. Then DG_1 is perpendicular to the tangent line to the circle Z_1 at D. Also the point G_1 lies on the axis of the circle, and that axis also passes through the point F_1. It follows that the line G_1F_1 is perpendicular to the plane Q_1 and is therefore perpendicular to the direction of the tangent line to the circle Z_1 at D. Two sides DG_1 and G_1F_1 of the triangle DG_1F_1 are therefore perpendicular to the direction of the tangent line to the circle Z_1 at the point D. The same is therefore true of the third side DF_1. Similarly the line DF_2 is perpendicular to the tangent line to the circle Z_2 at the point D.

Next we note that the tangent lines to the circles Z_1 and Z_2 at the point D and the lines DF_1 and DF_2 are all contained in the tangent plane to the sphere S at the point D. Moreover the line DF_1 is perpendicular to the tangent line to the circle Z_1 at D and the line DF_2 is perpendicular to the tangent line to the circle Z_2 at D. It follows that the angle between the tangent lines to the circles Z_1 and Z_2 at the point D is equal to the angle F_1DF_2.

Now the points of the circle Z_1 are equidistant from the point F_1, and so are the points of the circle W_1. Moreover the circles Z_1 and W_1 intersect at the points at which the circle Z_1 intersects the plane P. It follows that $|DF_1| = |EF_1|$ and $|DF_2| = |EF_2|$. The three sides of the triangle F_1DF_2 are thus equal to the corresponding three sides of the triangle F_1EF_2. It then follows from the SSS Congruence Rule (Euclid, *Elements*, I, 8) that the angles F_1DF_2 and F_2EF_2 are equal. But we have shown that the first of these two angles is the angle between the tangent lines to the circles Z_1 and Z_2 at the point D and the second is the angle between the tangent lines to the circles W_1 and W_2 at the point E. The result follows. ∎