Course MAU23203—Michaelmas Term 2022.
Assignment 2 — Worked Solutions.

1. *Let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function from $\mathbb{R}^2$ to $\mathbb{R}$ and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined so that $f(x,y) = \sqrt{x^2+y^2}\, g(x,y)$ for all $(x,y) \in \mathbb{R}^2$. Also suppose that $g(0,0) \neq 0$. Prove that the function f is not differentiable at the origin $(0,0)$.*

   Let $c = g(0,0)$. Suppose first that $c > 0$. Then there exists some positive real number δ such that $g(x,y) > \frac{1}{2}c$ whenever $\sqrt{x^2+y^2} < \delta$. Then $f(x,0) \geq \frac{1}{2}c|x|$ whenever $|x| < \delta$. If the partial derivative of $f(x,y)$ with respect to x were to exist at $(0,0)$, then it would be the case that

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{(x,y)=(0,0)} = \lim_{h\to 0+} \frac{f(h,0)}{h} = \lim_{h \to 0-} \frac{f(h,0)}{h}.$$

   But the inequality above would ensure that

$$\lim_{h\to 0+} \frac{f(h,0)}{h} \geq \tfrac{1}{2}c \quad \text{and} \quad \lim_{h\to 0-} \frac{f(h,0)}{h} \leq -\tfrac{1}{2}c.$$

   These one-sided limits would therefore be unequal, one being strictly positive and the other strictly negative, and thus a contradiction to the assumption of differentiability at $(0,0)$ has been established. Consequently the function f cannot be differentiable at $(0,0)$ in the case when $c > 0$. Replacing f by $-f$, we conclude that the function f cannot be differentiable at $(0,0)$ in the case when $c < 0$. Thus the function f cannot be differentiable at $(0,0)$ in all cases where $c \neq 0$.

2. *Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be the function of three real variables defined such that $f(0,0,0) = 0$ and*

$$f(x,y,z) = \frac{xy+yz+zx}{\sqrt{x^2+y^2+z^2}}$$

   for all points $(x,y,z) \in \mathbb{R}^3$ that are distinct from the origin $(0,0,0)$. Determine whether or not the function f is differentiable at the origin $(0,0,0)$, appropriately justifying your answer.

The function f is not differentiable at $(0,0,0)$. Note that $f(x,y,z) = 0$ when $y = z = 0$, when $z = x = 0$ and when $x = y = 0$. Consequently if the function f were to be differentiable at $(0,0,0)$ then the first order partial derivatives of this function at $(0,0,0)$ would all be zero, and therefore, in view of the definition of differentiability, it would be the case that

$$\lim_{(x,y,z)\to(0,0,0)} \frac{f(x,y,z)}{\sqrt{x^2+y^2+z^2}} = 0.$$

Consequently

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+zx}{x^2+y^2+z^2} = 0.$$

But it is clear that

$$\frac{xy+yz+zx}{x^2+y^2+z^2} = 1$$

whenever $x = y = z$ and $(x,y,z) \neq (0,0,0)$. Consequently it cannot be the case that

$$\lim_{(x,y,z)\to(0,0,0)} \frac{f(x,y,z)}{\sqrt{x^2+y^2+z^2}} = 0,$$

and therefore the function f is not differentiable at $(0,0,0)$.