Course MAU23203—Michaelmas Term 2022. Worked Solutions.

1. *Let n be a positive integer, and let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a bounded infinite sequence of points in $\mathbb{R}^n$. Suppose that the infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ does* not *converge to $\mathbf{0}$, where $\mathbf{0} = (0, 0, \ldots, 0)$. Prove that there exists a subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ of the given infinite sequence which converges to some point $\mathbf{p}$ satisfying the condition $\mathbf{p} \neq \mathbf{0}$.*

   [The required result may be proved by an application of the standard definition of convergence for infinite sequences in a Euclidean space, used in conjunction with a result, applicable to bounded sequences in Euclidean spaces, that is proved in the course notes.]

   The infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ does not converge to $\mathbf{0}$. Therefore there exists some positive real number ε with the property that, for all positive integers N, there exists some integer j satisfying $j \geq N$ for which $|\mathbf{x}_j| \geq \varepsilon$. The number of positive integers j for which $|\mathbf{x}_j| \geq \varepsilon$ is therefore infinite. These values of j are then the indices of a subsequence of the given subsequence. This subsequence must then have a convergent subsequence. Let $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ be that convergent subsequence, and the point $\mathbf{p}$ be the limit of that subsequence. Then $\mathbf{x}_{k_j} \in F$ for all positive integers j, where

   $$F = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \geq \varepsilon\}.$$

   Now the set F is a closed subset of $\mathbb{R}^n$. It follows that $\mathbf{p} \in F$, where

   $$\mathbf{p} = \lim_{j \to +\infty} \mathbf{x}_{k_j}.$$

   But $\mathbf{0} \notin F$. Consequently $\mathbf{p} \neq \mathbf{0}$. The required result has therefore been established.

2. *Throughout this question, let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the real-valued function on $\mathbb{R}^2$ defined such that*

   $$f(x, y) = \begin{cases} \dfrac{x^2}{y} & \textit{whenever } y \neq 0; \\ 0 & \textit{whenever } y = 0. \end{cases}$$

   (a) *Is it the case that, for all $(u, v) \in \mathbb{R}^2$,*

   $$\lim_{t \to 0} f(tu, tv) = 0?$$

   [Your answer should be appropriately justified.]

Yes. If $v = 0$ then $f(tu, tv) = 0$ for all real numbers t and therefore $\lim_{t \to 0} f(tu, tv) = 0$. If $v \neq 0$ then $f(tu, tv) = tu^2/v$, and thus $f(tu, tv)$ is a constant multiple of the variable t. Consequently $\lim_{t \to 0} f(tu, tv) = 0$ in the case where $v \neq 0$.

(b) *Is it the case that that function f is continuous at $(0, 0)$? [Your answer should be appropriately justified.]*

No. To prove that the function f is not continuous at $(0, 0)$ let $h: \mathbb{R} \to \mathbb{R}^2$ be the continuous function defined such that $h(t) = (t, t^2)$ for all real numbers t. Then $f(h(t)) = 1$ whenever $t \neq 0$, but $f(h(0)) = 0$. Consequently the composition function $f \circ h$ not continuous at 0. If the function f were continuous at $(0, 0)$ then the composition function $f \circ h$ would be continuous at 0). But this is not the case. Therefore the function f cannot be continuous at $(0, 0)$.

ALTERNATIVE SOLUTION. Choose $\varepsilon > 0$. Then given any $\delta > 0$, we can take $x = \frac{1}{2}\delta$ and then choose y satisfying $0 < y < \frac{1}{2}\delta$ close enough to zero to ensure that $x^2/y > \varepsilon$. Then $|(x, y) - (0, 0)| < \delta$ but $f(x, y) > \varepsilon$. Consequently there cannot exist any positive real number δ that is such as to ensure that $|f(x, y)| < \varepsilon$ for all $(x, y) \in \mathbb{R}^2$ satisfying $|(x, y) - (0, 0)| < \delta$. Consequently the function f is not continuous at $(0, 0)$.