MAU23203: Analysis in Several Real Variables
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Disquisition IV: Further Examples concerning Open And Closed Sets

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Example Consider the set

$$\{(x,y,z) \in \mathbb{R}^3 : \sin(x^2+y^2) < \cos(y^2+z^2)\}.$$

This set is open. Indeed function $f\colon \mathbb{R}^3 \to \mathbb{R}$ defined such that

$$f(x,y,z) = \cos(y^2+z^2) - \sin(x^2+y^2)$$

for all $(x,y,z) \in \mathbb{R}^3$ is continuous, and the given set is the preimage of the set of positive real numbers under this function, and thus is the preimage of an open set under a continuous function. The set is not closed. The point $(0, \frac{1}{2}\sqrt{\pi}, 0)$ does not belong to the set, but every open ball of positive radius about this point contains points $(0,y,0)$ where $y < \frac{1}{2}\sqrt{\pi}$, and such points do not belong to the set.

We conclude therefore that the set

$$\{(x,y,z) \in \mathbb{R}^3 : \sin(x^2+y^2) < \cos(y^2+z^2)\}.$$

is open, and not closed.

Example Consider the set

$$\{(x,y,z) \in \mathbb{R}^3 : z > 0 \text{ and } z(x^2+y^2) = 1\}.$$

This set is closed. Indeed can be written in the form

$$\{(x,y,z) \in \mathbb{R}^3 : z \geq 0 \text{ and } z(x^2+y^2) = 1\}.$$

The set $\{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$ is

$$\{(x, y, z) \in \mathbb{R}^3 : z(x^2 + y^2) = 1\}$$

is also a closed set as it is the preimage of the closed set $\{1\}$ under a continuous function of x, y and z. The intersection of these two closed sets is the given set. The set is not open: the point $(1, 0, 1)$ belongs to the set, but no open ball of positive radius about this point is contained in the set.

We conclude therefore that the set

$$\{(x, y, z) \in \mathbb{R}^3 : z > 0 \text{ and } z(x^2 + y^2) = 1\}.$$

is closed, and not open.

Example Let $g: \mathbb{R} \to \mathbb{R}$ and $h: \mathbb{R} \to \mathbb{R}$ be continuous real-valued functions on $\mathbb{R}$. Consider the set

$$\{(x, y) \in \mathbb{R}^2 : g(x) < y < h(x)\}$$

We show that this set is open in $\mathbb{R}^2$. To this end, let $\varphi: \mathbb{R}^2 \to \mathbb{R}$ and $\psi: \mathbb{R}^2 \to \mathbb{R}$ be defined so that $\varphi(x, y) = y - g(x)$ and $\psi(x, y) = h(x) - y$ for all $(x, y) \in \mathbb{R}^2$. Also let $P = \{t \in \mathbb{R} : t > 0\}$. Then P is open in $\mathbb{R}$. Now the set X in question satisfies $X = X_1 \cap X_2$, where

$$\begin{aligned} X_1 &= \{(x, y) \in \mathbb{R}^3 : g(x) < y\} = \{(x, y) \in \mathbb{R}^3 : \varphi(x, y) > 0\} = \varphi^{-1}(P), \\ X_2 &= \{(x, y) \in \mathbb{R}^3 : y < h(x)\} = \{(x, y) \in \mathbb{R}^3 : \psi(x, y) > 0\} = \psi^{-1}(P), \end{aligned}$$

It follows from Proposition 5.7 that the sets X_1 and X_2 are open in $\mathbb{R}^2$. Thus the given set X is the intersection of two open sets, and is thus itself open in $\mathbb{R}^2$.

Example Consider the subset Y of $\mathbb{R}^2$ defined such that

$$Y = \{(x, y) \in \mathbb{R}^2 : \text{there exists } n \in \mathbb{Z} \text{ such that } (x - n)^2 + y^2 < 1\}.$$

(In other words, a point (x, y) of $\mathbb{R}^2$ belongs to Y if and only if some integer n, depending on the values of x and y, can be found for which $(x-n)^2+y^2 < 1$.)

The set Y is open in $\mathbb{R}^2$. It is the union of open balls centred on the points $(n, 0)$ as n ranges over the set $\mathbb{Z}$ of integers. Any open ball is an open set, and any union of open sets is an open set. Therefore the set Y is an open set in $\mathbb{R}^2$.

The set Y is not closed in $\mathbb{R}^2$. The point $(0,1)$ belongs to the complement of the set Y, but any open ball of radius δ about this point contains points $(0,u)$ of $\mathbb{R}^2$ with $0 \leq u < 1$ and $u > 1-\delta$, and such points do belong to the set Y. Thus no open ball of positive radius centred on the point $(0,1)$ is contained within the complement of the set Y. Thus the complement $\mathbb{R}^2 \setminus Y$ of Y is not open, and therefore the set Y itself is not closed.

Example Consider the subset Z of $\mathbb{R}^2$ defined such that

$$Z = \{(x,y) \in \mathbb{R}^2 : \text{there exists } n \in \mathbb{Z} \text{ such that } (x-n)^2 + y^2 \leq 1\}.$$

The set Z is not open in $\mathbb{R}^2$. Indeed point $(0,1)$ belongs to the set Z, but any open ball of radius δ about this point contains points $(0,u)$ of $\mathbb{R}^2$ with $0 \leq u < 1$ and $u > 1-\delta$, and such points do not belong to the set Z. Thus no open ball of positive radius centred on the point $(0,1)$ is contained within the set Z.

The set Z is closed in $\mathbb{R}^2$. Let (u,v) be a point of the complement of Z. An integer M can be found large enough to ensure that $-M+2 < u < M-2$. If n is an integer satisfying $|n| \geq M$ then the closed ball of radius 1 about the point $(n,0)$ does not intersect the open ball of radius 1 about the point (u,v). Let F be the union of the closed balls of radius 1 about the points $(n,0)$ for which n is an integer satisfying $|n| < M$. The set F is then a finite union of closed balls. Moreover closed balls are closed sets, and any finite union of closed sets is closed. It follows that F is a closed set. Therefore some real number δ satisfying $0 < \delta < 1$ can be found so that the open ball of radius δ about (u,v) does not intersect the set F. Then the open ball of radius δ about (u,v) does not intersect the set Z. This shows that the complement of the set Z is open, and therefore the set Z itself is closed in $\mathbb{R}^2$.

Proposition A. *A subset X of $\mathbb{R}^n$ is closed in $\mathbb{R}^n$ if and only if the limit of every convergent sequence of points in X belongs to X.*

Proof First let X be a closed set in $\mathbb{R}^n$. Then the limit of every convergent sequence of points of X belongs to the set X (see Lemma 4.7).

Now let X be a subset of $\mathbb{R}^n$ with the property that, for every convergent sequence of points of X, the limit of that sequence belongs to the set X. Let Let $\mathbf{p}$ be a point of $\mathbb{R}^n$ that does not belong to the set X. Suppose that the complement of the set $\mathbb{R}^n$ were not an open set. Suppose that there did not exist any positive number δ with the property that the open ball of radius δ centred on the point $\mathbf{p}$ was wholly contained within the complement of the set X. Then every open ball of positive radius centered on the point $\mathbf{p}$ would have non-empty intersection with the set X, and therefore there would exist an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of the set X with the property that $|\mathbf{x}_j - \mathbf{p}| < 1/j$ for all positive integers j. This infinite sequence would converge to the point $\mathbf{p}$. But the point $\mathbf{p}$ has been chosen so that it lies outside the set X, and the set X in question has the property that every convergent infinite sequence within it converges to a point of the set X. Thus a contradiction would arise unless there exists some strictly positive real number δ with the property that the open ball of radius δ centred on the point $\mathbf{p}$ is wholly contained within the complement of the set X. This argument establishes that the complement of the set X is open in $\mathbb{R}^n$, because, given any point of the complement of X, there exists some open ball of positive radius centred on that point that is wholly contained within the complement of the set X. Consequently the set X is closed in $\mathbb{R}^n$, as required. ∎

Example We resume discussion of the subset Z of $\mathbb{R}^2$ defined so that

$$Z = \{(x, y) \in \mathbb{R}^2 : \text{there exists } n \in \mathbb{Z} \text{ such that } (x - n)^2 + y^2 \leq 1\}.$$

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a convergent sequence of points in Z, and let $\mathbf{p}$ be the limit of this sequence. We show that $\mathbf{p} \in Z$. Now the infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is bounded, because all convergent sequences are bounded. Therefore there exists a positive integer M such that $\mathbf{x}_j \in \bigcup_{n=-M}^{M} \overline{B}((n, 0), 1)$, where $\overline{B}((n, 0), 1)$ denotes the closed ball of radius 1 about the point $(n, 0)$. Now closed balls are closed sets, and therefore $\bigcup_{n=-M}^{M} \overline{B}((n, 0), 1)$, being a finite union of closed sets must itself be a closed set. It follows from this that $\mathbf{p} \in \bigcup_{n=-M}^{M} \overline{B}((n, 0), 1)$, and therefore $\mathbf{p} \in Z$. We conclude therefore that the limit of every convergent sequence of points belonging to Z must itself belong to the set Z. Applying Proposition A above, we conclude that (as was shown above by a different method) the set Z is closed in $\mathbb{R}^2$.