Course MAU23203—Michaelmas Term 2021.
Worked Solutions.

1. *Let X be a subset of n-dimensional Euclidean space, let $f\colon X \to \mathbb{R}$ and $h\colon X \to \mathbb{R}$ be real-valued functions on $\mathbb{R}$, and let $\mathbf{p}$ be a limit point of the set X. We say that the function h* remains bounded *as the point $\mathbf{x}$ tends to the point $\mathbf{p}$ in X if there exist positive constants M and δ such that $|h(\mathbf{x})| \leq M$ for all points $\mathbf{x}$ of the set X that satisfy the condition $0 < |\mathbf{x} - \mathbf{p}| < \delta$.*

   Suppose that $\lim\limits_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = 0$ and that the function h remains bounded as the point $\mathbf{x}$ tends to the point $\mathbf{p}$ in the set X. Prove that

   $$\lim_{\mathbf{x}\to\mathbf{p}} (h(\mathbf{x})f(\mathbf{x})) = 0.$$

   Let some positive real number ε be given. Now there exist positive real numbers M and δ_1 with the property that $|h(\mathbf{x})| \leq M$ for all points $\mathbf{x}$ of X that satisfy $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$. There then exists some positive real number δ_2 with the property that $|f(\mathbf{x})| < \varepsilon/M$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. If $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then $|h(\mathbf{x})f(\mathbf{x})| < \varepsilon$. Consequently $\lim\limits_{\mathbf{x}\to\mathbf{p}} (h(\mathbf{x})f(\mathbf{x})) = 0$, as required.

2. *Let X be an open set in m-dimensional space $\mathbb{R}^m$, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be real-valued functions on X, and let $\mathbf{p}$ be a point belonging to the set X. Also let $u\colon X \to \mathbb{R}$ and $v\colon X \to \mathbb{R}$ be the real-valued functions on X $u(\mathbf{p}) = 0$, $v(\mathbf{p}) = 0$,*

   $$f(\mathbf{x}) = f(\mathbf{p}) + (\nabla f)_{\mathbf{p}} \,.\, (\mathbf{x} - \mathbf{p}) + |\mathbf{x} - \mathbf{p}|\, u(\mathbf{x})$$

   for all $\mathbf{x} \in X$ and

   $$g(\mathbf{x}) = g(\mathbf{p}) + (\nabla g)_{\mathbf{p}} \,.\, (\mathbf{x} - \mathbf{p}) + |\mathbf{x} - \mathbf{p}|\, v(\mathbf{x})$$

   for all $\mathbf{x} \in X$. Let $w\colon X \to \mathbb{R}$ be the real-valued function on X that is uniquely characterized by the properties that $w(\mathbf{p}) = 0$ and

   $$\begin{aligned} f(\mathbf{x})g(\mathbf{x}) &= f(\mathbf{p})g(\mathbf{p}) + g(\mathbf{p})\,(\nabla f)_{\mathbf{p}} \,.\, (\mathbf{x} - \mathbf{p}) + f(\mathbf{p})\,(\nabla g)_{\mathbf{p}} \,.\, (\mathbf{x} - \mathbf{p}) \\ &\quad + |\mathbf{x} - \mathbf{p}|\, w(\mathbf{x}). \end{aligned}$$

   Now the definition of differentiability ensures that the function f is differentiable at the point $\mathbf{p}$ if and only if $\lim\limits_{\mathbf{x}\to\mathbf{p}} u(\mathbf{x}) = 0$. Similarly the function g is differentiable at the point $\mathbf{p}$ if and only if $\lim\limits_{\mathbf{x}\to\mathbf{p}} v(\mathbf{x}) = 0$.

Suppose that the functions f and g are differentiable at the point p. Find a formula that expresses the the value of the function w at each point $\mathbf{x}$ of X in terms of the points $\mathbf{x}$ and $\mathbf{p}$, the functions f, g u, v, and the gradients $(\nabla f)_{\mathbf{p}}$ and $(\nabla g)_{\mathbf{p}}$ of the functions f and g at the point $\mathbf{p}$. Then prove that $\lim_{\mathbf{x}\to\mathbf{p}} w(\mathbf{x}) = 0$. (This result ensures that the product of the functions f and g is differentiable at the point $\mathbf{p}$, with gradient $g(\mathbf{p})\,(\nabla f)_{\mathbf{p}} + f(\mathbf{p})\,(\nabla g)_{\mathbf{p}}$.)

Multiplying out, we find that

$$\begin{aligned} f(\mathbf{x})g(\mathbf{x}) &= f(\mathbf{p})g(\mathbf{p}) + g(\mathbf{p})\,(\nabla f)_{\mathbf{p}}\,.\,(\mathbf{x}-\mathbf{p}) + f(\mathbf{p})\,(\nabla g)_{\mathbf{p}}\,.\,(\mathbf{x}-\mathbf{p}) \\ &\quad + |\mathbf{x}-\mathbf{p}|\,g(\mathbf{p})u(\mathbf{x}) + |\mathbf{x}-\mathbf{p}|\,f(\mathbf{p})v(\mathbf{x}) \\ &\quad + |\mathbf{x}-\mathbf{p}|\,v(\mathbf{x})\,(\nabla f)_{\mathbf{p}}\,.\,(\mathbf{x}-\mathbf{p}) \\ &\quad + |\mathbf{x}-\mathbf{p}|\,u(\mathbf{x})\,(\nabla g)_{\mathbf{p}}\,.\,(\mathbf{x}-\mathbf{p}) \\ &\quad + |\mathbf{x}-\mathbf{p}|^2 u(\mathbf{v})v(\mathbf{x}) \\ &\quad + ((\nabla f)_{\mathbf{p}}\,.\,(\mathbf{x}-\mathbf{p}))\,(\,(\nabla g)_{\mathbf{p}}\,.\,(\mathbf{x}-\mathbf{p}))\,. \end{aligned}$$

Consequently

$$\begin{aligned} w(\mathbf{x}) &= g(\mathbf{p})u(\mathbf{x}) + f(\mathbf{p})v(\mathbf{x}) \\ &\quad + v(\mathbf{x})\,(\nabla f)_{\mathbf{p}}\,.\,(\mathbf{x}-\mathbf{p}) \\ &\quad + u(\mathbf{x})\,(\nabla g)_{\mathbf{p}}\,.\,(\mathbf{x}-\mathbf{p}) \\ &\quad + |\mathbf{x}-\mathbf{p}|u(\mathbf{v})v(\mathbf{x}) \\ &\quad + \frac{1}{|\mathbf{x}-\mathbf{p}|}\,((\nabla f)_{\mathbf{p}}\,.\,(\mathbf{x}-\mathbf{p}))\,(\,(\nabla g)_{\mathbf{p}}\,.\,(\mathbf{x}-\mathbf{p}))\,. \end{aligned}$$

Now Schwarz's Inequality ensures that

$$|(\nabla f)_{\mathbf{p}}\,.\,(\mathbf{x}-\mathbf{p})| \le |(\nabla f)_{\mathbf{p}}|\,|\mathbf{x}-\mathbf{p})|$$

$$|(\nabla g)_{\mathbf{p}}\,.\,(\mathbf{x}-\mathbf{p})| \le |(\nabla g)_{\mathbf{p}}|\,|\mathbf{x}-\mathbf{p})|.$$

Consequently

$$\frac{1}{|\mathbf{x}-\mathbf{p}|}\,|(\nabla f)_{\mathbf{p}}\,.\,(\mathbf{x}-\mathbf{p})|\,|\,(\nabla g)_{\mathbf{p}}\,.\,(\mathbf{x}-\mathbf{p})| \le |(\nabla f)_{\mathbf{p}}|\,|(\nabla g)_{\mathbf{p}}|\,|\mathbf{x}-\mathbf{p})|,$$

and therefore

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{1}{|\mathbf{x}-\mathbf{p}|}\,((\nabla f)_{\mathbf{p}}\,.\,(\mathbf{x}-\mathbf{p}))\,(\,(\nabla g)_{\mathbf{p}}\,.\,(\mathbf{x}-\mathbf{p})) = 0.$$

Also

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{p})u(\mathbf{x}) &= g(\mathbf{p}) \lim_{\mathbf{x}\to\mathbf{p}} u(\mathbf{x}) = 0, \\
\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{p})v(\mathbf{x}) &= f(\mathbf{p}) \lim_{\mathbf{x}\to\mathbf{p}} v(\mathbf{x}) = 0, \\
\lim_{\mathbf{x}\to\mathbf{p}} v(\mathbf{x})\,(\nabla f)_{\mathbf{p}} \,.\, (\mathbf{x}-\mathbf{p}) &= \lim_{\mathbf{x}\to\mathbf{p}} v(\mathbf{x}) \lim_{\mathbf{x}\to\mathbf{p}} ((\nabla f)_{\mathbf{p}} \,.\, (\mathbf{x}-\mathbf{p}))) = 0, \\
\lim_{\mathbf{x}\to\mathbf{p}} u(\mathbf{x})\,(\nabla g)_{\mathbf{p}} \,.\, (\mathbf{x}-\mathbf{p}) &= \lim_{\mathbf{x}\to\mathbf{p}} u(\mathbf{x}) \lim_{\mathbf{x}\to\mathbf{p}} ((\nabla g)_{\mathbf{p}} \,.\, (\mathbf{x}-\mathbf{p}))) = 0, \\
\lim_{\mathbf{x}\to\mathbf{p}} (|\mathbf{x}-\mathbf{p}|u(\mathbf{v})v(\mathbf{x})) &= \left(\lim_{\mathbf{x}\to\mathbf{p}} |\mathbf{x}-\mathbf{p}|\right) \left(\lim_{\mathbf{x}\to\mathbf{p}} u(\mathbf{v})\right) \left(\lim_{\mathbf{x}\to\mathbf{p}} v(\mathbf{x})\right) \\
&= 0.
\end{aligned}$$

Consequently $\lim_{\mathbf{x}\to\mathbf{p}} w(\mathbf{x}) = 0$, as required.