Course MAU23203—Michaelmas Term 2021.
Worked Solutions.

1. *Let X be a subset of n-dimensional Euclidean space. A subset W of X is open in X if and only if there exists some open set V in $\mathbb{R}^n$ for which $W = V \cap X$ (see Proposition 4.5 in the MAU23203 course notes for Michaelmas Term 2021). A subset G of X is said to be* closed in X *if and only if its complement $X \setminus G$ in X is open in X. (Here the complement $X \setminus G$ consists of all points of the set X that do not belong to the set G.)*

   (a) *Prove that a subset G of X is closed in X if and only if there exists a closed set F in $\mathbb{R}^n$ for which $G = F \cap X$. [Your proof should be logically accurate, and ought not to exceed ten carefully-chosen sentences.]*

   First suppose that G is closed in X. Then $X \setminus G$ is open in X and consequently (by a proposition proved in the course notes) there exists some subset V of X open in X for which $X \setminus G = X \cap V$. Let $F = \mathbb{R}^n \setminus V$. Then F is closed in $\mathbb{R}^n$, and

   $$X \cap F = X \cap (\mathbb{R}^n \setminus V) = X \setminus V = X \setminus (X \cap V) = X \setminus (X \setminus G) = G.$$

   Now suppose that G is a subset of X with the property that there exists a closed set F in $\mathbb{R}^n$ for which $G = X \cap F$. Let $V = \mathbb{R}^n \setminus F$. Then $X \cap V = X \setminus F = X \setminus G$, and moreover $X \cap V$ is open in X. Then $X \setminus G = X \cap V$, and thus $X \setminus G$ is open in X, and therefore G is closed in X, as required.

   (b) *Give a short proof, in no more than six sentences, of the result that if X is a closed set in $\mathbb{R}^n$, and if G is a subset of X that is closed in X, then the set G is closed in n-dimensional Euclidean space $\mathbb{R}^n$.*

   It follows from the result of (a) that there exists some subset F of $\mathbb{R}^n$ that is closed in $\mathbb{R}^n$ and satisfies $X \cap F = G$. It then follows that the set G is the intersection of two subsets of $\mathbb{R}^n$ that are closed in $\mathbb{R}^n$, and therefore the set G is itself closed in $\mathbb{R}^n$.

(c) *Let X be a closed set in $\mathbb{R}^n$ and let $f\colon X \to \mathbb{R}$ be a real-valued function on X that is continuous on X. Give a short proof, in no more than six sentences that, for any real number c, the set*

$$\{\mathbf{x} \in X : f(\mathbf{x}) \geq c\}$$

is a closed set in n-dimensional space $\mathbb{R}^n$. [You may apply, without proof, the result of any lemma, proposition, theorem or corollary in the MAU23203 course notes for Michaelmas Term 2021. Indeed, where you can apply directly a result that is already proved in the notes, you should not, in the context of a homework assignment, incorporate the steps of the given proof included in the course notes into your own proof.]

Note that

$$f^{-1}(\{t \in \mathbb{R} : t > c\}) = X \setminus A,$$

where $A = \{\mathbf{x} \in X : f(\mathbf{x}) \geq c\}$, and consequently the complement $X \setminus A$ of A in X is open in X (by an immediate application of a proposition proved in the course notes). It follows that the set A is closed in X. But X is itself closed in $\mathbb{R}^n$. It now follows, on applying the result of (b), that A is closed in $\mathbb{R}^n$, as required.

2. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a real-valued function defined over the set of real numbers, and let v be a real number. We say that the real number v is the* limit *of the function f as t increases to infinity, and write $\lim\limits_{t\to+\infty} f(t) = v$, if, given any positive real number ε, there exists some real number L that is large enough to ensure that $|f(t) - v| < \varepsilon$ for all real numbers t for which $t > L$.*

   Now Z be the subset of two-dimensional space $\mathbb{R}^2$ defined so that

$$Z = \{(0,0)\} \cup \{(e^{-t}\cos t, e^{-t}\sin t\} : t \in \mathbb{R}\}.$$

   (Thus the set Z is the union of the singleton set consisting of the origin and the set of points that lie on a given spiral in the plane that swirls in towards the origin.) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a real-valued function on the set of real numbers, let v be a real number, and let $g\colon Z \to \mathbb{R}$ be the real-valued function on the set Z defined so that $g(0,0) = v$ and $g(e^{-t}\cos t, e^{-t}\sin t) = f(t)$ for all real numbers t. Prove that the function g is continuous at the origin $(0,0)$ if and only if the real number v is the limit of $f(t)$ is t increases to infinity.

First note that

$$|(e^{-t}\cos t, e^{-t}\sin t)| = e^{-t}$$

for all real numbers t.

Now suppose that $v = \lim_{t \to +\infty} f(t)$. Let some positive real number ε be given. Then there exists some real number L such that $|f(t) - v| < \varepsilon$ whenever $t > L$. Set $\delta = e^{-L}$. Then

$$|g(x, y) - g(0, 0)| < \delta$$

for all $(x, y) \in Z$ for which $|(x, y)| < \delta$, and therefore the function g is continuous at $(0, 0)$.

Conversely suppose that the function g is continuous at $(0, 0)$. Let some positive real number ε be given. Then there exists some positive real number δ such that $|g(x, y) - v| < \varepsilon$ whenever $(x, y) \in Z$ satisfies $|(x, y)| < \delta$, where $v = g(0, 0)$. Let $L = -\log \delta$. Then $|f(t) - v| < \varepsilon$ whenever $t > L$, and therefore $v = \lim_{t \to +\infty} f(t)$.

We conclude that $v = \lim_{t \to +\infty} f(t)$ if and only if the function g is continuous at $(0, 0)$, as required.