# Course MAU23203
# Analysis in Several Real Variables.
# Michaelmas Term 2021.
# Assignment 2.

**To be submitted on Blackboard on or before 11pm on Tuesday 30th November, 2021.**

Assignment submissions should be uploaded to Blackboard as a *single* file in PDF format. (Thus submissions should neither be constituted of multiple files, nor of photographic image files in a format such as JPEG or PNG.) It is recommended that a suitable scanner app be used.

Students are advised to retain a copy of their submission. It is anticipated that the assignment submissions would not be printed at any stage. Nor would they be annotated with hand-written comments: any feedback would most likely take the form of marks and comments forwarded separately to students to be interpreted in conjunction with copies of assignment submissions in their possession.

**Students are reminded that they must comply with College policies with regard to plagiarism, which are published on the website located at the following URL:**

`http://tcd-ie.libguides.com/plagiarism`

**Please complete the cover sheet on the back of this page and attach it to the front of your completed assignment script, in particular signing the declaration with regard to plagiarism. Please make sure also that you include both name and student number on work handed in.**

**At an alternative to printing off the plagiarism declaration, it is acceptable to copy the wording of the plagiarism declaration on a sheet of paper and include the signed copy of the declaration, with name and student number, as a page included with the submission.**

**Solutions to problems should be expressed in appropriately concise and correct logical language. Attempted solutions that are incoherent, unclear, overlong or logically confused are unlikely to gain substantial credit.**

**Module MAU22200—Analysis in Several Real Variables.**
**Michaelmas Term 2021.**
**Assignment 2.**

**Name (please print): . . . . . . . . . . . . . . . . . . . . . . . . . . . . .**

**Student number: . . . . . . . . . . . . . . . . . . . . . . . . . . . . .**

**Date submitted: . . . . . . . . . . . . . . . . . . . . . . . . . . . . .**

**I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar for the current year, found at**

`http://www.tcd.ie/calendar`

**I have also completed the Online Tutorial on avoiding plagiarism *Ready Steady Write*, located at**

`http://tcd-ie.libguides.com/plagiarism/ready-steady-write`

**Signed:**

**. . . . . . . . . . . . . . . . . . . . . . . . . . . . .**

## Course MAU23203: Michaelmas Term 2021.
## Assignment 2.

1. Let $X$ be a subset of $n$-dimensional Euclidean space, let $f\colon X \to \mathbb{R}$ and $h\colon X \to \mathbb{R}$ be real-valued functions on $\mathbb{R}$, and let $\mathbf{p}$ be a limit point of the set $X$. We say that the function $h$ *remains bounded* as the point $\mathbf{x}$ tends to the point $\mathbf{p}$ in $X$ if there exist positive constants $M$ and $\delta$ such that $|h(\mathbf{x})| \leq M$ for all points $\mathbf{x}$ of the set $X$ that satisfy the condition $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

   Suppose that $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = 0$ and that the function $h$ remains bounded as the point $\mathbf{x}$ tends to the point $\mathbf{p}$ in the set $X$. Prove that

   $$\lim_{\mathbf{x}\to\mathbf{p}} (h(\mathbf{x})f(\mathbf{x})) = 0.$$

2. Let $X$ be an open set in $m$-dimensional space $\mathbb{R}^m$, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be real-valued functions on $X$, and let $\mathbf{p}$ be a point belonging to the set $X$. Also let $u\colon X \to \mathbb{R}$ and $v\colon X \to \mathbb{R}$ be the real-valued functions on $X$ $u(\mathbf{p}) = 0$, $v(\mathbf{p}) = 0$,

   $$f(\mathbf{x}) = f(\mathbf{p}) + (\nabla f)_{\mathbf{p}} \,.\, (\mathbf{x} - \mathbf{p}) + |\mathbf{x} - \mathbf{p}|\, u(\mathbf{x})$$

   for all $\mathbf{x} \in X$ and

   $$g(\mathbf{x}) = g(\mathbf{p}) + (\nabla g)_{\mathbf{p}} \,.\, (\mathbf{x} - \mathbf{p}) + |\mathbf{x} - \mathbf{p}|\, v(\mathbf{x})$$

   for all $\mathbf{x} \in X$. Let $w\colon X \to \mathbb{R}$ be the real-valued function on $X$ that is uniquely characterized by the properties that $w(\mathbf{p}) = 0$ and

   $$\begin{aligned} f(\mathbf{x})g(\mathbf{x}) &= f(\mathbf{p})g(\mathbf{p}) + g(\mathbf{p})\,(\nabla f)_{\mathbf{p}} \,.\, (\mathbf{x} - \mathbf{p}) + f(\mathbf{p})\,(\nabla g)_{\mathbf{p}} \,.\, (\mathbf{x} - \mathbf{p}) \\ &\quad + |\mathbf{x} - \mathbf{p}|\, w(\mathbf{x}). \end{aligned}$$

   Now the definition of differentiability ensures that the function $f$ is differentiable at the point $\mathbf{p}$ if and only if $\lim_{\mathbf{x}\to\mathbf{p}} u(\mathbf{x}) = 0$. Similarly the function $g$ is differentiable at the point $\mathbf{p}$ if and only if $\lim_{\mathbf{x}\to\mathbf{p}} v(\mathbf{x}) = 0$.

   Suppose that the functions $f$ and $g$ are differentiable at the point $p$. Find a formula that expresses the the value of the function $w$ at each point $\mathbf{x}$ of $X$ in terms of the points $\mathbf{x}$ and $\mathbf{p}$, the functions $f$, $g$ $u$, $v$, and the gradients $(\nabla f)_{\mathbf{p}}$ and $(\nabla g)_{\mathbf{p}}$ of the functions $f$ and $g$ at the point $\mathbf{p}$. Then prove that $\lim_{\mathbf{x}\to\mathbf{p}} w(\mathbf{x}) = 0$. (This result ensures that the product of the functions $f$ and $g$ is differentiable at the point $\mathbf{p}$, with gradient $g(\mathbf{p})\,(\nabla f)_{\mathbf{p}} + f(\mathbf{p})\,(\nabla g)_{\mathbf{p}}$.)