Course MAU23203

Analysis in Several Real Variables.

Michaelmas Term 2021.

Assignment 1.

To be submitted on Blackboard on or before 11pm on Thursday 4th November, 2021.

Assignment submissions should be uploaded to Blackboard as a *single* file in PDF format. (Thus submissions should neither be constituted of multiple files, nor of photographic image files in a format such as JPEG or PNG.) It is recommended that a suitable scanner app be used.

Students are advised to retain a copy of their submission. It is anticipated that the assignment submissions would not be printed at any stage. Nor would they be annotated with hand-written comments: any feedback would most likely take the form of marks and comments forwarded separately to students to be interpreted in conjunction with copies of assignment submissions in their possession.

Students are reminded that they must comply with College policies with regard to plagiarism, which are published on the website located at the following URL:

`http://tcd-ie.libguides.com/plagiarism`

Please complete the cover sheet on the back of this page and attach it to the front of your completed assignment script, in particular signing the declaration with regard to plagiarism. Please make sure also that you include both name and student number on work handed in.

At an alternative to printing off the plagiarism declaration, it is acceptable to copy the wording of the plagiarism declaration on a sheet of paper and include the signed copy of the declaration, with name and student number, as a page included with the submission.

Solutions to problems should be expressed in appropriately concise and correct logical language. Attempted solutions that are incoherent, unclear, overlong or logically confused are unlikely to gain substantial credit.

Module MAU22200—Analysis in Several Real Variables.
Michaelmas Term 2021.
Assignment I.

Name (please print): . . . .

Student number: . . . .

Date submitted: . . . .

I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar for the current year, found at

`http://www.tcd.ie/calendar`

I have also completed the Online Tutorial on avoiding plagiarism *Ready Steady Write*, located at

`http://tcd-ie.libguides.com/plagiarism/ready-steady-write`

Signed:

. . . .

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1. Let X be a subset of n-dimensional Euclidean space. A subset W of X is open in X if and only if there exists some open set V in $\mathbb{R}^n$ for which $W = V \cap X$ (see Proposition 4.5 in the MAU23203 course notes for Michaelmas Term 2021). A subset G of X is said to be *closed in* X if and only if its complement $X \setminus G$ in X is open in X. (Here the complement $X \setminus G$ consists of all points of the set X that do not belong to the set G.)

   (a) Prove that a subset G of X is closed in X if and only if there exists a closed set F in $\mathbb{R}^n$ for which $G = F \cap X$. [Your proof should be logically accurate, and ought not to exceed ten carefully-chosen sentences.]

   (b) Give a short proof, in no more than six sentences, of the result that if X is a closed set in $\mathbb{R}^n$, and if G is a subset of X that is closed in X, then the set G is closed in n-dimensional Euclidean space $\mathbb{R}^n$.

   (c) Let X be a closed set in $\mathbb{R}^n$ and let $f\colon X \to \mathbb{R}$ be a real-valued function on X that is continuous on X. Give a short proof, in no more than six sentences that, for any real number c, the set

$$\{\mathbf{x} \in X : f(\mathbf{x}) \geq c\}$$

   is a closed set in n-dimensional space $\mathbb{R}^n$. [You may apply, without proof, the result of any lemma, proposition, theorem or corollary in the MAU23203 course notes for Michaelmas Term 2021. Indeed, where you can apply directly a result that is already proved in the notes, you should not, in the context of a homework assignment, incorporate the steps of the given proof included in the course notes into your own proof.]

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a real-valued function defined over the set of real numbers, and let v be a real number. We say that the real number v is the *limit* of the function f as t increases to infinity, and write $\lim_{t \to +\infty} f(t) = v$, if, given any positive real number ε, there exists some real number L that is large enough to ensure that $|f(t) - v| < \varepsilon$ for all real numbers t for which $t > L$.

   Now Z be the subset of two-dimensional space $\mathbb{R}^2$ defined so that

$$Z = \{(0,0)\} \cup \{(e^{-t} \cos t, e^{-t} \sin t\} : t \in \mathbb{R}\}.$$

(Thus the set Z is the union of the singleton set consisting of the origin and the set of points that lie on a given spiral in the plane that swirls in towards the origin.) Let $f: \mathbb{R} \to \mathbb{R}$ be a real-valued function on the set of real numbers, let v be a real number, and let $g: Z \to \mathbb{R}$ be the real-valued function on the set Z defined so that $g(0,0) = v$ and $g(e^{-t} \cos t, e^{-t} \sin t) = f(t)$ for all real numbers t. Prove that the function g is continuous at the origin $(0,0)$ if and only if the real number v is the limit of $f(t)$ is t increases to infinity.