MAU23203: Analysis in Several Real Variables
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Disquisition III: Examples relating to Limits and Continuity

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Example Consider the function $f: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

This function f is continuous at all points of $\mathbb{R}^2$ that are distinct from the origin $(0,0)$. Indeed it follows immediately from the definition of continuity that if the restriction of a function of several real variables to some open subset of its domain is continuous on that open subset, then the function itself is continuous at each point of that open subset. In the present example, the function f is expressed as the quotient of two continuous functions, the denominator being non-zero throughout the complement of the origin in $\mathbb{R}^2$. Consequently the restriction of the function f to the complement of the origin in $\mathbb{R}^2$ is continuous on that open set, and therefore the function f itself is continuous at all points of the complement of the origin in two-dimensional space $\mathbb{R}^2$.

Accordingly we investigate the behaviour of the function $f: \mathbb{R}^2 \to \mathbb{R}$ around the origin. To carry through such an investigation, it is advisable to seek to construct some mental picture of the qualitative behaviour of this function around the origin. Let us in particular investigate the behaviour of this function on a circle of radius r centred on the origin.

Now

$$-(x^2+y^2) \leq 2xy \leq x^2+y^2$$

for all real numbers x and y, because $(x-y)^2 \geq 0$ and $(x+y)^2 \geq 0$. Moreover $2xy = x^2+y^2$ if and only if $x = y$, and $2xy = -(x^2+y^2)$ if and only if

$x = -y$. It follows that the function f takes values between -1 and 1 on the circle of radius r for each positive real number r. Note that this function is not continuous at $(0,0)$. There are various ways of showing this. For this particular function, considering the behaviour of the function on a circle of positive radius centred on the origin, it should be clear that, whilst the function f has the value 0 at those points on such a circle where the circle crosses the coordinate axes $y = 0$ and $x = 0$, the function f takes the value 1 at those points where the circle crosses the line $x = y$, and takes the value -1 at those points where the circle crosses the line $x = -y$. Therefore, for this particular function, a formal proof that this function is not continuous at the origin can be obtained by considering the behaviour of the function along the line $x = y$.

The function f thus takes the value 1 at all points of the line $x = y$ with the exception of the point $(0,0)$, where the function takes the value 0. It should be pretty obvious that such a function could not possibly be continuous at $(0,0)$. We consider various way by which the existence of the discontinuity can be formally established.

We can establish the existence of the discontinuity of the function f at $(0,0)$ by applying the epsilon-delta criterion for continuity at this point with ε chosen to be equal to $\frac{1}{2}$. Given any positive real number δ, some real number t can be chosen for which $0 < t < \delta/\sqrt{2}$. Then $|(t,t) - (0,0)| < \delta$ but $|f(t,t) - f(0,0)| = 1$. Thus no positive real number δ can be chosen to ensure that $|f(x,y) - f(0,0)| < \frac{1}{2}$ whenever $|(x,y) - (0,0)| < \delta$. The function f therefore has a discontinuity at $(0,0)$.

It is not however necessary to apply the epsilon-delta criterion directly in order to show that the function f is discontinuous at $(0,0)$. Consider the infinite sequence $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots$ of points of $\mathbb{R}^2$, where $\mathbf{p}_j = \left(\frac{1}{j}, \frac{1}{j}\right)$ for all positive integers j. Then $f(\mathbf{p}_j) = 1$ for all positive integers j, and therefore $\lim_{j\to+\infty} f(\mathbf{p}_j) = 1$. But $\lim_{j\to+\infty} \mathbf{p}_j = (0,0)$ and $f(0,0) = 0$. Thus $\lim_{j\to+\infty} f(\mathbf{p}_j) \neq f\left(\lim_{j\to+\infty} \mathbf{p}_j\right)$. Consequently the function f must have a discontinuity at $(0,0)$.

One can also formally show the existence of the discontinuity as follows. Let $q\colon \mathbb{R} \to \mathbb{R}^2$ be the function defined so that $q(t) = (t,t)$ for all real numbers t. The function q is continuous. If the function f were continuous at $(0,0)$ then the composition function $f\colon q$ would be continuous at 0. But $f(q(0)) = 0$ and $f(q(t)) = 1$ for all non-zero real numbers t. Thus composition function then fails to be continuous at 0, and consquently the function f must have a discontinuity at $(0,0)$.

But, although the function has a discontinuity at the origin, the partial derivatives

$$\frac{\partial f(x,y)}{\partial x} \text{ and } \frac{\partial f(x,y)}{\partial y}$$

of the function f with respect to the variables x and y exist at all points of the plane $\mathbb{R}^2$. In particular those partial derivatives exist at the origin $(0,0)$.

The values of the partial derivatives of the function f can be determined as functions of x and y away from the origin by the usual methods of multivariable calculus. At the origin $(0,0)$ of Cartesian coordinates itself, we find that

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{(x,y)=(0,0)} = 0, \qquad \left.\frac{\partial f(x,y)}{\partial y}\right|_{(x,y)=(0,0)} = 0$$

on account of the fact that $f(x,0) = f(0,y) = 0$ for all $x, y \in \mathbb{R}$.

Example Consider the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(x,y) = \begin{cases} \dfrac{4xy}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Note that this function is not continuous at $(0,0)$. Indeed, for each positive real number r, the values of the function on the circle of radius r centred on the origin range between $-1/r^2$ and $1/r^2$, with the maximum value $1/r^2$ being achieved at those points on the circle of radius r centred on the origin at which $x = y$, and the minimum value $-1/r^2$ being achieved at those points on that circle at which $x = -y$.

Accordingly consider in particular the behaviour of the function g along the line $x = y$. Now $g(t,t) = 1/t^2$ for all non-zero real numbers t, and therefore $g(t,t) \to +\infty$ as $t \to 0$, yet $g(x,0) = g(0,y) = 0$ for all $x, y \in \mathbb{R}$. This function g accordingly has a discontinuity at $(0,0)$. Moreover the values of the function g on a circle of radius r take on all real values between $-1/r^2$ and $1/r^2$. It follows easily from this that, no matter how small the value of the positive real number δ, every single real number is the value taken on by the function g at some point of the open ball of radius δ centred on the origin.

Nevertheless the partial derivatives

$$\frac{\partial g(x,y)}{\partial x} \text{ and } \frac{\partial g(x,y)}{\partial y}$$

exist everywhere on $\mathbb{R}^2$, even at $(0,0)$. Indeed the partial derivatives of the function g away from the origin can be computed by the standard methods of multivariable calculus. Furthermore at the origin we find that

$$\left.\frac{\partial g(x,y)}{\partial x}\right|_{(x,y)=(0,0)} = 0, \qquad \left.\frac{\partial g(x,y)}{\partial y}\right|_{(x,y)=(0,0)} = 0$$

on account of the fact that $g(x,0) = g(0,y) = 0$ for all $x, y \in \mathbb{R}$.

Example Consider the function $h\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$h(x,y) = \begin{cases} \dfrac{2xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Given real numbers b and c, let $u_{b,c}\colon \mathbb{R} \to \mathbb{R}$ be defined so that $u_{b,c}(t) = h(bt, ct)$ for all $t \in \mathbb{R}$. If $b = 0$ or $c = 0$ then $u_{b,c}(t) = 0$ for all $t \in \mathbb{R}$. If $b \neq 0$ and $c \neq 0$ then

$$u_{b,c}(t) = \frac{2bc^2t^3}{b^2t^2 + c^4t^4} = \frac{2bc^2t}{b^2 + c^2t^2}.$$

Also, for non-zero constants b and c, the function $u_{b,c}$ satisfies the equation $u_{b,c}(t) = h(bt, ct) = \dfrac{2bc^2t}{b^2 + c^2t^2}$ for all real numbers t.

Given non-zero constants b and c, the standard rules of one-variable calculus enable us to differentiate the function $u_{b,c}$ any number of times. Accordingly this function $u_{b,c}\colon \mathbb{R} \to \mathbb{R}$ is a real-valued function of a single real variable that has derivatives of all orders.

Also, for non-zero constants b and c, the function $u_{b,c}$ satisfies the equation $u_{b,c}(t) = h(bt, ct) = \dfrac{2bc^2t}{b^2 + c^2t^2}$ for all real numbers t.

Moreover the first derivative $u'_{b,c}(0)$ of $u_{b,c}(t)$ at $t = 0$ is given by the formula

$$u'_{b,c}(0) = \begin{cases} \dfrac{2c^2}{b} & \text{if } b \neq 0; \\ 0 & \text{if } b = 0. \end{cases}$$

We have shown that the restriction of the function $h\colon \mathbb{R}^2 \to \mathbb{R}$ to any line passing through the origin determines a function that may be differentiated any number of times with respect to distance along the line. Analogous arguments show that the restriction of the function h to any other line in the plane also determines a function that may be differentiated any number of times with respect to distance along the line.

Now $h(x,y) = 1$ for all $(x,y) \in \mathbb{R}^2$ satisfying $x > 0$ and $y = \pm\sqrt{x}$, and similarly $h(x,y) = -1$ for all $(x,y) \in \mathbb{R}^2$ satisfying $x < 0$ and $y = \pm\sqrt{-x}$. It follows that every open disk centred on the origin $(0,0)$ contains some points at which the function h takes the value 1, and other points at which the function takes the value -1, and indeed the function h will take on all real values between -1 and 1 on any open disk centred on the origin, no matter how small the disk. Therefore the function $h\colon \mathbb{R}^2 \to \mathbb{R}$ is not continuous at zero, even though the partial derivatives of the function h with respect to x and y exist at each point of $\mathbb{R}^2$.