Module MAU23203: Analysis in Several Real Variables
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Section 1: The Real Number System

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1 The Real Number System

1.1 Basic Properties of the Real Number System

The real numbers $\mathbb{R}$ constitute a *field* with respect to the usual operations of addition and multiplication. In other words, the real number system satisfies all of the following properties: the operations of addition and multiplication satisfy the usual commutative, associative and distributive laws, so that $x + y = y + x$, $xy = yx$, $(x + y) + z = x + (y + z)$, $(xy)z = x(yz)$ and $(x + y)z = xz + yz$ for all real numbers x, y and z; there exist real numbers 0 and 1 characterized by the properties that $0 + x = x$ and $1x = x$ for all real numbers x; given any real number x, there exists a real number $-x$ characterized by the property that $x + (-x) = 0$; given any non-zero real number x there exists a real number x^{-1} characterized by the property that $xx^{-1} = 1$.

In the field $\mathbb{R}$ of real numbers operations of subtraction of real numbers, and division of real numbers by non-zero real numbers are defined so that $x - y = x + (-y)$ and $x/z = xz^{-1}$ for all real numbers x and y and for all non-zero real numbers z. A variety of other algebraic identities and properties follow as consequences of those just stated: analogous identities and properties are valid in any field.

The real numbers $\mathbb{R}$ constitute an *ordered field* with respect to the usual operations of addition and multiplication and the usual ordering. This statement amounts to asserting that, in addition to having operations of addition and multiplication satisfying the properties already described, there is an ordering $<$ on the real numbers which satisfies all the following properties:—

- given two real numbers x and y, exactly one of the ordering relations $x < y$, $x = y$, $y < x$ must hold for x and y (*Trichotomy Law*);
- if real numbers x, y and z satisfy both $x < y$ and $y < z$ then $x < z$ (*Transitivity Law*);
- if x and y are real numbers that satisfy $x < y$ then $x + z < y + z$ for all real numbers z;
- if x and y are real numbers satisfying $x > 0$ and $y > 0$ then $xy > 0$.

The statement that the real numbers, with the usual operations of addition and multiplication and the usual ordering, constitute an ordered field is not in itself sufficient to characterize the real number system completely. Indeed the rational numbers, with the usual operations of addition and multiplication and the usual ordering, constitute in themselves an ordered field that is embedded as a subfield of the real numbers. Thus the basic properties, or axioms, that characterize ordered fields are not in themselves sufficient to enable one to prove properties of the real number system that are not shared by the rational number system.

In particular, the real number system should contain an element, $\sqrt{2}$, that is characterized by the properties that $\sqrt{2} > 0$ and $(\sqrt{2})^2 = 2$. Now, already by the time of Plato, the ancient Greeks knew that there are no positive integers m and n that satisfy the equation $m^2 = 2n^2$. Accordingly there cannot exist any rational number whose square is equal to the number 2. Consequently, it cannot be possible to prove the existence of the real number $\sqrt{2}$ using only the basic properties, or axioms, that characterize ordered fields.

There are also some basic properties, shared by the systems of rational numbers and real numbers, that do not follow as logical consequences of the ordered field axioms. One such property is the *Archimedean Property*: given any real (or rational) number x, there exists a positive integer that is greater than x. It follows easily from the Archimedean property that, given any positive real number ε, no matter how small, there exists some positive integer n with the property that $1/n < \varepsilon$.

In order to complete the axiomatic characterization of the real number system, we introduce the *Least Upper Bound Principle* (or *Least Upper Bound Axiom*). Before stating this principle, we establish some basic terminology.

Let S be a subset of the set $\mathbb{R}$ of real numbers. A real number u is said to be an *upper bound* of the set S if $x \leq u$ for all $x \in S$. The set S is said to be *bounded above* if such an upper bound exists. Similarly a real number l is said to be a *lower bound* of the set S if $x \geq l$ for all $x \in S$. The set S is said to be *bounded below* if such a lower bound exists.

Definition Let S be a subset of the set $\mathbb{R}$ of real numbers that is bounded above. A real number s is said to be the *least upper bound* (or *supremum*) of S (and is denoted by $\sup S$) if s is an upper bound of S and $s \leq u$ for all upper bounds u of S.

Definition Let S be a subset of the set $\mathbb{R}$ of real numbers that is bounded below. A real number t is said to be the *greatest lower bound* (or *infimum*) of S (and is denoted by $\inf S$) if t is a lower bound of S and $t \geq l$ for all lower bounds l of S.

Note that there is no requirement that the least upper bound of a set of real numbers, where it exists, either must belong or else must not belong to the set which it bounds. Indeed the number 2 is the least upper bound of the sets $\{x \in \mathbb{Q} : x \leq 2\}$ and $\{x \in \mathbb{Q} : x < 2\}$. Note that the first of these sets contains its least upper bound, whereas the second set does not.

The Least Upper Bound Principle may now be stated as follows.

> **Least Upper Bound Principle.** Given any non-empty set S of real numbers that is bounded above, there exists a real number $\sup S$ that is the least upper bound for the set S.

It follows as a consequence of the Least Upper Bound Principle that, given any non-empty set S of real numbers that is bounded below, there exists a real number $\inf S$ that is the greatest lower bound for the set S. Indeed, given any non-empty set S of real numbers that is bounded below, let $T = \{-x : x \in S\}$. Then the set T is non-empty and bounded above, and therefore there exists a least upper bound $\sup T$ for the set T. It is then a straightforward exercise to verify that $\inf S = -\sup T$.

1.2 Dedekind-complete Ordered Fields

Given a subset of an ordered field, the concepts of *bounded above*, *bounded below*, *upper bound*, *lower bound*, *least upper bound* and *greatest lower bound* can be defined in the obvious fashion so as to generalize the definitions previously given in the particular case of subsets of the set of real numbers. An ordered field is said to be *Dedekind-complete* if, given any subset of that ordered field that is bounded above, there exists, within the ordered field, a least upper bound for that set. The real number system can be characterized by the statement that the real numbers, with the usual operations of addition and multiplication and the usual ordering, constitute a Dedekind-complete ordered field.

Moreover it can be shown that any two Dedekind-complete ordered fields are isomorphic as ordered fields. Thus, given any two Dedekind-complete ordered fields, there exists a one-to-one correspondence between elements of one field and elements of the other which respects the algebraic operations, so that sums correspond to sums and products correspond to products, and also respects the ordering, so that if, in one of the two ordered fields, a first element is less than a second, the element of the other field corresponding to the first element is less than that corresponding to the second.

In consequence, there is no essential difference between any two Dedekind-complete ordered fields with regard to algebraic and ordering properties. One Dedekind-complete ordered field is as good as another for the purpose of providing a model of the real number system.

In 1872 the mathematicians Richard Dedekind and Georg Cantor each constructed models of the real number system that proved adequate for the purpose of providing foundations for the theorems and constructions of real analysis.

In Dedekind's construction, each irrational number is represented as a decomposition of the collection of rational numbers into two classes (or sets) L and R, where each rational number belongs to exactly one of the two classes L and R, and where each rational number belonging to L is less than all the rational numbers belonging to R. Each such decomposition of the collection of rational numbers is referred to as a *Dedekind section*.

In Cantor's construction, expressed in more contemporary language, each real number is constructed as an equivalence class of Cauchy sequences of rational numbers. An infinite sequence $q_1, q_2, q_3, \ldots$ of rational numbers is a Cauchy sequence if, given any positive integer m, there exists some positive integer N such that $|q_j - q_k| < 1/m$ whenever $j \geq N$ and $k \geq N$. Two such Cauchy sequences of rational numbers $q_1, q_2, q_3, \ldots$ and $r_1, r_2, r_3, \ldots$ are said to be equivalent if, given any positive integer m, there exists some positive integer N such that $|q_j - r_j| < 1/m$ whenever $j \geq N$. (Note that, in order to avoid circularity, in phrasing these definitions, it is necessary to use definitions where quantities are made less than the reciprocal $1/m$ of some positive integer m in place of the "positive real number ε".) One can show that the definition of equivalence of Cauchy sequences previously stated is an equivalence relation. The resulting equivalence classes of Cauchy sequences are identified with real numbers in Cantor's construction of the real number system.

The constructions published by Dedekind and Cantor in 1872 each yield a set of mathematical objects that can be provided with appropriately-defined operations of addition and multiplication, together with a natural ordering. It can be shown that these constructions of Dedekind and Cantor each give rise to a Dedekind-complete ordered field. Their methods are thus equally viable for the purpose of constructing sets whose elements can be regarded as representing real numbers satisfying all the properties required in order to build the theory of real analysis on a secure foundation.

1.3 Convergence of Infinite Sequences of Real Numbers

An *infinite sequence* $x_1, x_2, x_3, \ldots$ of real numbers associates to each positive integer j a corresponding real number x_j.

Definition An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to *converge* to some real number p if and only if the following criterion is satisfied:

given any strictly positive real number ε, there exists some positive integer N such that $|x_j - p| < \varepsilon$ for all positive integers j satisfying $j \geq N$.

If an infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers converges to some real number p, then p is said to be the *limit* of the sequence, and we can indicate the convergence of the infinite sequence to p by writing '$x_j \to p$ as $j \to +\infty$', or by writing '$\lim_{j \to +\infty} x_j = p$'.

Let x and p be real numbers, and let ε be a strictly positive real number. Then $|x - p| < \varepsilon$ if and only if both $x - p < \varepsilon$ and $p - x < \varepsilon$. It follows that $|x - p| < \varepsilon$ if and only if $p - \varepsilon < x < p + \varepsilon$. The condition $|x - p| < \varepsilon$ essentially requires that the value of the real number x should agree with p to within an error of at most ε. An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers converges to some real number p if and only if, given any positive real number ε, there exists some positive integer N such that $p - \varepsilon < x_j < p + \varepsilon$ for all positive integers j satisfying $j \geq N$.

Definition We say that an infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is *bounded above* if there exists some real number B such that $x_j \leq B$ for all positive integers j. Similarly we say that this sequence is *bounded below* if there exists some real number A such that $x_j \geq A$ for all positive integers j. A sequence is said to be *bounded* if it is bounded above and bounded below. Thus the sequence $x_1, x_2, x_3, \ldots$ is bounded if and only if there exist real numbers A and B such that $A \leq x_j \leq B$ for all positive integers j.

Lemma 1.1 *Every convergent sequence of real numbers is bounded.*

Proof Let $x_1, x_2, x_3, \ldots$ be a sequence of real numbers converging to some real number p. On applying the formal definition of convergence (with $\varepsilon = 1$), we deduce the existence of some positive integer N such that $p - 1 < x_j < p + 1$ for all $j \geq N$. But then $A \leq x_j \leq B$ for all positive integers j, where A is the minimum of $x_1, x_2, \ldots, x_{N-1}$ and $p - 1$, and B is the maximum of $x_1, x_2, \ldots, x_{N-1}$ and $p + 1$. ∎

1.4 Monotonic Sequences

An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to be *strictly increasing* if $x_{j+1} > x_j$ for all positive integers j, *strictly decreasing* if $x_{j+1} < x_j$ for all positive integers j, *non-decreasing* if $x_{j+1} \geq x_j$ for all positive integers j, *non-increasing* if $x_{j+1} \leq x_j$ for all positive integers j. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

Theorem 1.2 *Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.*

Proof Let $x_1, x_2, x_3, \ldots$ be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Axiom that there exists a least upper bound p for the set $\{x_j : j \in \mathbb{N}\}$. We claim that the sequence converges to p.

Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j - p| < \varepsilon$ whenever $j \geq N$. Now $p - \varepsilon$ is not an upper bound for the set $\{x_j : j \in \mathbb{N}\}$ (since p is the least upper bound), and therefore there must exist some positive integer N such that $x_N > p - \varepsilon$. But then $p - \varepsilon < x_j \leq p$ whenever $j \geq N$, since the sequence is non-decreasing and bounded above by p. Thus $|x_j - p| < \varepsilon$ whenever $j \geq N$. Therefore $x_j \to p$ as $j \to +\infty$, as required.

If the sequence $x_1, x_2, x_3, \ldots$ is non-increasing and bounded below then the sequence $-x_1, -x_2, -x_3, \ldots$ is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence $x_1, x_2, x_3, \ldots$ is also convergent. ∎

1.5 Subsequences of Sequences of Real Numbers

Definition Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. A *subsequence* of this infinite sequence is a sequence of the form $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$ where $j_1, j_2, j_3, \ldots$ is an infinite sequence of positive integers with

$$j_1 < j_2 < j_3 < \cdots.$$

Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. The following sequences are examples of subsequences of this sequence:—

$$x_1, x_3, x_5, x_7, \ldots$$
$$x_1, x_4, x_9, x_{16}, \ldots$$

1.6 The Bolzano-Weierstrass Theorem

Theorem 1.3 (Bolzano-Weierstrass) *Every bounded sequence of real numbers has a convergent subsequence.*

Proof Let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers. We define a *peak index* to be a positive integer j with the property that $a_j \geq a_k$ for all positive integers k satisfying $k \geq j$. Thus a positive integer j is a peak index if and only if the jth member of the infinite sequence $a_1, a_2, a_3, \ldots$ is greater than or equal to all succeeding members of the sequence. Let S be the set of all peak indices. Then

$$S = \{j \in \mathbb{N} : a_j \geq a_k \text{ for all } k \geq j\}.$$

First let us suppose that the set S of peak indices is infinite. Arrange the elements of S in increasing order so that $S = \{j_1, j_2, j_3, j_4, \ldots\}$, where $j_1 < j_2 < j_3 < j_4 < \cdots$. It follows from the definition of peak indices that $a_{j_1} \geq a_{j_2} \geq a_{j_3} \geq a_{j_4} \geq \cdots$. Thus $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a non-increasing subsequence of the original sequence $a_1, a_2, a_3, \ldots$. This subsequence is bounded below (since the original sequence is bounded). It follows from Theorem 1.2 that $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a convergent subsequence of the original sequence.

Now suppose that the set S of peak indices is finite. Choose a positive integer j_1 which is greater than every peak index. Then j_1 is not a peak index. Therefore there must exist some positive integer j_2 satisfying $j_2 > j_1$ such that $a_{j_2} > a_{j_1}$. Moreover j_2 is not a peak index (because j_2 is greater than j_1 and j_1 in turn is greater than every peak index). Therefore there must exist some positive integer j_3 satisfying $j_3 > j_2$ such that $a_{j_3} > a_{j_2}$. We can continue in this way to construct (by induction on j) a strictly increasing subsequence $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ of our original sequence. This increasing subsequence is bounded above (since the original sequence is bounded) and thus is convergent, by Theorem 1.2. This completes the proof of the Bolzano-Weierstrass Theorem. ∎