Module MAU23203: Analysis in Several Real Variables
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Section 5: Limits and Continuity for Functions of Several Variables

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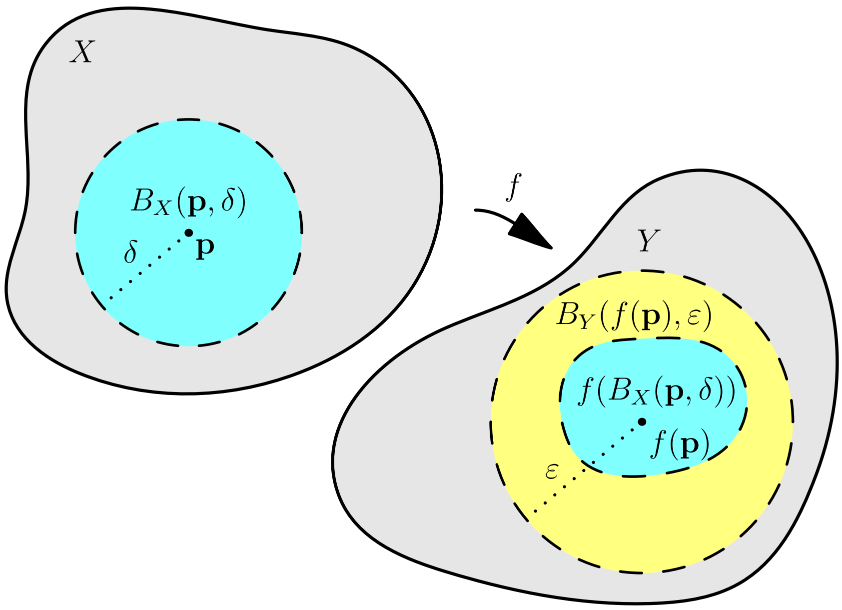
Contents

5 Limits and Continuity for Functions of Several Variables

5.1 Continuity of Functions of Several Real Variables

Definition Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A function $f: X \to Y$ from X to Y is said to be *continuous* at a point $\mathbf{p}$ of X if and only if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$.

The function $f: X \to Y$ is said to be continuous on X if and only if it is continuous at every point $\mathbf{p}$ of X.



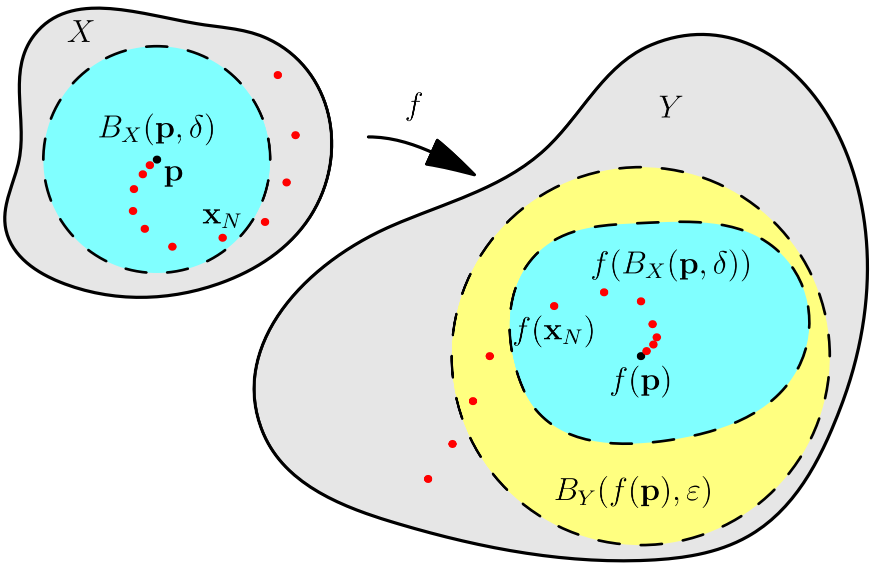
Lemma 5.1 *Let X, Y and Z be subsets of Euclidean spaces, let $f: X \to Y$ be a function from X to Y and let $g: Y \to Z$ be a function from Y to Z. Suppose that f is continuous at some point $\mathbf{p}$ of X and that g is continuous at $f(\mathbf{p})$. Then the composition function $g \circ f: X \to Z$ is continuous at $\mathbf{p}$.*

Proof Let some positive real number ε be given. Then there exists some positive real number η such that $|g(\mathbf{y}) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - f(\mathbf{p})| < \eta$. But then there exists some positive real number δ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \eta$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that

$|g(f(\mathbf{x})) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus $g \circ f$ is continuous at $\mathbf{p}$, as required. ∎

Lemma 5.2 *Let X and Y be subsets of Euclidean spaces, and let $f: X \to Y$ be a continuous function from X to Y. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be an infinite sequence of points of X which converges to some point $\mathbf{p}$ of X. Then the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$.*

Proof Let some positive real number ε be given. The function f is continuous at $\mathbf{p}$, and therefore there exists some positive real number δ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Also the infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to the point $\mathbf{p}$, and therefore there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$. It follows that if $j \geq N$ then $|f(\mathbf{x}_j) - f(\mathbf{p})| < \varepsilon$. Thus the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$, as required. ∎



Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f: X \to Y$ be a function from X to Y. Then

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

for all $\mathbf{x} \in X$, where $f_1, f_2, \ldots, f_n$ are functions from X to $\mathbb{R}$, referred to as the *components* of the function f.

Proposition 5.3 *Let X and Y be subsets of Euclidean spaces, and let $\mathbf{p} \in X$. A function $f\colon X \to Y$ is continuous at the point $\mathbf{p}$ if and only if its components are all continuous at $\mathbf{p}$.*

Proof Let Y be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. Note that the ith component f_i of f is given by $f_i = \pi_i \circ f$, where $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ is the continuous function which maps $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ onto its ith component y_i. Now any composition of continuous functions is continuous, by Lemma 5.1. Thus if f is continuous at $\mathbf{p}$, then so are the components of f.

Conversely suppose that the components of f are continuous at $\mathbf{p} \in X$. Let some positive real number ε be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $|f_i(\mathbf{x}) - f_i(\mathbf{p})| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - f(\mathbf{p})|^2 = \sum_{i=1}^{n} |f_i(\mathbf{x}) - f_i(\mathbf{p})|^2 < \varepsilon^2,$$

and hence $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$. Thus the function f is continuous at $\mathbf{p}$, as required. ∎

Lemma 5.4 *The functions $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $m\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $s(x, y) = x + y$ and $m(x, y) = xy$ are continuous.*

Proof Let $(u, v) \in \mathbb{R}^2$. We first show that $s\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). Let some positive real number ε be given. Let $\delta = \frac{1}{2}\varepsilon$. If (x, y) is any point of $\mathbb{R}^2$ whose distance from (u, v) is less than δ then $|x - u| < \delta$ and $|y - v| < \delta$, and hence

$$|s(x, y) - s(u, v)| = |x + y - u - v| \le |x - u| + |y - v| < 2\delta = \varepsilon.$$

This shows that $s\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v).

Next we show that $m\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). Let some positive real number ε be given. Now

$$m(x, y) - m(u, v) = xy - uv = (x - u)(y - v) + u(y - v) + (x - u)v.$$

for all points (x, y) of $\mathbb{R}^2$. Thus if the distance from (x, y) to (u, v) is less than δ then $|x - u| < \delta$ and $|y - v| < \delta$, and hence $|m(x, y) - m(u, v)| < \delta^2 + (|u| + |v|)\delta$. Let some positive real number ε be given. If the positive real number δ is chosen to be the minimum of 1 and $\varepsilon/(1 + |u| + |v|)$ then $\delta^2 + (|u| + |v|)\delta \le (1 + |u| + |v|)\delta \le \varepsilon$, and thus $|m(x, y) - m(u, v)| < \varepsilon$ for all points (x, y) of $\mathbb{R}^2$ whose distance from (u, v) is less than δ. This shows that $m\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). ∎

Proposition 5.5 *Let X be a subset of $\mathbb{R}^n$, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be continuous functions from X to $\mathbb{R}$. Then the functions $f + g$, $f - g$ and $f \cdot g$ are continuous. If in addition $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ then the quotient function f/g is continuous.*

Proof Note that $f + g = s \circ h$ and $f \cdot g = m \circ h$, where $h\colon X \to \mathbb{R}^2$, $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $m\colon \mathbb{R}^2 \to \mathbb{R}$ are given by $h(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$, $s(u, v) = u + v$ and $m(u, v) = uv$ for all $\mathbf{x} \in X$ and $u, v \in \mathbb{R}$. It follows from Proposition 5.3, Lemma 5.4 and Lemma 5.1 that $f + g$ and $f \cdot g$ are continuous, being compositions of continuous functions. Now $f - g = f + (-g)$, and both f and $-g$ are continuous. Therefore $f - g$ is continuous.

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$. Note that $1/g = r \circ g$, where $r\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is the reciprocal function, defined by $r(t) = 1/t$. Now the reciprocal function r is continuous. Thus the function $1/g$ is a composition of continuous functions and is thus continuous. But then, using the fact that a product of continuous real-valued functions is continuous, we deduce that f/g is continuous. ∎

Example Consider the function $f\colon \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}^2$ defined by

$$f(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right).$$

The continuity of the components of the function f follows from straightforward applications of Proposition 5.5. It then follows from Proposition 5.3 that the function f is continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Lemma 5.6 *Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a continuous function mapping X into $\mathbb{R}^n$, and let $|f|\colon X \to \mathbb{R}$ be the real-valued function on X defined such that $|f|(\mathbf{x}) = |f(\mathbf{x})|$ for all $\mathbf{x} \in X$. Then the real-valued function $|f|$ is continuous on X.*

Proof Let $\mathbf{x}$ and $\mathbf{p}$ be points of X. Then

$$|f(\mathbf{x})| = |(f(\mathbf{x}) - f(\mathbf{p})) + f(\mathbf{p})| \leq |f(\mathbf{x}) - f(\mathbf{p})| + |f(\mathbf{p})|$$

and

$$|f(\mathbf{p})| = |(f(\mathbf{p}) - f(\mathbf{x})) + f(\mathbf{x})| \leq |f(\mathbf{x}) - f(\mathbf{p})| + |f(\mathbf{x})|,$$

and therefore

$$\Big| |f(\mathbf{x})| - |f(\mathbf{p})| \Big| \leq |f(\mathbf{x}) - f(\mathbf{p})|.$$

The result now follows on applying the definition of continuity, using the above inequality. Indeed let $\mathbf{p}$ be a point of X, and let some positive real number ε be given. Then there exists a positive real number δ small enough to ensure that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then

$$\Big| |f(\mathbf{x})| - |f(\mathbf{p})| \Big| \leq |f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus the function $|f|$ is continuous, as required. ∎

5.2 Limits of Functions of Several Real Variables

Definition Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f: X \to \mathbb{R}^n$ be a function mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set X, and let $\mathbf{q}$ be a point in $\mathbb{R}^n$. The point $\mathbf{q}$ is said to be the *limit* of $f(\mathbf{x})$, as $\mathbf{x}$ tends to $\mathbf{p}$ in X, if and only if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f: X \to \mathbb{R}^n$ be a function mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set X, and let $\mathbf{q}$ be a point of $\mathbb{R}^n$. If $\mathbf{q}$ is the limit of $f(\mathbf{x})$ as $\mathbf{x}$ tends to $\mathbf{p}$ in X then we can denote this fact by writing $\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$.

Proposition 5.7 *Let X be a subset of $\mathbb{R}^m$, let $\mathbf{p}$ be a limit point of X, and let $\mathbf{q}$ be a point of $\mathbb{R}^n$. A function $f: X \to \mathbb{R}^n$ has the property that*

$$\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

if and only if

$$\lim_{\mathbf{x} \to \mathbf{p}} f_i(\mathbf{x}) = q_i$$

for $i = 1, 2, \ldots, n$, where $f_1, f_2, \ldots, f_n$ are the components of the function f and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$.

Proof Suppose that $\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$. Let i be an integer between 1 and n, and let some positive real number ε be given. Then there exists some positive real number δ such that $|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta$. It then follows from the definition of the Euclidean norm that

$$|f_i(\mathbf{x}) - q_i| \leq |f(\mathbf{x}) - \mathbf{q}| < \varepsilon$$

whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta$. Thus if $\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$ then $\lim_{\mathbf{x} \to \mathbf{p}} f_i(\mathbf{x}) = q_i$ for $i = 1, 2, \ldots, n$.

Conversely suppose that

$$\lim_{\mathbf{x} \to \mathbf{p}} f_i(\mathbf{x}) = q_i$$

for $i = 1, 2, \ldots, n$. Let some positive real number ε be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $|f_i(\mathbf{x}) - q_i| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - \mathbf{q}|^2 = \sum_{i=1}^{n} |f_i(\mathbf{x}) - q_i|^2 < \varepsilon^2,$$

and hence $|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$. Thus

$$\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q},$$

as required. ∎

Proposition 5.8 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ and $g\colon X \to \mathbb{R}^n$ be functions mapping X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of X, and let $\mathbf{q}$ and $\mathbf{r}$ be points of $\mathbb{R}^n$. Suppose that*

$$\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and

$$\lim_{\mathbf{x} \to \mathbf{p}} g(\mathbf{x}) = \mathbf{r}.$$

Then

$$\lim_{\mathbf{x} \to \mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) = \mathbf{q} + \mathbf{r}.$$

Proof Let some strictly positive real number ε be given. Then there exist strictly positive real numbers δ_1 and δ_2 such that

$$|f(\mathbf{x}) - \mathbf{q}| < \tfrac{1}{2}\varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$ and

$$|g(\mathbf{x}) - \mathbf{r}| < \tfrac{1}{2}\varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and if $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - \mathbf{q}| < \tfrac{1}{2}\varepsilon$$

and

$$|g(\mathbf{x}) - \mathbf{r}| < \tfrac{1}{2}\varepsilon,$$

and therefore

$$\begin{aligned} |f(\mathbf{x}) + g(\mathbf{x}) - (\mathbf{q} + \mathbf{r})| &\leq |f(\mathbf{x}) - \mathbf{q}| + |g(\mathbf{x}) - \mathbf{r}| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

It follows that

$$\lim_{\mathbf{x} \to \mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) = \mathbf{q} + \mathbf{r},$$

as required. ∎

Lemma 5.9 *Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, let $\mathbf{p}$ be a limit point of X, let $\mathbf{q}$ be a point of Y, let $f: X \to Y$ be a function satisfying $f(X) \subset Y$, and let $g: Y \to \mathbb{R}^k$ be a function from Y to $\mathbb{R}^k$. Suppose that*

$$\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and that the function g is continuous at $\mathbf{q}$. Then

$$\lim_{\mathbf{x} \to \mathbf{p}} g(f(\mathbf{x})) = g(\mathbf{q}).$$

Proof Let some positive real number ε be given. Then there exists some positive real number η such that $|g(\mathbf{y}) - g(\mathbf{q})| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - \mathbf{q}| < \eta$, because the function g is continuous at $\mathbf{q}$. But then there exists some positive real number δ such that $|f(\mathbf{x}) - \mathbf{q}| < \eta$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|g(f(\mathbf{x})) - g(\mathbf{q})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$, and thus

$$\lim_{\mathbf{x} \to \mathbf{p}} g(f(\mathbf{x})) = g(\mathbf{q}),$$

as required. ∎

Proposition 5.10 *Let X be a subset of $\mathbb{R}^m$, let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be real-valued functions on X, and let $\mathbf{p}$ be a limit point of the set X.*

Suppose that $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})$ *and* $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})$ *both exist. Then so do* $\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})+g(\mathbf{x}))$, $\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})-g(\mathbf{x}))$ *and* $\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x}))$, *and moreover*

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})+g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}),\\
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})-g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) - \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}),\\
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) \times \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}),
\end{aligned}$$

If moreover $g(\mathbf{x}) \neq 0$ *for all* $\mathbf{x} \in X$ *and* $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) \neq 0$ *then*

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})}{\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})}.$$

Proof Let $q = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})$ and $r = \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})$, and let $h: X \to \mathbb{R}^2$ be defined such that

$$h(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$$

for all $\mathbf{x} \in X$. Then

$$\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x}) = (q, r)$$

(see Proposition 5.7).

Let $s: \mathbb{R}^2 \to \mathbb{R}$ and $m: \mathbb{R}^2 \to \mathbb{R}$ be the functions from $\mathbb{R}^2$ to $\mathbb{R}$ defined such that $s(u,v) = u + v$ and $m(u,v) = uv$ for all $u, v \in \mathbb{R}$. Then the functions s and m are continuous (see Lemma 5.4). Also $f + g = s \circ h$ and $f \cdot g = m \circ h$. It follows from this that

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})+g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} s(f(\mathbf{x}), g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} s(h(\mathbf{x}))\\
&= s\left(\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x})\right) = s(q, r) = q + r,
\end{aligned}$$

(see Lemma 5.9), and

$$\lim_{\mathbf{x}\to\mathbf{p}}(-g(\mathbf{x})) = -r.$$

It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})-g(\mathbf{x})) = q - r.$$

Similarly, when taking limits of products of functions,

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} m(f(\mathbf{x}), g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} m(h(\mathbf{x}))\\
&= m\left(\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x})\right) = m(q, r) = qr
\end{aligned}$$

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ and that $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) \neq 0$. Representing the function sending $\mathbf{x} \in X$ to $1/g(\mathbf{x})$ as the composition of the function g and the reciprocal function $e\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, where $e(t) = 1/t$ for all non-zero real numbers t, we find, as in the first proof, that the function sending each point $\mathbf{x}$ of X to

$$\lim_{\mathbf{x}\to\mathbf{p}} \left(\frac{1}{g(\mathbf{x})}\right) = \frac{1}{r}.$$

It then follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{q}{r},$$

as required. ∎

Proposition 5.11 *Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a function mapping the set X into $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of the set X that is also a limit point of X. Then the function f is continuous at the point $\mathbf{p}$ if and only if $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = f(\mathbf{p})$.*

Proof The result follows directly on comparing the relevant definitions. ∎

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of the set X. Suppose that the point $\mathbf{p}$ is not a limit point of the set X. Then there exists some strictly positive real number δ_0 such that $|\mathbf{x}-\mathbf{p}| \geq \delta_0$ for all $\mathbf{x} \in X$ satisfying $\mathbf{x} \neq \mathbf{p}$. The point $\mathbf{p}$ is then said to be an *isolated point* of X.

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$. The definition of continuity then ensures that any function $f\colon X \to \mathbb{R}^n$ mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$ is continuous at any isolated point of its domain X.

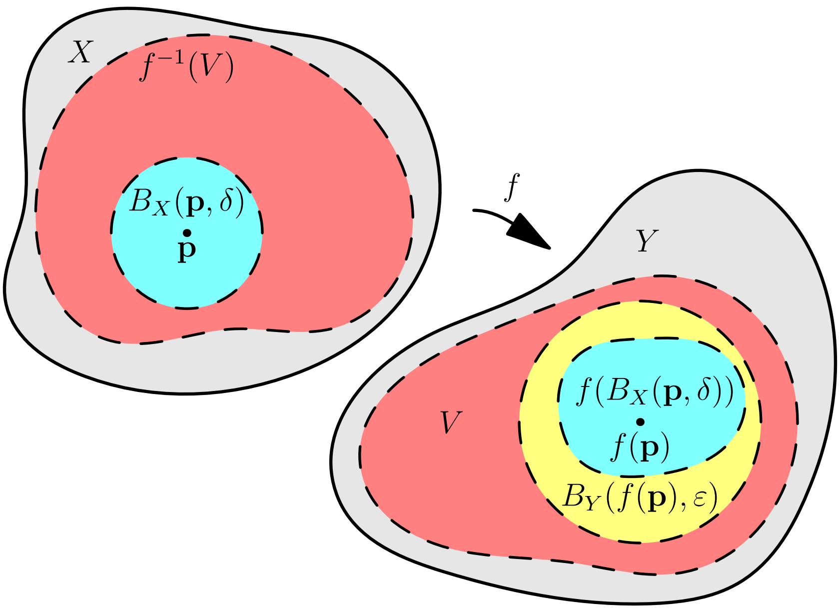
5.3 Continuous Functions and Open Sets

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f\colon X \to Y$ be a function from X to Y. We recall that the function f is continuous at a point $\mathbf{p}$ of X if, given any positive real number ε, there exists some positive real number δ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x}-\mathbf{p}| < \delta$. Thus the function $f\colon X \to Y$ is continuous at $\mathbf{p}$ if and only if, given any positive real number ε, there exists some positive real number δ such that the function f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (where $B_X(\mathbf{p}, \delta)$ and $B_Y(f(\mathbf{p}), \varepsilon)$ denote the open balls in X and Y of radius δ and ε about $\mathbf{p}$ and $f(\mathbf{p})$ respectively).

Given any function $f\colon X \to Y$, we denote by $f^{-1}(V)$ the *preimage* of a subset V of Y under the map f, defined by $f^{-1}(V) = \{\mathbf{x} \in X : f(\mathbf{x}) \in V\}$.

Proposition 5.12 *Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f\colon X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(V)$ is open in X for every open subset V of Y.*

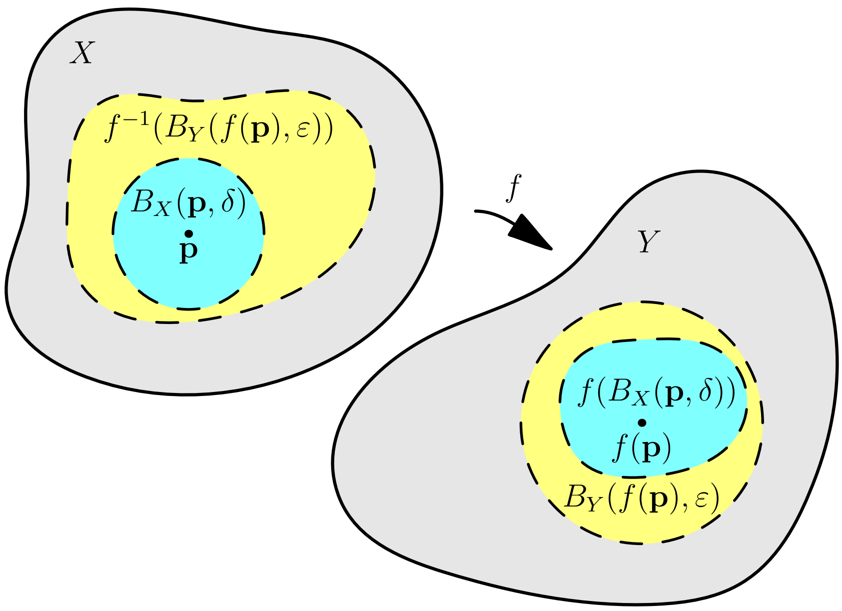
Proof Suppose that $f\colon X \to Y$ is continuous. Let V be an open set in Y. We must show that $f^{-1}(V)$ is open in X. Let $\mathbf{p} \in f^{-1}(V)$. Then $f(\mathbf{p}) \in V$. But V is open, hence there exists some positive real number ε with the property that $B_Y(f(\mathbf{p}), \varepsilon) \subset V$. But f is continuous at $\mathbf{p}$. Therefore there exists some positive real number δ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (see the remarks above). Thus $f(\mathbf{x}) \in V$ for all $\mathbf{x} \in B_X(\mathbf{p}, \delta)$, showing that $B_X(\mathbf{p}, \delta) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is open in X for every open set V in Y.



Conversely suppose that $f\colon X \to Y$ is a function with the property that $f^{-1}(V)$ is open in X for every open set V in Y. Let $\mathbf{p} \in X$. We must show that f is continuous at $\mathbf{p}$. Let some positive real number ε be given. Then $B_Y(f(\mathbf{p}), \varepsilon)$ is an open set in Y, by Lemma 4.1, hence $f^{-1}\left(B_Y(f(\mathbf{p}), \varepsilon)\right)$ is an open set in X which contains $\mathbf{p}$. It follows that there exists some positive real number δ such that $B_X(\mathbf{p}, \delta) \subset f^{-1}\left(B_Y(f(\mathbf{p}), \varepsilon)\right)$. Thus, given any positive real number ε, there exists some positive real number δ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$. We conclude that f is continuous at $\mathbf{p}$, as required. ∎

Let X be a subset of $\mathbb{R}^n$, let $f\colon X \to \mathbb{R}$ be continuous, and let c be some real number. Then the sets

$$\{\mathbf{x} \in X : f(\mathbf{x}) > c\}$$



and

$$\{\mathbf{x} \in X : f(\mathbf{x}) < c\}$$

are open in X, and, given real numbers a and b satisfying $a < b$, the set

$$\{\mathbf{x} \in X : a < f(\mathbf{x}) < b\}$$

is open in X.

Again let X be a subset of $\mathbb{R}^n$, let $f\colon X \to \mathbb{R}$ be continuous, and let c be some real number. Now a subset of X is closed in X if and only if its complement is open in X. Consequently the sets

$$\{\mathbf{x} \in X : f(\mathbf{x}) \leq c\}$$

and

$$\{\mathbf{x} \in X : f(\mathbf{x}) \geq c\},$$

being the complements in X of sets that are open in X, must themselves be closed in X. It follows that that set

$$\{\mathbf{x} \in X : f(\mathbf{x}) = c\},$$

being the intersection of two subsets X that are closed in X, must itself be closed in X.