Module MAU23203: Analysis in Several Real Variables
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Section 3: Convergence in Euclidean Spaces

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3 Convergence in Euclidean Spaces

3.1 Convergence of Sequences in Euclidean Spaces

Definition An infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to *converge* to a point $\mathbf{p}$ if and only if, given strictly positive real number ε, there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$.

Given a convergent infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$, the point $\mathbf{p}$ to which the sequence converges is referred to as the *limit* of the infinite sequence, and may be denoted by $\lim_{j \to +\infty} \mathbf{x}_j$.

Lemma 3.1 *Let $\mathbf{p}$ be a point of $\mathbb{R}^n$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. Then an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to $\mathbf{p}$ if and only if the ith components of the elements of this sequence converge to p_i for $i = 1, 2, \ldots, n$.*

Proof For each positive integer j, let $(\mathbf{x}_j)_i$ denote the ith component of $\mathbf{x}_j$. Then $|(\mathbf{x}_j)_i - p_i| \leq |\mathbf{x}_j - \mathbf{p}|$ for $i = 1, 2, \ldots, n$ and for all positive integers j. It follows directly from the definition of convergence that if $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ then $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$.

Conversely suppose that, for each integer i between 1 and n, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$. Let some positive real number ε be given. Then there exist positive integers $N_1, N_2, \ldots, N_n$ such that $|(\mathbf{x}_j)_i - p_i| < \varepsilon/\sqrt{n}$ whenever $j \geq N_i$. Let N be the maximum of $N_1, N_2, \ldots, N_n$. If $j \geq N$ then $j \geq N_i$ for $i = 1, 2, \ldots, n$, and therefore

$$|\mathbf{x}_j - \mathbf{p}|^2 = \sum_{i=1}^{n} ((\mathbf{x}_j)_i - p_i)^2 < n \left(\frac{\varepsilon}{\sqrt{n}} \right)^2 = \varepsilon^2.$$

Thus $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$, as required. ∎

3.2 The Multidimensional Bolzano-Weierstrass Theorem

Theorem 3.2 (Multidimensional Bolzano-Weierstrass Theorem) *Every bounded sequence of points in a Euclidean space has a convergent subsequence.*

Proof The theorem is proved by induction on the dimension n of the space $\mathbb{R}^n$ within which the points reside. When $n = 1$, the required result is the one-dimensional case of the Bolzano-Weierstrass Theorem, and the result has already been established in this case (see Theorem 1.3).

When $n > 1$, the result is proved in dimension n asssuming the result in dimensions $n - 1$ and 1. Consequently the result is established successively in dimensions $2, 3, 4, \ldots$, and therefore is valid for bounded sequences in $\mathbb{R}^n$ for all positive integers n.

It has been shown that every bounded infinite sequence of real numbers has a convergent subsequence (Theorem 1.3). Let n be an integer greater than one, and suppose, as an induction hypothesis, that, in cases where $n > 2$, all bounded sequences of points in $\mathbb{R}^{n-1}$ have convergent subsequences. Let $S\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ and $T\colon \mathbb{R}^n \to \mathbb{R}$ and be the linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$ and $\mathbb{R}$ respectively defined such that

$$S(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{n-1})$$

and

$$T(x_1, x_2, \ldots, x_n) = x_n$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a infinite bounded sequence of points (or vectors) in $\mathbb{R}^n$, and let some strictly positive real number ε be given. Now the infinite sequence

$$S\mathbf{x}_1, S\mathbf{x}_2, S\mathbf{x}_3, \ldots$$

of points of $\mathbb{R}^{n-1}$ is a bounded infinite sequence. In the case when $n = 2$ we can apply the one-dimensional Bolzano-Weierstrass Theorem (Theorem 1.3) to conclude that this sequence of real numbers has a convergent subsequence. In cases where $n > 2$, we are supposing as our induction hypothesis that any bounded sequence in $\mathbb{R}^{n-1}$ has a convergent subsequence. Thus, assuming this induction hypothesis in cases where $n > 2$, we can conclude, in all cases with $n > 1$, that the bounded infinite sequence $S\mathbf{x}_1, S\mathbf{x}_2, S\mathbf{x}_3, \ldots$ of points in $\mathbb{R}^{n-1}$ has a convergent subsequence. Let that convergent subsequence be

$$S\mathbf{x}_{m_1}, S\mathbf{x}_{m_2}, S\mathbf{x}_{m_3}, \ldots,$$

where $m_1, m_2, m_3, \ldots$ is a strictly increasing infinite sequence of positive integers, and let $\mathbf{q} = \lim_{j \to +\infty} S\mathbf{x}_{m_j}$. There then exists some positive integer L such that

$$|S\mathbf{x}_{m_j} - \mathbf{q}| < \tfrac{1}{2}\varepsilon$$

for all positive integers j for which $m_j \geq L$. (Indeed the definition of convergence ensures the existence of a positive integer N that is large enough to ensure that $|S\mathbf{x}_{m_j} - \mathbf{q}| < \frac{1}{2}\varepsilon$ whenever $j \geq N$. Taking $L = m_N$ then ensures that $j \geq N$ whenever $m_j \geq L$.)

Now the one-dimensional Bolzano-Weierstrass Theorem ensures that the bounded infinite sequence

$$T\mathbf{x}_{m_1}, T\mathbf{x}_{m_2}, T\mathbf{x}_{m_3}, \ldots$$

of real numbers has a convergent subsequence (where, for each $\mathbf{x} \in \mathbb{R}^n$, $T\mathbf{x}$ is defined to be the final Cartesian component of $\mathbf{x}$). It follows that there is a strictly increasing infinite sequence $k_1, k_2, k_3, \ldots$ of positive integers, where each k_j is equal to one of the positive integers $m_1, m_2, m_3, \ldots$, such that the infinite sequence

$$T\mathbf{x}_{k_1}, T\mathbf{x}_{k_2}, T\mathbf{x}_{k_3}, \ldots$$

is convergent. (Here, for $j = 1, 2, 3, \ldots$, the infinite sequence of points $\mathbf{x}_{k_j}$ is a subsequence of the infinite sequence of points $\mathbf{x}_{m_j}$, which sequence is in turn a subsequence of the infinite sequence of points $\mathbf{x}_j$.) Let $r = \lim_{j \to +\infty} T\mathbf{x}_{k_j}$. There then exists some positive integer M such that $M \geq L$ and

$$|T\mathbf{x}_{k_j} - r| < \tfrac{1}{2}\varepsilon$$

for all positive integers j for which $k_j \geq M$. It follows that if $k_j \geq M$ then

$$|S\mathbf{x}_{k_j} - \mathbf{q}| < \tfrac{1}{2}\varepsilon \quad \text{and} \quad |T\mathbf{x}_{k_j} - r| < \tfrac{1}{2}\varepsilon.$$

Now there is a point $\mathbf{p}$ of $\mathbb{R}^n$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, determined so that

$$\mathbf{q} = S\mathbf{p} = (p_1, p_2, \ldots, p_{n-1}) \quad \text{and} \quad r = T\mathbf{p} = p_n.$$

Also it follows from the definition of the Euclidean norm that

$$|\mathbf{x} - \mathbf{p}|^2 = |S\mathbf{x} - \mathbf{q}|^2 + |T\mathbf{x} - r|^2$$

for all $\mathbf{x} \in \mathbb{R}^n$. It follows that

$$|\mathbf{x}_{k_j} - \mathbf{p}|^2 = |S\mathbf{x}_{k_j} - \mathbf{q}|^2 + |T\mathbf{x}_{k_j} - r|^2 < \tfrac{1}{2}\varepsilon^2$$

whenever $k_j \geq M$. But then $|\mathbf{x}_{k_j} - \mathbf{p}| < \varepsilon$ for all positive integers j for which $k_j \geq M$. It follows that $\lim_{j \to +\infty} \mathbf{x}_{k_j} = \mathbf{p}$. We conclude therefore that the bounded infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ does indeed have a convergent subsequence. This completes the proof of the Bolzano-Weierstrass Theorem in dimension n for all positive integers n. ∎

3.3 Cauchy Sequences in Euclidean Spaces

Definition A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of n-dimensional Euclidean space $\mathbb{R}^n$ is said to be a *Cauchy sequence* if, given any strictly positive real number ε, there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < \varepsilon$ for all positive integers j and k satisfying $j \geq N$ and $k \geq N$.

Lemma 3.3 *Every Cauchy sequence of points of n-dimensional Euclidean space $\mathbb{R}^n$ is bounded.*

Proof Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a Cauchy sequence of points in $\mathbb{R}^n$. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < 1$ whenever $j \geq N$ and $k \geq N$. In particular, $|\mathbf{x}_j| \leq |\mathbf{x}_N| + 1$ whenever $j \geq N$. Therefore $|\mathbf{x}_j| \leq R$ for all positive integers j, where R is the maximum of the real numbers $|\mathbf{x}_1|, |\mathbf{x}_2|, \ldots, |\mathbf{x}_{N-1}|$ and $|\mathbf{x}_N| + 1$. Thus the sequence is bounded, as required. ∎

Theorem 3.4 (Cauchy's Criterion for Convergence) *An infinite sequence of points of n-dimensional Euclidean space $\mathbb{R}^n$ is convergent if and only if it is a Cauchy sequence.*

Proof First we show that convergent sequences in $\mathbb{R}^n$ are Cauchy sequences. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a convergent sequence of points in $\mathbb{R}^n$, and let $\mathbf{p} = \lim\limits_{j \to +\infty} \mathbf{x}_j$. Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$ for all $j \geq N$. Thus if $j \geq N$ and $k \geq N$ then $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$ and $|\mathbf{x}_k - \mathbf{p}| < \frac{1}{2}\varepsilon$, and hence

$$|\mathbf{x}_j - \mathbf{x}_k| = |(\mathbf{x}_j - \mathbf{p}) - (\mathbf{x}_k - \mathbf{p})| \leq |\mathbf{x}_j - \mathbf{p}| + |\mathbf{x}_k - \mathbf{p}| < \varepsilon.$$

Thus the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is a Cauchy sequence.

Conversely we must show that any Cauchy sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ in $\mathbb{R}^n$ is convergent. Now Cauchy sequences are bounded, by Lemma 3.3. The sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ therefore has a convergent subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$, by the multidimensional Bolzano-Weierstrass Theorem (Theorem 3.2). Let $\mathbf{p} = \lim\limits_{j \to +\infty} \mathbf{x}_{k_j}$. We claim that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ itself converges to $\mathbf{p}$.

Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < \frac{1}{2}\varepsilon$ whenever $j \geq N$ and $k \geq N$ (since the sequence is a Cauchy sequence). Let m be chosen large enough to ensure that $k_m \geq N$ and $|\mathbf{x}_{k_m} - \mathbf{p}| < \frac{1}{2}\varepsilon$. Then

$$|\mathbf{x}_j - \mathbf{p}| \leq |\mathbf{x}_j - \mathbf{x}_{k_m}| + |\mathbf{x}_{k_m} - \mathbf{p}| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

whenever $j \geq N$. It follows that $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$, as required. ∎