Module MAU23203: Analysis in Several Real Variables
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Section 6: Continuous Functions on Closed Bounded Sets

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6 Continuous Functions on Closed Bounded Sets

6.1 The Multidimensional Extreme Value Theorem

Lemma 6.1 *Let X be a closed bounded set in $\mathbb{R}^m$, and let $f\colon X \to \mathbb{R}$ be a continuous real-valued function defined on X. Suppose that the set of values of the function f on X is bounded below. Then there exists a point $\mathbf{u}$ of X such that $f(\mathbf{x}) \geq f(\mathbf{u})$ for all $\mathbf{x} \in X$.*

Proof Let

$$m = \inf\{f(\mathbf{x}) : \mathbf{x} \in X\}.$$

Then there exists an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ in X such that

$$f(\mathbf{x}_j) < m + \frac{1}{j}$$

for all positive integers j. It follows from the multidimensional Bolzano-Weierstrass Theorem (Theorem 3.2) that this sequence has a subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ which converges to some point $\mathbf{u}$ of $\mathbb{R}^m$.

Now the point $\mathbf{u}$ belongs to X because X is closed (see Lemma 4.7). Also

$$m \leq f(\mathbf{x}_{k_j}) < m + \frac{1}{k_j}$$

for all positive integers j. It follows that $\displaystyle\lim_{j \to +\infty} f(\mathbf{x}_{k_j}) = m$. Consequently

$$f(\mathbf{u}) = f\left(\lim_{j \to +\infty} \mathbf{x}_{k_j}\right) = \lim_{j \to +\infty} f(\mathbf{x}_{k_j}) = m$$

(see Lemma 5.2). It follows therefore that $f(\mathbf{x}) \geq f(\mathbf{u})$ for all $\mathbf{x} \in X$, Thus the function f attains a minimum value at the point $\mathbf{u}$ of X, which is what we were required to prove. ∎

Lemma 6.2 *Let X be a closed bounded set in $\mathbb{R}^m$, and let $\varphi\colon X \to \mathbb{R}^n$ be a continuous function mapping X into $\mathbb{R}^n$. Then there exists a positive real number M with the property that $|\varphi(\mathbf{x})| \leq M$ for all $\mathbf{x} \in X$.*

Proof Let $g\colon X \to \mathbb{R}$ be defined such that

$$g(\mathbf{x}) = \frac{1}{1 + |\varphi(\mathbf{x})|}$$

for all $\mathbf{x} \in X$. Now the real-valued function mapping each $\mathbf{x} \in X$ to $|\varphi(\mathbf{x})|$ is continuous (see Lemma 5.6) and quotients of continuous real-valued functions are continuous where they are defined (see Lemma 5.5). It follows that the function $g: X \to \mathbb{R}$ is continuous. Moreover the values of this function are bounded below by zero. Consequently there exists some point $\mathbf{w}$ of X with the property that $g(\mathbf{x}) \geq g(\mathbf{w})$ for all $\mathbf{x} \in X$ (see Lemma 6.1). Let $M = |\varphi(\mathbf{w})|$. Then $|\varphi(\mathbf{x})| \leq M$ for all $\mathbf{x} \in X$. The result follows. ∎

Theorem 6.3 (The Multidimensional Extreme Value Theorem) *Let X be a closed bounded set in $\mathbb{R}^m$, and let $f: X \to \mathbb{R}$ be a continuous real-valued function defined on X. Then there exist points $\mathbf{u}$ and $\mathbf{v}$ of X such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.*

Proof It follows from Lemma 6.2 that there exists positive real number M with the property that $-M \leq f(\mathbf{x}) \leq M$ for all $\mathbf{x} \in X$. Thus the set of values of the function f is bounded above and below on X. Consequently there exist points $\mathbf{u}$ and $\mathbf{v}$ where the functions f and $-f$ respectively attain their minimum values on the set X (see Lemma 6.1). The result follows. ∎

6.2 Uniform Continuity for Functions of Several Real Variables

Definition Let X be a subset of $\mathbb{R}^m$. A function $\varphi: X \to \mathbb{R}^n$ from X to $\mathbb{R}^n$ is said to be *uniformly continuous* if, given any positive real number ε, there exists some positive real number δ (whose value does not depend on either $\mathbf{y}$ or $\mathbf{z}$) such that $|\varphi(\mathbf{y}) - \varphi(\mathbf{z})| < \varepsilon$ for all points $\mathbf{y}$ and $\mathbf{z}$ of X satisfying $|\mathbf{y} - \mathbf{z}| < \delta$.

Theorem 6.4 *Let X be a subset of $\mathbb{R}^m$ that is both closed and bounded. Then any continuous function $\varphi: X \to \mathbb{R}^n$ is uniformly continuous.*

Proof Let some positive real number ε be given. Suppose that there did not exist any positive real number δ small enough to ensure that $|\varphi(\mathbf{y}) - \varphi(\mathbf{z})| < \varepsilon$ for all points $\mathbf{y}$ and $\mathbf{z}$ of the set X satisfying $|\mathbf{y} - \mathbf{z}| < \delta$. Then, for each positive integer j, there would exist points $\mathbf{u}_j$ and $\mathbf{v}_j$ in X such that $|\mathbf{u}_j - \mathbf{v}_j| < 1/j$ and $|\varphi(\mathbf{u}_j) - \varphi(\mathbf{v}_j)| \geq \varepsilon$. But the sequence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ would be bounded, since X is bounded, and thus would possess a subsequence $\mathbf{u}_{k_1}, \mathbf{u}_{k_2}, \mathbf{u}_{k_3}, \ldots$ converging to some point $\mathbf{p}$ (Theorem 3.2). Moreover $\mathbf{p} \in X$, because X is closed in $\mathbb{R}^n$. The sequence $\mathbf{v}_{k_1}, \mathbf{v}_{k_2}, \mathbf{v}_{k_3}, \ldots$ would also converge to $\mathbf{p}$, because

$$\lim_{j \to +\infty} |\mathbf{v}_{k_j} - \mathbf{u}_{k_j}| = 0.$$

But then the sequences

$$\varphi(\mathbf{u}_{k_1}), f(\mathbf{u}_{k_2}), f(\mathbf{u}_{k_3}), \ldots$$

and

$$\varphi(\mathbf{v}_{k_1}), f(\mathbf{v}_{k_2}), f(\mathbf{v}_{k_3}), \ldots$$

would both converge to $\varphi(\mathbf{p})$, because φ is continuous (see Lemma 5.2). Therefore

$$\lim_{j \to +\infty} \left|\varphi(\mathbf{u}_{k_j}) - \varphi(\mathbf{v}_{k_j})\right| = 0.$$

But, assuming that no positive real number δ could be found satisfying the stated requirements, the points $\mathbf{u}_j$ and $\mathbf{v}_j$ had been chosen for all positive integers j so that $|\mathbf{u}_j - \mathbf{v}_j| < 1/j$ and $|\varphi(\mathbf{u}_j) - \varphi(\mathbf{v}_j)| \geq \varepsilon$. Consequently $\varphi(\mathbf{u}_{k_j})$ and $\varphi(\mathbf{v}_{k_j})$ could not both converge to $\varphi(\mathbf{p})$ as j increases to infinity. Thus the assumption that no positive real number δ would have the required property would lead to a contradiction. We conclude therefore that, in order to avoid arriving at this contradiction, there must exist some positive real number δ such that $|\varphi(\mathbf{y}) - \varphi(\mathbf{z})| < \varepsilon$ for all points $\mathbf{y}$ and $\mathbf{z}$ of the set X satisfying $|\mathbf{y} - \mathbf{z}| < \delta$, as required. ∎