Course MAU23203—Michaelmas Term 2020.

Throughout this assignment, let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the real-valued function on the plane $\mathbb{R}^2$ defined so that $f(0,0) = 0$ and

$$f(x,y) = \frac{x^4 + y^4}{x^2 + y^2} \quad \textit{whenever} \quad (x,y) \neq (0,0).$$

Moreover, for all points (p,q) of $\mathbb{R}^2$, let

$$f_x(p,q), \quad f_y(p,q), \quad f_{xx}(p,q), \quad f_{xy}(p,q), \quad f_{yx}(p,q), \quad f_{yy}(p,q)$$

denote the values of the functions

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial x\,\partial y}, \quad \frac{\partial^2 f}{\partial y\,\partial x}, \quad \frac{\partial^2 f}{\partial y^2}$$

respectively at the point (p,q).

1. *Show that the second order partial derivatives of the function f exist at the point $(0,0)$, and determine the values of $f_x(0,0)$, $f_y(0,0)$, $f_{xx}(0,0)$, $f_{xy}(0,0)$, $f_{yx}(0,0)$ and $f_{yy}(0,0)$.*

   The function f satisfies $f(x,0) = x^2$ and $f(0,y) = y^2$ for all real numbers x and y. It follows from the definitions of the relevant partial derivatives that $f_x(x,0) = 2x$ and $f_y(0,y) = 2y$ for all real numbers x and y. Differentiating with respect to x and y respectively at $(0,0)$, we find that $f_{xx}(0,0) = 2$ and $f_{yy}(0,0) = 2$.

   Now, on taking first order partial derivatives, we find that

$$\begin{aligned} f_x(x,y) &= \frac{(x^2+y^2)4x^3 - (x^4+y^4)2x}{(x^2+y^2)^2} = \frac{2x^5 + 4x^3y^2 - 2xy^4}{(x^2+y^2)^2} \\ f_y(x,y) &= \frac{(x^2+y^2)4y^3 - (x^4+y^4)2y}{(x^2+y^2)^2} = \frac{2y^5 + 4y^3x^2 - 2yx^4}{(x^2+y^2)^2} \end{aligned}$$

   whenever $(x,y) \neq (0,0)$. Consequently

$$f_y(x,0) = 0 \quad \text{and} \quad f_x(0,y) = 0$$

   for all non-zero values of x and y. These identities hold also when $x = 0$ and when $y = 0$. Consequently $f_y(x,0) = 0$ and $f_x(0,y)$ for all real numbers x and y. It follows that $f_{xy}(0,0) = 0$ and $f_{yx}(0,0) = 0$.

2. *Write down expressions for the second order partial derivatives f_{xx}, f_{xy}, f_{yx} and f_{yy} at all points of the plane other than $(0,0)$.*

[You are simply asked to write down the relevant expressions. You are free to use software packages and or websites such as Wolfram Alpha to determine the values of the second order partial derivatives of the function at points other than $(0,0)$. ***Do not show your working, if you determine these second order partial derivatives without the assistance of software. Do not submit your rough working. Simply make sure that your expressions are correct, and write down the correct expressions.****]*

[Note, the following working is not required and should not be submitted. The the first two expressions may be verified by successively entering the query strings

```
d^2 / dx^2 (x^4 + y^4) / (x^2 + y^2)
d^2 / dx dy (x^4 + y^4) / (x^2 + y^2)
```

into Wolfram Alpha.]

$$\begin{aligned}
f_{xx}(x,y) &= \frac{1}{(x^2+y^2)^3}\Big((x^2+y^2)\frac{d}{dx}(2x^5+4x^3y^2-2xy^4) \\
&\qquad - 2(2x^5+4x^3y^2-2xy^4)\frac{\partial}{\partial x}(x^2+y^2)\Big) \\
&= \frac{1}{(x^2+y^2)^3}\Big((x^2+y^2)(10x^4+12x^2y^2-2y^4) \\
&\qquad - 4x(2x^5+4x^3y^2-2xy^4)\Big) \\
&= \frac{10x^6+22x^4y^2+10x^2y^4-2y^6-8x^6-16x^4y^2+8x^2y^4}{(x^2+y^2)^3} \\
&= \frac{2x^6+6x^4y^2+18x^2y^4-2y^6}{(x^2+y^2)^3}
\end{aligned}$$

$$\begin{aligned}
f_{yx}(x,y) &= \frac{1}{(x^2+y^2)^3}\Big((x^2+y^2)\frac{d}{dy}(2x^5+4x^3y^2-2xy^4) \\
&\qquad - 2(2x^5+4x^3y^2-2xy^4)\frac{\partial}{\partial y}(x^2+y^2)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(x^2+y^2)^3}\Big((x^2+y^2)(8x^3y - 8xy^3) \\
&\qquad - 4y(2x^5 + 4x^3y^2 - 2xy^4)\Big) \\
&= \frac{8x^5y - 8xy^5 - 8x^5y - 16x^3y^3 + 8xy^5}{(x^2+y^2)^3} \\
&= \frac{-16x^3y^3}{(x^2+y^2)^3}
\end{aligned}
$$

when $(x, y) \neq (0, 0)$.

Taking account of the fact that the expression defining $f(x, y)$ is unchanged when the variables x and y are interchanged, we deduce that

$$f_{xy}(x, y) = \frac{-16x^3y^3}{(x^2+y^2)^3}$$

and

$$f_{yy}(x, y) = \frac{2y^6 + 6y^4x^2 + 18y^2x^4 - 2x^6}{(x^2+y^2)^3}$$

whenever $(x, y) \neq (0, 0)$.

3. *Are the second order partial derivatives of the function f continuous at the point $(0, 0)$? [Briefly justify your answer.]*

   The second order partial derivatives of the function f are not continuous at the point $(0, 0)$. As is apparent from the expressions obtained in question 2, the values of these second order partial derivatives depend on the angle which the line segment joining $(0, 0)$ to (x, y) makes with the x-axis.

   In particular, note that $f_{xx}(0, y) = -2$ for $y \neq 0$, whereas $f_{xx}(x, 0) = 2$ for $x \neq 0$, hence f_{xx} is discontinuous at $(0, 0)$. Also $f_{yy}(x, 0) = -2$ for $x \neq 0$, whereas $f_{yy}(0, y) = 2$ for $x \neq 0$, hence f_{yy} is discontinuous at $(0, 0)$. Also $f_{xy}(t, t) = -2$ and $f_{yx}(t, t) = -2$ when $t \neq 0$, (and $f_{xy}(t, -t) = 2$ and $f_{yx}(t, -t) = 2$ when $t \neq 0$), $f_{xy}(0, 0) = 0$ and $f_{yx}(0, 0) = 0$.

   Alternatively

   In particular, note that $f_{xx}(t, t) = 3$, $f_{yy}(t, t) = 3$ $f_{xy}(t, t) = -2$ and $f_{yx} = -2$ when $t \neq 0$, but $f_{xx}(0, 0) = 2$, $f_{yy} = 2$ $f_{xy}(0, 0) = 0$ and $f_{yx}(0, 0) = 0$.

4. *Let*

$$\begin{aligned} g(x,y) &= f(x,y) - f(0,0) - x f_x(0,0) - y f_y(0,0) \\ &\quad - \tfrac{1}{2}x^2 f_{xx}(0,0) - \tfrac{1}{2}xy(f_{xy}(0,0) + f_{yx}(0,0)) - \tfrac{1}{2}y^2 f_{yy}(0,0). \end{aligned}$$

for all points (x,y) of $\mathbb{R}^2$. Determine an expression that specifies the value of the function g at all points (x,y) of the plane other than the origin.

It follows from the results and calculations of question 1 that $f_x(0,0) = 0$, $f_y(0,0) = 0$, $f_{xx}(0,0) = 2$, $f_{yy}(0,0) = 2$, $f_{xy}(0,0) = 0$ and $f_{yx}(0,0) = 0$. Also $f(0,0) = 0$. Consequently

$$\begin{aligned} g(x,y) &= f(x,y) - x^2 - y^2 \\ &= \frac{x^4 + y^4}{x^2 + y^2} - x^2 - y^2 \\ &= \frac{x^4 + y^4 - (x^2 + y^2)^2}{x^2 + y^2} \\ &= -\frac{-2x^2y^2}{x^2 + y^2} \end{aligned}$$

5. *Is it the case that the function g determined in the previous question has the property that*

$$\lim_{(x,y)\to(0,0)} \frac{g(x,y)}{x^2 + y^2} = 0?$$

[Briefly justify your answer.]

It is not the case that this limit is equal to zero. Indeed

$$\frac{g(x,y)}{x^2 + y^2} = \frac{-2x^2y^2}{(x^2 + y^2)^2}$$

when $(x,y) \neq 0$. If it were the case that

$$\lim_{(x,y)\to(0,0)} \frac{g(x,y)}{x^2 + y^2} = 0$$

then it would follow that

$$\lim_{t\to 0} \frac{g(t,t)}{2t^2} = 0.$$

But $g(t,t)/(2t^2)$ has the value $-\frac{1}{2}$ whenever $t \neq 0$, and therefore cannot tend to zero as t tends to zero.