Course MAU23203—Michaelmas Term 2020.
Worked Solutions for Assignment I.

1. *Let X be a closed subset of $\mathbb{R}^2$, let $f\colon X \to \mathbb{R}$ be a continuous real-valued function on X, and let*

$$Y = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in X \text{ and } z = f(x, y)\}.$$

Prove that the set Y is a closed subset of $\mathbb{R}^3$.

First Solution

Let (p, q, r) be a point of $\mathbb{R}^3 \setminus Y$. Consider first the case in which $(p, q) \notin X$. In that case there exists some positive real number δ such that the open disk in $\mathbb{R}^2$ of radius δ centred on (p, q) is contained in the complement of X, because X is closed in $\mathbb{R}^2$. Then, for all points (x, y, z) of the open ball of radius δ about the point (p, q, r), the point (x, y) belongs to $\mathbb{R}^2 \setminus X$. Consequently the open ball of radius δ about the point (p, q, r) is contained in the complement of Y.

Next consider the case in which $(p, q, r) \in \mathbb{R}^3 \setminus Y$ and $(p, q) \in X$. Then $r \neq f(p, q)$. The continuity of the function f ensures the existence of a positive number η small enough to ensure that $|f(x, y) - f(p, q)| < \frac{1}{2}|r - f(p, q)|$ whenever $(x, y) \in X$ and $\sqrt{(x-p)^2 + (y-q)^2} < \eta$. Let δ be the minimum of η and $\frac{1}{2}|r - f(p, q)|$. Then $\delta > 0$, and if the point (x, y, z) lies within a distance δ of (p, q, r) and if $(x, y) \in X$ then $|f(x, y) - f(p, q)| < \frac{1}{2}|r - f(p, q)|$. Also $|z - r| < \delta$, and consequently

$$|z - f(p, q)| > |r - f(p, q)| - \delta \geq \tfrac{1}{2}|r - f(p, q)|.$$

It follows that $z \neq f(x, y)$. Consequently the open ball of radius δ about the point (p, q, r) is contained in the complement of the set Y. Combining the results of the two cases discussed above, we conclude that the complement of the set Y is open, and therefore the set Y itself is closed.

Second Solution

Let

$$Z = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in X\}.$$

Then Z is closed in $\mathbb{R}^3$, because it is the preimage of the closed set X under the continuous function from $\mathbb{R}^3$ to $\mathbb{R}^2$ that maps each point

(x, y, z) of $\mathbb{R}^3$ to (x, y). The set Y is a subset of the closed set Z. It follows that, given any point (p, q, r) of $\mathbb{R}^3 \setminus Z$, there exists some open ball of positive radius centred on that point that is wholly contained within the complement of the set Z and is therefore wholly contained within the complement of the set Y.

Now let (p, q, r) be a point of $Z \setminus Y$. Then $r \neq f(p, q)$. First suppose that $r > f(p, q)$. Let $c = \frac{1}{2}(r + f(p, q))$. Then $f(p, q) < c < r$. Now

$$\{(x, y) \in X : f(x, y) < c\}$$

is open in X, because it is the preimage of the open interval $(-\infty, c)$ under the continuous map f. Consequently there exists some positive real number η such that $f(x, y) < c$ for all points (x, y) of X that lie within a distance η of the the point (p, q). Let

$$V = \{(x, y, z) \in \mathbb{R}^3 : \sqrt{(x - p)^2 + (y - q)^2} < \eta \text{ and } z > c\}.$$

Then the set V is open in $\mathbb{R}^3$ because it is the intersection of the open set $\{(x, y, z) \in \mathbb{R}^3 : z > c\}$ with the preimage of the open disk in $\mathbb{R}^2$ of radius η about the point (p, q) under the continuous map from $\mathbb{R}^3$ to $\mathbb{R}^2$ that sends each point (x, y, z) of $\mathbb{R}^3$ to (x, y), and the preimage of the open disk of radius η centred on the point (p, q), being the preimage of an open set under a continuous map, must itself be open in $\mathbb{R}^3$. Also $V \cap Y = \emptyset$. The point (p, q, r) belongs to the open set V. Consequently there exists an open ball of positive radius centred on the point (p, q, r) that is wholly contained within the open set V. This open ball is then wholly contained in the complement of the set Y.

Now let (p, q, r) be a point of $Z \setminus Y$ for which $r < f(p, q)$. In this case take $c = \frac{1}{2}(r + f(p, q))$, as before. Then $r < c < f(p, q)$. There then exists some positive real number η such that $f(x, y) > c$ for all points (x, y) of X that lie within a distance η of the point (p, q). Take

$$W = \{(x, y, z) \in \mathbb{R}^3 : \sqrt{(x - p)^2 + (y - q)^2} < \eta \text{ and } z < c\}.$$

Then W is open in $\mathbb{R}^3$, and $W \cap Y = \emptyset$. The point (p, q, r) belongs to the open set W. Consequently there exists an open ball of positive radius centred on the point (p, q, r) that is wholly contained within the open set W. This open ball is then wholly contained in the complement of the set W.

We conclude therefore that, given any point (p, q, r) of the complement of the set Y, there exists an open ball of positive radius centred on that point that is wholly contained within the complement of the set Y. Consequently the complement of Y in $\mathbb{R}^3$ is an open set, and therefore the set Y itself is a closed subset of $\mathbb{R}^3$.

Third Solution

Let

$$Z = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in X\}.$$

Then Z is closed in $\mathbb{R}^3$, because it is the preimage of the closed set X under the continuous function from $\mathbb{R}^3$ to $\mathbb{R}^2$ that maps each point (x, y, z) of $\mathbb{R}^3$ to (x, y). The set Y is a subset of the closed set Z. It follows [as in the *Second Solution* above] that, given any point (p, q, r) of $\mathbb{R}^3 \setminus Z$, there exists some open ball of positive radius centred on that point that is wholly contained within the complement of the set Z and is therefore wholly contained within the complement of the set Y.

Now let $g\colon Z \to \mathbb{R}$ be the continuous function from Z to $\mathbb{R}$ defined so that $g(x, y, z) = z - f(x, y)$. Then the function g, being the difference of two continuous functions, is itself a continuous function. Then $Z \setminus Y = g^{-1}(\mathbb{R} \setminus \{0\})$. Now the set $\mathbb{R} \setminus \{0\}$ of non-zero real numbers is open in $\mathbb{R}$, and the preimage of an open set under a continuous function with domain Z is itself open in Z. Consequently $Z \setminus Y$ is open in Z and therefore, given any point of $\mathbf{p}$ of $Z \setminus Y$, there exists some open ball of positive radius centred on the point $\mathbf{p}$ whose intersection with the set Z is wholly contained in $Z \setminus Y$. This open ball is then contained in the complement $\mathbb{R}^3 \setminus Y$ in $\mathbb{R}^3$ of the set Y, because $Y \subset Z$.

It follows from these results that the complement of the set Y is an open subset of $\mathbb{R}^3$, and therefore the set Y itself is a closed subset of $\mathbb{R}^3$.

Fourth Solution

A subset of $\mathbb{R}^3$ is closed in $\mathbb{R}^3$ if and only if the limit of any convergent sequence whose members belong to the subset in question belongs to that subset.

For each positive integer j let (x_j, y_j, z_j) be the jth member of a convergent sequence of points of Y, and let $(p, q, r) = \lim_{j \to +\infty} (x_j, y_j, z_j)$. Then $(x_j, y_j) \in X$ for all positive integers j. The set X is closed and $(p, q) = \lim_{j \to +\infty} (x_j, y_j)$. It follows that $(p, q) \in X$. Moreover

$$r = \lim_{j \to +\infty} z_j = \lim_{j \to +\infty} f(x_j, y_j) = f\left(\lim_{j \to +\infty} x_j, \lim_{j \to +\infty} y_j \right) = f(p, q),$$

because the function f is continuous and the limit of the images of the members of a convergent sequence of points under a continuous function is the image of the limit of that convergent sequence of points. It follows that $(p, q, r) \in Y$. This result establishes that the set Y is closed in $\mathbb{R}^3$.

2. *Let $f\colon (0, +\infty) \to \mathbb{R}$ be a real-valued function defined over the set of positive real numbers with the properties that $f(x) > 0$ for all positive real numbers x and $\lim_{x \to 0^+} f(x) = 0$. (In other words, $f(x)$ tends to zero as x tends to zero in the set of positive real numbers.) Let*

$$X = \{(x, y) \in \mathbb{R}^2 : -f(x) \leq y \leq f(x)\},$$

let $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots$ be an infinite sequence of points that all belong to the set X, and let $\mathbf{p}_j = (x_j, y_j)$ for each positive integer j. Suppose that $\lim_{j \to +\infty} x_j = 0$. Prove that

$$\lim_{j \to +\infty} \mathbf{p}_j = (0, 0).$$

Solution

The function f is continuous, and the limit of the images of the members of a convergent sequence of points under a continuous function is the image of the limit of that convergent sequence of points. Now $\lim_{j \to +\infty} x_j = 0$ and $\lim_{x \to 0^+} f(x) = 0$. It follows that $\lim_{j \to +\infty} f(x_j) = 0$. Now $-f(x_j) \leq y_j \leq f(x_j)$ for all positive integers j, and

$$\lim_{j \to +\infty} -f(x_j) = -\lim_{j \to +\infty} f(x_j) = 0.$$

It follows from the *Squeeze Theorem* (or *Sandwich Theorem*) of calculus, or analysis, in a single real variable, that $\lim_{j \to +\infty} y_j = 0$. Also a sequence of points in 2-dimensional space $\mathbb{R}^2$ is convergent if and only if its components are convergent, in which case the components of the limit of the sequence of points is the point whose components are the limits of the corresponding sequences of components. Consequently

$$\lim_{j \to +\infty} \mathbf{p}_j = \lim_{j \to +\infty} (x_j, y_j) = \left(\lim_{j \to +\infty} x_j, \lim_{j \to +\infty} y_j \right) = (0, 0),$$

as required.