Module MAU23203: Analysis in Several Real Variables
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Part II (Sections 6 and 7)

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Contents

6 The Riemann Integral in One Dimension

6.1 Darboux Sums and the Riemann Integral

The approach to the theory of integration discussed below was developed by Jean-Gaston Darboux (1842–1917). The integral defined using lower and upper sums in the manner described below is sometimes referred to as the *Darboux integral* of a function on a given interval. However the class of functions that are integrable according to the definitions introduced by Darboux is the class of *Riemann-integrable* functions. Thus the approach using Darboux sums provides a convenient approach to define and establish the basic properties of the *Riemann integral*.

Definition A *partition* P of an interval $[a, b]$ is a set $\{u_0, u_1, u_2, \ldots, u_N\}$ of real numbers satisfying $a = u_0 < u_1 < u_2 < \cdots < u_{N-1} < u_N = b$.

Given any bounded real-valued function f on $[a, b]$, the *upper sum* (or *upper Darboux sum*) $U(P, f)$ of f for the partition P of $[a, b]$ is defined so that

$$U(P, f) = \sum_{i=1}^{N} M_i (u_i - u_{i-1}),$$

where $M_i = \sup\{f(x) : u_{i-1} \leq x \leq u_i\}$.

Similarly the *lower sum* (or *lower Darboux sum*) $L(P, f)$ of f for the partition P of $[a, b]$ is defined so that

$$L(P, f) = \sum_{i=1}^{N} m_i (u_i - u_{i-1}),$$

where $m_i = \inf\{f(x) : u_{i-1} \leq x \leq u_i\}$.

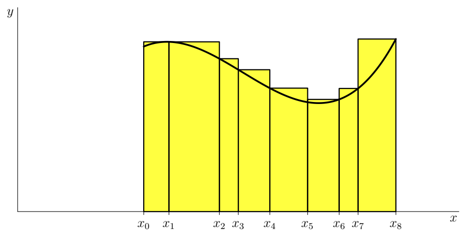
Clearly $L(P, f) \leq U(P, f)$. Moreover $\sum\limits_{i=1}^{N} (u_i - u_{i-1}) = b - a$, and therefore

$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a),$$

for any real numbers m and M satisfying $m \leq f(x) \leq M$ for all $x \in [a, b]$.

Definition Let f be a bounded real-valued function on the interval $[a, b]$, where $a < b$. The *upper Riemann integral* $\mathcal{U}\int_a^b f(x)\,dx$ (or *upper Darboux integral*) and the *lower Riemann integral* $\mathcal{L}\int_a^b f(x)\,dx$ (or *lower Darboux integral*) of the function f on $[a, b]$ are defined by

$$\begin{aligned} \mathcal{U}\int_a^b f(x)\,dx &= \inf\left\{U(P, f) : P \text{ is a partition of } [a, b]\right\}, \\ \mathcal{L}\int_a^b f(x)\,dx &= \sup\left\{L(P, f) : P \text{ is a partition of } [a, b]\right\}. \end{aligned}$$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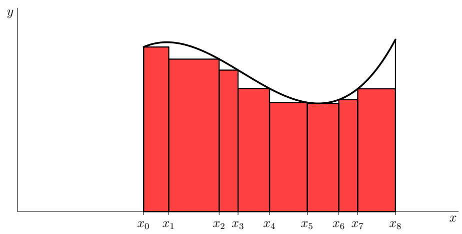
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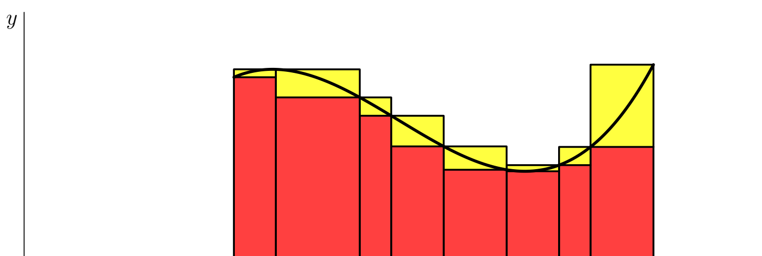
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The definition of upper and lower integrals thus requires that $\mathcal{U}\int_a^b f(x)\,dx$ be the infimum of the values of $U(P,f)$ and that $\mathcal{L}\int_a^b f(x)\,dx$ be the supremum of the values of $L(P,f)$ as P ranges over all possible partitions of the interval $[a,b]$.

Definition A bounded function $f\colon [a,b] \to \mathbb{R}$ on a closed bounded interval $[a,b]$ is said to be *Riemann-integrable* (or *Darboux-integrable*) on $[a,b]$ if

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

in which case the *Riemann integral* $\int_a^b f(x)\,dx$ (or *Darboux integral*) of f on $[a,b]$ is defined to be the common value of $\mathcal{U}\int_a^b f(x)\,dx$ and $\mathcal{L}\int_a^b f(x)\,dx$.

When $a > b$ we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

for all Riemann-integrable functions f on $[b,a]$. We set $\int_a^b f(x)\,dx = 0$ when $b = a$.

If f and g are bounded Riemann-integrable functions on the interval $[a,b]$, and if $f(x) \leq g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$, since $L(P,f) \leq L(P,g)$ and $U(P,f) \leq U(P,g)$ for all partitions P of $[a,b]$.

Definition Let P and R be partitions of $[a,b]$, given by $P = \{u_0, u_1, \ldots, u_N\}$ and $R = \{v_0, v_1, \ldots, v_L\}$. We say that the partition R is a *refinement* of P if $P \subset R$, so that, for each u_i in P, there is some v_j in R with $u_i = v_j$.

Lemma 6.1 *Let R be a refinement of some partition P of $[a,b]$. Then*

$$L(R,f) \geq L(P,f) \quad \textit{and} \quad U(R,f) \leq U(P,f)$$

for any bounded function $f\colon [a,b] \to \mathbb{R}$.

Proof Let $P = \{u_0, u_1, \ldots, u_N\}$ and $R = \{v_0, v_1, \ldots, v_L\}$, where $a = u_0 < u_1 < \cdots < u_N = b$ and $a = v_0 < v_1 < \cdots < v_L = b$. Now for each integer i between 0 and N there exists some integer $j(i)$ between 0 and L such that $u_i = v_{j(i)}$ for each i, since R is a refinement of P. Moreover $0 = j(0) < j(1) < \cdots < j(N) = L$. For each i, let R_i be the partition of

$[u_{i-1}, u_i]$ given by $R_i = \{v_j : j(i-1) \leq j \leq j(i)\}$. Then $L(R, f) = \sum_{i=1}^{N} L(R_i, f)$ and $U(R, f) = \sum_{i=1}^{N} U(R_i, f)$. Moreover

$$m_i(u_i - u_{i-1}) \leq L(R_i, f) \leq U(R_i, f) \leq M_i(u_i - u_{i-1}),$$

since $m_i \leq f(x) \leq M_i$ for all $x \in [u_{i-1}, u_i]$. On summing these inequalities over i, we deduce that $L(P, f) \leq L(R, f) \leq U(R, f) \leq U(P, f)$, as required. ∎

Given any two partitions P and Q of $[a, b]$ there exists a partition R of $[a, b]$ which is a refinement of both P and Q. For example, we can take $R = P \cup Q$. Such a partition is said to be a *common refinement* of the partitions P and Q.

Lemma 6.2 *Let f be a bounded real-valued function on the interval $[a, b]$. Then*

$$\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx.$$

Proof Let P and Q be partitions of $[a, b]$, and let R be a common refinement of P and Q. It follows from Lemma 6.1 that $L(P, f) \leq L(R, f) \leq U(R, f) \leq U(Q, f)$. Thus, on taking the supremum of the left hand side of the inequality $L(P, f) \leq U(Q, f)$ as P ranges over all possible partitions of the interval $[a, b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \leq U(Q, f)$ for all partitions Q of $[a, b]$. But then, taking the infimum of the right hand side of this inequality as Q ranges over all possible partitions of $[a, b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx$, as required. ∎

Example Let $f(x) = cx + d$, where $c \geq 0$. We shall show that f is Riemann-integrable on $[0, 1]$ and evaluate $\int_0^1 f(x)\,dx$ from first principles.

For each positive integer N, let P_N denote the partition of $[0, 1]$ into N subintervals of equal length. Thus $P_N = \{u_0, u_1, \ldots, u_N\}$, where $u_i = i/N$. Now the function f takes values between $(i-1)c/N + d$ and $ic/N + d$ on the interval $[u_{i-1}, u_i]$, and therefore

$$m_i = \frac{(i-1)c}{N} + d, \qquad M_i = \frac{ic}{N} + d$$

where $m_i = \inf\{f(x) : u_{i-1} \leq x \leq u_i\}$ and $M_i = \sup\{f(x) : u_{i-1} \leq x \leq u_i\}$. Thus

$$
\begin{aligned}
L(P_N, f) &= \sum_{i=1}^{N} m_i(u_i - u_{i-1}) = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{ci}{N} + d - \frac{c}{N}\right) \\
&= \frac{c(N+1)}{2N} + d - \frac{c}{N} = \frac{c}{2} + d - \frac{c}{2N}, \\
U(P_N, f) &= \sum_{i=1}^{N} M_i(u_i - u_{i-1}) = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{ci}{N} + d\right) \\
&= \frac{c(N+1)}{2N} + d = \frac{c}{2} + d + \frac{c}{2N}.
\end{aligned}
$$

It follows that

$$\lim_{N\to+\infty} L(P_N, f) = \frac{c}{2} + d$$

and

$$\lim_{N\to+\infty} U(P_N, f) = \frac{c}{2} + d$$

Now $L(P_N, f) \leq \mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx \leq U(P_N, f)$ for all positive integers N. It follows that $\mathcal{L}\int_a^b f(x)\,dx = \frac{1}{2}c + d = \mathcal{U}\int_a^b f(x)\,dx$. Thus f is Riemann-integrable on the interval $[0, 1]$, and $\int_0^1 f(x)\,dx = \frac{1}{2}c + d$.

Example Let $f\colon [0, 1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Let P be a partition of the interval $[0, 1]$ given by $P = \{u_0, u_1, u_2, \ldots, u_N\}$, where $0 = u_0 < u_1 < u_2 < \cdots < u_N = 1$. Then

$$\inf\{f(x) : u_{i-1} \leq x \leq u_i\} = 0, \qquad \sup\{f(x) : u_{i-1} \leq x \leq u_i\} = 1,$$

for $i = 1, 2, \ldots, N$, and thus $L(P, f) = 0$ and $U(P, f) = 1$ for all partitions P of the interval $[0, 1]$. It follows that $\mathcal{L}\int_0^1 f(x)\,dx = 0$ and $\mathcal{U}\int_0^1 f(x)\,dx = 1$, and therefore the function f is not Riemann-integrable on the interval $[0, 1]$.

6.2 Basic Properties of the Riemann Integral

Lemma 6.3 *Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function on a closed bounded interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$. Then the lower and upper Riemann integrals of f and $-f$ are related by the identities*

$$
\begin{aligned}
\mathcal{U}\int_a^b (-f(x))\,dx &= -\mathcal{L}\int_a^b f(x)\,dx, \\
\mathcal{L}\int_a^b (-f(x))\,dx &= -\mathcal{U}\int_a^b f(x)\,dx.
\end{aligned}
$$

Proof Let P be a partition of $[a, b]$, and let $P = \{u_0, u_1, u_2, \ldots, u_N\}$, where

$$a = u_0 < u_1 < u_2 < \cdots < u_N = b.$$

Also let

$$\begin{aligned} m_i[f] &= \inf\{f(x) : u_{i-1} \leq x \leq u_i\}, \\ M_i[f] &= \sup\{f(x) : u_{i-1} \leq x \leq u_i\}, \\ m_i[-f] &= \inf\{-f(x) : u_{i-1} \leq x \leq u_i\}, \\ M_i[-f] &= \sup\{-f(x) : u_{i-1} \leq x \leq u_i\} \end{aligned}$$

for $i = 1, 2, \ldots, N$. Then $m_i[-f] = -M_i[f]$ and $M_i[-f] = -m_i[f]$, and therefore

$$\begin{aligned} L(P, -f) &= \sum_{i=1}^{N} m_i[-f](u_i - u_{i-1}) = -\sum_{i=1}^{N} M_i[f](u_i - u_{i-1}) \\ &= -U(P, f). \end{aligned}$$

Thus $L(P, -f) = -U(P, f)$ for all partitions P of the interval $[a, b]$. Similarly $U(P, -f) = -L(P, f)$ for all partitions P of that interval. It follows from the definition of the upper and lower integrals that

$$\begin{aligned} \mathcal{U}\int_a^b (-f(x))\,dx &= \inf\{U(P, -f) : P \text{ is a partition of } [a, b]\} \\ &= \inf\{-L(P, f) : P \text{ is a partition of } [a, b]\} \\ &= -\sup\{L(P, f) : P \text{ is a partition of } [a, b]\} \\ &= -\mathcal{L}\int_a^b f(x)\,dx \end{aligned}$$

Similarly

$$\begin{aligned} \mathcal{L}\int_a^b (-f(x))\,dx &= \sup\{L(P, -f) : P \text{ is a partition of } [a, b]\} \\ &= \sup\{-U(P, f) : P \text{ is a partition of } [a, b]\} \\ &= -\inf\{U(P, f) : P \text{ is a partition of } [a, b]\} \\ &= -\mathcal{U}\int_a^b f(x)\,dx. \end{aligned}$$

This completes the proof. ∎

Proposition 6.4 *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$. Then the functions $f+g$ and $f-g$ are Riemann-integrable on $[a,b]$, and moreover*

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx,$$

and

$$\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

Proof Let some strictly positive real number ε be given. The definition of Riemann-integrability and the Riemann integral ensures that there exist partitions P_1, P_2, P_3 and P_4 of $[a,b]$ for which

$$L(P_1,f) > \int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon,$$

$$U(P_2,f) < \int_a^b f(x)\,dx + \tfrac{1}{2}\varepsilon,$$

$$L(P_3,g) > \int_a^b g(x)\,dx - \tfrac{1}{2}\varepsilon$$

and

$$U(P_4,g) < \int_a^b g(x)\,dx + \tfrac{1}{2}\varepsilon.$$

Let P be a common refinement of P_1, P_2, P_3 and P_4. Applying Lemma 6.1, we see that

$$L(P,f) \geq L(P_1,f) > \int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon,$$

$$U(P,f) \leq U(P_2,f) < \int_a^b f(x)\,dx + \tfrac{1}{2}\varepsilon,$$

$$L(P,g) \geq L(P_3,g) > \int_a^b g(x)\,dx - \tfrac{1}{2}\varepsilon$$

and

$$U(P,g) \leq U(P_4,g) < \int_a^b g(x)\,dx + \tfrac{1}{2}\varepsilon.$$

Let $P = \{u_0, u_1, \ldots, u_N\}$, where

$$a = u_0 < u_1 < \cdots < u_N = b,$$

and let

$$\begin{aligned}
M_i[f] &= \sup\{f(x) : u_{i-1} \leq x \leq u_i\}, \\
m_i[f] &= \inf\{f(x) : u_{i-1} \leq x \leq u_i\}, \\
M_i[g] &= \sup\{g(x) : u_{i-1} \leq x \leq u_i\}, \\
m_i[g] &= \inf\{g(x) : u_{i-1} \leq x \leq u_i\}, \\
M_i[f+g] &= \sup\{f(x)+g(x) : u_{i-1} \leq x \leq u_i\}, \\
m_i[f+g] &= \inf\{f(x)+g(x) : u_{i-1} \leq x \leq u_i\}.
\end{aligned}$$

Now the inequalities

$$m_i[f] + m_i[g] \leq f(x) + g(x) \leq M_i[f] + M_i[g]$$

are satisfied for $i = 1, 2, \ldots, N$ and for all $x \in [u_{i-1}, u_i]$. It follows from the definitions of $M_i[f+g]$ and $m_i[f+g]$ as the least upper bound and greatest lower bound respectively of the values of $f(x)+g(x)$ on the interval $[u_{i-1}, u_i]$ that

$$m_i[f] + m_i[g] \leq m_i[f+g] \leq M_i[f+g] \leq M_i[f] + M_i[g]$$

for $i = 1, 2, \ldots, N$. Multiplying these inequalities by the lengths $u_i - u_{i-1}$ of the subintervals determined by the partition P, and then summing over $i = 1, 2, \ldots, N$, we deduce that

$$L(P,f) + L(P,g) \leq L(P,f+g) \leq U(P,f+g) \leq U(P,f) + U(P,g).$$

Now inequalities satisfied by the Darboux upper and lower sums for the partition P guaranteed by the choice of P (as described above) then ensure that

$$L(P,f) + L(P,g) > \int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \varepsilon$$

and

$$U(P,f) + U(P,g) < \int_a^b f(x)\,dx + \int_a^b g(x)\,dx + \varepsilon.$$

It follows that

$$\begin{aligned}
\int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \varepsilon &< L(P,f+g) \leq U(P,f+g) \\
&< \int_a^b f(x)\,dx + \int_a^b g(x)\,dx + \varepsilon.
\end{aligned}$$

But

$$\begin{aligned} L(P, f+g) &\leq \mathcal{L}\int_a^b (f(x)+g(x))\,dx \\ &\leq \mathcal{U}\int_a^b (f(x)+g(x))\,dx \leq U(P, f+g). \end{aligned}$$

It follows therefore that

$$\begin{aligned} \int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \varepsilon &< \mathcal{L}\int_a^b (f(x)+g(x))\,dx \\ &\leq \mathcal{U}\int_a^b (f(x)+g(x))\,dx \\ &< \int_a^b f(x)\,dx + \int_a^b g(x)\,dx + \varepsilon. \end{aligned}$$

These latter inequalities must hold for all positive real numbers ε, no matter how small. It follows that

$$\begin{aligned} \int_a^b f(x)\,dx + \int_a^b g(x)\,dx &\leq \mathcal{L}\int_a^b (f(x)+g(x))\,dx \\ &\leq \mathcal{U}\int_a^b (f(x)+g(x))\,dx \\ &\leq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx. \end{aligned}$$

The extreme left hand and extreme right hand sides of the above chain of inequalities are equal. Therefore

$$\begin{aligned} \mathcal{L}\int_a^b (f(x)+g(x))\,dx &= \mathcal{U}\int_a^b (f(x)+g(x))\,dx \\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx. \end{aligned}$$

We conclude therefore that the function $f+g$ is Riemann-integrable and that the value of the Riemann integral of this function is the sum of the integrals of the functions f and g on the interval $[a,b]$.

On replacing g by $-g$, we may deduce the corresponding result for the function $f-g$, thereby completing the proof. ∎

Proposition 6.5 *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$. Then the* *function f is Riemann-integrable on $[a, b]$ if and only if, given any positive real number ε, there exists a partition P of $[a, b]$ with the property that*

$$U(P, f) - L(P, f) < \varepsilon.$$

Proof First suppose that $f\colon [a, b] \to \mathbb{R}$ is Riemann-integrable on $[a, b]$. Let some positive real number ε be given. Then

$$\int_a^b f(x)\,dx$$

is equal to the common value of the lower and upper integrals of the function f on $[a, b]$, and therefore there exist partitions Q and R of $[a, b]$ for which

$$L(Q, f) > \int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon$$

and

$$U(R, f) < \int_a^b f(x)\,dx + \tfrac{1}{2}\varepsilon.$$

Let P be a common refinement of the partitions Q and R. Now

$$L(Q, f) \le L(P, f) \le U(P, f) \le U(R, f).$$

(see Lemma 6.1). It follows that

$$U(P, f) - L(P, f) \le U(R, f) - L(Q, f) < \varepsilon.$$

Now suppose that $f\colon [a, b] \to \mathbb{R}$ is a bounded function on $[a, b]$ with the property that, given any positive real number ε, there exists a partition P of $[a, b]$ for which $U(P, f) - L(P, f) < \varepsilon$. Let $\varepsilon > 0$ be given. Then there exists a partition P of $[a, b]$ for which $U(P, f) - L(P, f) < \varepsilon$. Now it follows from the definitions of the upper and lower integrals that

$$L(P, f) \le \mathcal{L}\int_a^b f(x)\,dx \le \mathcal{U}\int_a^b f(x)\,dx \le U(P, f),$$

and therefore

$$\mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx < U(P, f) - L(P, f) < \varepsilon.$$

Thus the difference between the values of the upper and lower integrals of f on $[a, b]$ must be less than every strictly positive real number ε, and therefore

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx.$$

This completes the proof. ∎

Lemma 6.6 *Let $f_1, f_2, \ldots, f_s$ and h be bounded real-valued functions on a closed bounded interval $[a, b]$, where a and b are real numbers satisfying $a < b$. Suppose that there exists a positive constant K with the property that*

$$|h(v) - h(w)| \leq K \sum_{j=1}^{s} |f_j(v) - f_j(w)|$$

for all $v, w \in [a, b]$. Then the upper and lower Darboux sums of these real-valued functions satisfy the inequality

$$U(P, h) - L(P, h) \leq K \sum_{j=1}^{s} (U(P, f_j) - L(P, f_j))$$

for all partitions P of the interval $[a, b]$.

Proof Let P be a partition of $[a, b]$, and let $P = \{u_0, u_1, \ldots, u_N\}$, where

$$a = u_0 < u_1 < \cdots < u_N = b,$$

and let

$$\begin{aligned} M_i[f_j] &= \sup\{f_j(x) : u_{i-1} \leq x \leq u_i\}, \\ m_i[f_j] &= \inf\{f_j(x) : u_{i-1} \leq x \leq u_i\} \end{aligned}$$

for $j = 1, 2, \ldots, s$ and $i = 1, 2, \ldots, N$, and

$$\begin{aligned} M_i[h] &= \sup\{h(x) : u_{i-1} \leq x \leq u_i\}, \\ m_i[h] &= \inf\{h(x) : u_{i-1} \leq x \leq u_i\} \end{aligned}$$

for $i = 1, 2, \ldots, N$.

Let i be an integer between 1 and N. The definitions of $M_i[h]$ and $m_i[h]$ ensure that, given any positive real number δ, there exist $v_i, w_i \in [u_{i-1}, u_i]$ such that $h(v_i) > M_i[h] - \delta$ and $h(w_i) < m_i[h] + \delta$. But then

$$\begin{aligned} M_i[h] - m_i[h] - 2\delta &< h(v_i) - h(w_i) \leq K \sum_{j=1}^{s} |f_j(v_i) - f_j(w_i)| \\ &\leq K \sum_{j=1}^{s} (M_i[f_j] - m_i[f_j]). \end{aligned}$$

The inequality

$$M_i[h] - m_i[h] - 2\delta < K \sum_{j=1}^{s} (M_i[f_j] - m_i[f_j])$$

therefore holds for all positive values of the real number δ, no matter how small, and therefore

$$M_i[h] - m_i[h] \leq K \sum_{j=1}^{s} (M_i[f_j] - m_i[f_j]).$$

Multiplying both sides of this inequality by the length $u_i - u_{i-1}$ of the ith subinterval of $[a, b]$ determined by the partition P, and summing for $i = 1, 2, \ldots, N$, we find that

$$\begin{aligned} U(P, h) - L(P, h) &= \sum_{i=1}^{N} (M_i[h] - m_i[h])(u_i - u_{i-1}) \\ &\leq K \sum_{j=1}^{s} \sum_{i=1}^{N} (M_i[f_j] - m_i[f_j])(u_i - u_{i-1}) \\ &\leq K \sum_{j=1}^{s} (U(P, f_j) - L(P, f_j)). \end{aligned}$$

We conclude therefore that

$$U(P, h) - L(P, h) \leq K \sum_{j=1}^{s} (U(P, f_j) - L(P, f_j))$$

for all partitions P of the interval $[a, b]$, as required. ∎

Proposition 6.7 *Let $f_1, f_2, \ldots, f_s$ be bounded Riemann-integrable real-valued functions on a closed bounded interval $[a, b]$, where a and b are real numbers satisfying $a < b$, and let h be a bounded real-valued function on $[a, b]$. Suppose that there exists a positive constant K with the property that*

$$|h(v) - h(w)| \leq K \sum_{j=1}^{s} |f_j(v) - f_j(w)|$$

for all $u, v \in [a, b]$. Then the function h is Riemann-integrable on $[a, b]$.

Proof Given any positive real number ε, there exist partitions $P_1, P_2, \ldots, P_s$ of $[a, b]$ with the property that

$$U(P_j, f_j) - L(P_j, f_j) < \frac{\varepsilon}{sK}$$

for $j = 1, 2, \ldots, s$ (see Proposition 6.5). Let P be a common refinement of the partitions $P_1, P_2, \ldots, P_s$. Then

$$U(P, f_j) - L(P, f_j) \leq U(P_j, f_j) - L(P_j, f_j) < \frac{\varepsilon}{sK}$$

for $j = 1, 2, \ldots, s$ (see Lemma 6.1). It then follows from Lemma 6.6 that

$$U(P, h) - L(P, h) \leq K \sum_{j=1}^{s} (U(P, f_j) - L(P, f_j)) < \varepsilon.$$

On applying Proposition 6.5, we therefore conclude that the function h is Riemann-integrable on $[a, b]$, as required. ∎

Proposition 6.8 *Let $f\colon [a, b] \to \mathbb{R}$ and $g\colon [a, b] \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed bounded interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$. Then the function $f \cdot g$ is Riemann-integrable on $[a, b]$, where $(f \cdot g)(x) = f(x)g(x)$ for all $x \in [a, b]$.*

Proof The functions f and g are bounded on $[a, b]$, and therefore there exists some positive real number K with the property that $|f(x)| \leq K$ and $|g(x)| \leq K$ for all $x \in [a, b]$. But then

$$\begin{aligned} |f(v)g(v) - f(w)g(w)| &= |f(v)(g(v) - g(w)) + (f(v) - f(w))g(w)| \\ &\leq |f(v)(g(v) - g(w))| + |(f(v) - f(w))g(w)| \\ &\leq K\,(|g(v) - g(w)| + |f(v) - f(w)|) \end{aligned}$$

for all $v, w \in [a, b]$. The result therefore follows directly on applying Proposition 6.7. ∎

Proposition 6.9 *Let $f\colon [a, b] \to \mathbb{R}$ be a bounded Riemann-integrable function on a closed interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$, and let $|f|\colon [a, b] \to \mathbb{R}$ be the function defined such that $|f|(x) = |f(x)|$ for all $x \in [a, b]$. Then the function $|f|$ is Riemann-integrable on $[a, b]$, and*

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

Proof Let some positive real number ε be given. It follows from Proposition 6.5 that there exists a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon.$$

$$\Big||f(v)| - |f(w)|\Big| \leq |f(v) - f(w)|$$

for all $v, w \in [a, b]$. Applying Lemma 6.6, we conclude that

$$U(P, |f|) - L(P, |f|) \leq U(P, f) - L(P, f) < \varepsilon.$$

Proposition 6.5 then ensures that the function $|f|$ is Riemann-integrable on $[a, b]$,

Now $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in [a, b]$. It follows that

$$-\int_a^b |f(x)|\, dx \leq \int_a^b f(x)\, dx \leq \int_a^b |f(x)|\, dx.$$

It follows that

$$\left|\int_a^b f(x)\, dx\right| \leq \int_a^b |f(x)|\, dx,$$

as required. ∎

Proposition 6.10 *Let f be a bounded real-valued function on the interval $[a, c]$. Suppose that f is Riemann-integrable on the intervals $[a, b]$ and $[b, c]$, where $a < b < c$. Then f is Riemann-integrable on $[a, c]$, and*

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx.$$

Proof Let some positive real number ε be given. The function f is Riemann-integrable on the interval $[a, b]$ and therefore there exists a partition Q of $[a, b]$ such that the lower Darboux sum $L(Q, f)$ of f on $[a, b]$ with respect to the partition Q of $[a, b]$ satisfies

$$L(Q, f) > \int_a^b f(x)\, dx - \tfrac{1}{2}\varepsilon.$$

Similarly there exists a partition R of $[b, c]$ of $[a, b]$ such that the lower Darboux sum $L(Q, f)$ of f on $[b, c]$ with respect to the partition R of $[b, c]$ satisfies

$$L(R, f) > \int_b^c f(x)\, dx - \tfrac{1}{2}\varepsilon.$$

Now the partitions Q and R combine to give a partition P of the interval $[a, c]$, where $P = Q \cup R$. Indeed $Q = \{v_0, v_1, \ldots, v_L\}$, where $v_0, v_1, \ldots, v_L$ are real numbers satisfying

$$a = v_0 < v_1 < v_2 < \cdots v_{L-1} < v_L = b,$$

and $R = \{w_0, w_1, \ldots, w_N\}$, where $w_0, w_1, \ldots, w_N$ are real numbers satisfying

$$b = w_0 < w_1 < w_2 < \cdots w_{N-1} < w_N = c.$$

Then

$$P = \{a, v_1, v_2, \ldots, v_{L-1}, b, w_1, w_2, \ldots, w_{N-1}, c\}.$$

It follows directly from the definition of Darboux lower sums that

$$L(P, f) = L(Q, f) + L(R, f).$$

The choice of the partitions Q and R then ensures that

$$L(P, f) > \int_a^b f(x)\,dx + \int_b^c f(x)\,dx - \varepsilon.$$

The lower Riemann integral $\mathcal{L}\int_a^c f(x)\,dx$ is by definition the least upper bound of the lower Darboux sums of f on the interval $[a, c]$. It follows that

$$\mathcal{L}\int_a^c f(x)\,dx > \int_a^b f(x)\,dx + \int_b^c f(x)\,dx - \varepsilon.$$

Moreover this inequality holds for all values of the positive real number ε. It follows that

$$\mathcal{L}\int_a^c f(x)\,dx \geq \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Applying this result with the function f replaced by $-f$ yields the inequality

$$\mathcal{L}\int_a^c (-f(x))\,dx \geq -\int_a^b f(x)\,dx - \int_b^c f(x)\,dx.$$

But

$$\mathcal{L}\int_a^c (-f(x))\,dx = -\mathcal{U}\int_a^c f(x)\,dx$$

(see Lemma 6.3). It follows that

$$\mathcal{U}\int_a^c f(x)\,dx \leq \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \leq \mathcal{L}\int_a^c f(x)\,dx.$$

But

$$\mathcal{L}\int_a^c f(x)\,dx \leq \mathcal{U}\int_a^c f(x)\,dx.$$

It follows that

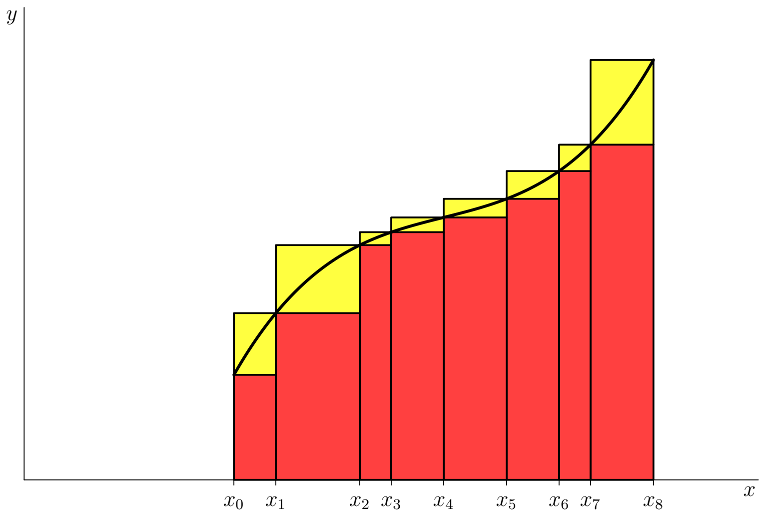
$$\mathcal{L}\int_a^c f(x)\,dx = \mathcal{U}\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

The result follows. ∎

6.3 Integrability of Monotonic Functions

Let a and b be real numbers satisfying $a < b$. A real-valued function $f\colon [a,b] \to \mathbb{R}$ defined on the closed bounded interval $[a,b]$ is said to be *non-decreasing* if $f(v) \leq f(w)$ for all real numbers v and w satisfying $a \leq v \leq w \leq b$. Similarly $f\colon [a,b] \to \mathbb{R}$ is said to be *non-increasing* if $f(v) \geq f(w)$ for all real numbers v and w satisfying $a \leq v \leq w \leq b$. The function $f\colon [a,b] \to \mathbb{R}$ is said to be *monotonic* on $[a,b]$ if either it is non-decreasing on $[a,b]$ or else it is non-increasing on $[a,b]$.

Proposition 6.11 *Let a and b be real numbers satisfying $a < b$. Then every monotonic function on the interval $[a,b]$ is Riemann-integrable on $[a,b]$.*



Proof Let $f\colon [a,b] \to \mathbb{R}$ be a non-decreasing function on the closed bounded interval $[a,b]$. Then $f(a) \leq f(x) \leq f(b)$ for all $x \in [a,b]$, and therefore the function f is bounded on $[a,b]$. Let some positive real number ε be given. Let δ be some strictly positive real number for which $(f(b) - f(a))\delta < \varepsilon$, and let P be a partition of $[a,b]$ of the form $P = \{u_0, u_1, u_2, \ldots, u_N\}$, where

$$a = u_0 < u_1 < u_2 < \cdots < u_{N-1} < u_N = b$$

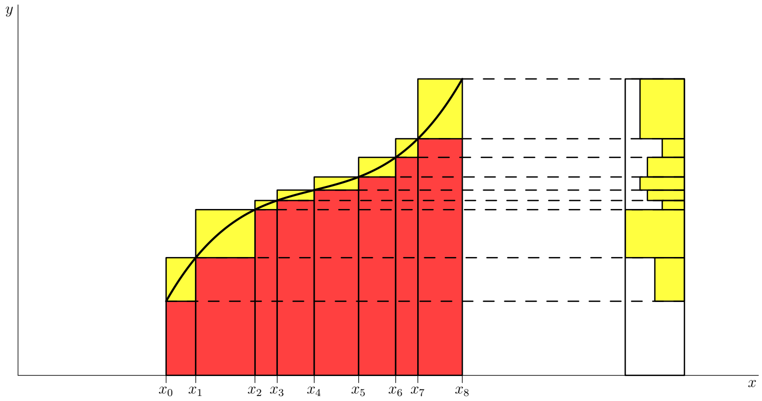
and $u_i - u_{i-1} < \delta$ for $i = 1, 2, \ldots, N$. The maximum and minimum values of $f(x)$ on the interval $[u_{i-1}, u_i]$ are attained at u_i and u_{i-1} respectively, and therefore the upper sum $U(P,f)$ and $L(P,f)$ of f for the partition P satisfy

$$U(P,f) = \sum_{i=1}^{N} f(u_i)(u_i - u_{i-1})$$

and

$$L(P, f) = \sum_{i=1}^{N} f(u_{i-1})(u_i - u_{i-1}).$$

Now $f(u_i) - f(u_{i-1}) \geq 0$ for $i = 1, 2, \ldots, N$. It follows that



$$\begin{aligned} & U(P, f) - L(P, f) \\ = & \sum_{i=1}^{N} (f(u_i) - f(u_{i-1}))(u_i - u_{i-1}) \\ < & \delta \sum_{i=1}^{N} (f(u_i) - f(u_{i-1})) = \delta(f(b) - f(a)) < \varepsilon. \end{aligned}$$

We have thus shown that

$$\mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx < \varepsilon$$

for all strictly positive numbers ε. But

$$\mathcal{U}\int_a^b f(x)\,dx \geq \mathcal{L}\int_a^b f(x)\,dx.$$

It follows that

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

and thus the function f is Riemann-integrable on $[a, b]$.

Now let $f\colon [a, b] \to \mathbb{R}$ be a non-increasing function on $[a, b]$. Then $-f$ is a non-decreasing function on $[a, b]$ and it follows from what we have just shown that $-f$ is Riemann-integrable on $[a, b]$. It follows that the function f itself must be Riemann-integrable on $[a, b]$, as required. ∎

Corollary 6.12 *Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function on the interval $[a,b]$, where a and b are real numbers satisfying $a < b$. Suppose that there exist real numbers $u_0, u_1, \ldots, u_N$, where*

$$a = u_0 < u_1 < u_2 < \cdots < u_{N-1} < u_N = b,$$

such that the function f restricted to the interval $[u_{i-1}, u_i]$ is monotonic on $[u_{i-1}, u_i]$ for $i = 1, 2, \ldots, N$. Then f is Riemann-integrable on $[a,b]$.

Proof The result follows immediately on applying the results of Proposition 6.10 and Proposition 6.11. ∎

Remark The result and proof-strategy of Proposition 6.11 are to be found in their essentials in Isaac Newton, *Philosophiae naturalis principia mathematica* (1686), Book 1, Section 1, Lemmas 2 and 3.

6.4 Integrability of Continuous functions

Theorem 6.13 *Let a and b be real numbers satisfying $a < b$. Then any continuous real-valued function on the interval $[a,b]$ is Riemann-integrable.*

Proof Let f be a continuous real-valued function on $[a,b]$. Then f is bounded above and below on the interval $[a,b]$, and moreover $f\colon [a,b] \to \mathbb{R}$ is uniformly continuous on $[a,b]$. (These results follow from Theorem 4.21 and Theorem 4.22.) Therefore there exists some strictly positive real number δ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in [a,b]$ satisfy $|x - y| < \delta$.

Choose a partition P of the interval $[a,b]$ such that each subinterval in the partition has length less than δ. Write $P = \{u_0, u_1, \ldots, u_N\}$, where $a = u_0 < u_1 < \cdots < u_N = b$. Now if $u_{i-1} \leq x \leq u_i$ then $|x - u_i| < \delta$, and hence $f(u_i) - \varepsilon < f(x) < f(u_i) + \varepsilon$. It follows that

$$f(u_i) - \varepsilon \leq m_i \leq M_i \leq f(u_i) + \varepsilon \qquad (i = 1, 2, \ldots, N),$$

where $m_i = \inf\{f(x) : u_{i-1} \leq x \leq u_i\}$ and $M_i = \sup\{f(x) : u_{i-1} \leq x \leq u_i\}$. Therefore

$$\begin{aligned} &\sum_{i=1}^{N} f(u_i)(u_i - u_{i-1}) - \varepsilon(b-a) \\ &\qquad \leq \quad L(P,f) \leq U(P,f) \\ &\qquad \leq \quad \sum_{i=1}^{N} f(u_i)(u_i - u_{i-1}) + \varepsilon(b-a), \end{aligned}$$

where $L(P, f)$ and $U(P, f)$ denote the lower and upper sums of the function f for the partition P.

We have now shown that

$$0 \leq \mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx \leq U(P, f) - L(P, f) \leq 2\varepsilon(b - a).$$

But this inequality must be satisfied for any strictly positive real number ε. Therefore

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

and thus the function f is Riemann-integrable on $[a, b]$. ∎

6.5 The Fundamental Theorem of Calculus

Let a and b be real numbers satisfying $a < b$. One can show that all continuous functions on the interval $[a, b]$ are Riemann-integrable (see Theorem 6.13). However the task of calculating the Riemann integral of a continuous function directly from the definition is difficult if not impossible for all but the simplest functions. Thus to calculate such integrals one makes use of the Fundamental Theorem of Calculus.

Theorem 6.14 (The Fundamental Theorem of Calculus) *Let f be a continuous real-valued function on the interval $[a, b]$, where $a < b$. Then*

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

for all x satisfying $a < x < b$.

Proof Let some strictly positive real number ε be given, and let ε_0 be a real number chosen so that $0 < \varepsilon_0 < \varepsilon$. (For example, one could choose $\varepsilon_0 = \frac{1}{2}\varepsilon$.) Now the function f is continuous at x, where $a < x < b$. It follows that there exists some strictly positive real number δ such that

$$f(x) - \varepsilon_0 \leq f(t) \leq f(x) + \varepsilon_0$$

for all $t \in [a, b]$ satisfying $x - \delta < t < x + \delta$. Let $F(s) = \int_a^s f(t)\,dt$ for all $s \in (a, b)$. Then

$$\begin{aligned} F(x + h) &= \int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt \\ &= F(x) + \int_x^{x+h} f(t)\,dt \end{aligned}$$

whenever $x + h \in [a, b]$. Also

$$\frac{1}{h} \int_x^{x+h} f(x)\,dt = \frac{f(x)}{h} \int_x^{x+h} dt = f(x),$$

because $f(x)$ is constant as t varies between x and $x + h$. It follows that

$$\frac{F(x+h) - F(x)}{h} - f(x) = \frac{1}{h} \int_x^{x+h} (f(t) - f(x))\,dt$$

whenever $x + h \in [a, b]$. But if $0 < |h| < \delta$ and $x + h \in [a, b]$ then

$$-\varepsilon_0 \leq f(t) - f(x) \leq \varepsilon_0$$

for all real numbers t belonging to the closed interval with endpoints x and $x + h$, and therefore

$$-\varepsilon_0 |h| \leq \int_x^{x+h} (f(t) - f(x))\,dt \leq \varepsilon_0 |h|.$$

It follows that

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| \leq \varepsilon_0 < \varepsilon$$

whenever $x + h \in [a, b]$ and $0 < |h| < \delta$. We conclude that

$$\frac{d}{dx} \left(\int_a^x f(t)\,dt \right) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x),$$

as required. ∎

7 Multiple Integrals

7.1 Multiple Integrals of Bounded Continuous Functions

We considers integrals of continuous real-valued functions of several real variables over regions that are products of closed bounded intervals. Any subset of n-dimensional Euclidean space $\mathbb{R}^n$ that is a product of closed bounded intervals is a closed bounded set in $\mathbb{R}^n$. It follows from the Extreme Value Theorem (Theorem 4.21) that any continuous real-valued function on a product of closed bounded intervals is necessarily bounded on that product of intervals. It is also uniformly continuous on that product of intervals (see Theorem 4.22)

Proposition 7.1 *Let n be an integer greater than 1, let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers, where $a_i < b_i$ for $i = 1, 2, \ldots, n$, let $f\colon [a_1, b_1] \times \cdots \times [a_n, b_n] \to \mathbb{R}$ be a continuous real-valued function, and let*

$$g(x_1, x_2, \ldots, x_{n-1}) = \int_{a_n}^{b_n} f(x_1, x_2, \ldots, x_{n-1}, t)\, dt.$$

for all $(n-1)$-tuples $(x_1, x_2, \ldots, x_{n-1})$ of real numbers satisfying $a_i \leq x_i \leq b_i$ for $i = 1, 2, \ldots, n-1$. Then the function

$$g\colon [a_1, b_1] \times [a_2, b_2] \cdots \times [a_{n-1}, b_{n-1}] \to \mathbb{R}$$

is continuous.

Proof Let some positive real number ε be given, and let ε_0 be chosen so that $0 < (b_n - a_n)\varepsilon_0 < \varepsilon$. The function f is uniformly continuous on $[a_1, b_1] \times [a_2, b_2] \cdots \times [a_n, b_n]$ (see Theorem 4.22). Therefore there exists some positive real number δ such that

$$|f(x_1, x_2, \ldots, x_{n-1}, t) - f(u_1, u_2, \ldots, u_{n-1}, t)| < \varepsilon_0$$

for all real numbers $x_1, x_2, \ldots, x_{n-1}$, $u_1, u_2, \ldots, u_{n-1}$ and t satisfying $a_i \leq x_i \leq b_i$, $a_i \leq u_i < b_i$ and $|x_i - u_i| < \delta$ for $i = 1, 2, \ldots, n-1$ and $a_n \leq t \leq b_n$. Applying Proposition 6.9, we see that

$$\begin{aligned}
&|g(x_1, x_2, \ldots, x_{n-1}) - g(u_1, u_2, \ldots, u_{n-1})| \\
&\qquad = \left| \int_{a_n}^{b_n} (f(x_1, x_2, \ldots, x_{n-1}, t) - f(u_1, u_2, \ldots, u_{n-1}, t))\, dt \right| \\
&\qquad \leq \int_{a_n}^{b_n} |f(x_1, x_2, \ldots, x_{n-1}, t) - f(u_1, u_2, \ldots, u_{n-1}, t)|\, dt \\
&\qquad \leq \varepsilon_0(b_n - a_n) < \varepsilon
\end{aligned}$$

whenever $a_i \le x_i \le b_i$, $a_i \le u_i < b_i$ and $|x_i - u_i| < \delta$ for $i = 1, 2, \ldots, n-1$. The result follows. ∎

Proposition 7.1 ensures that, given a continuous real-valued function $f\colon [a_1, b_1] \times \cdots \times [a_n, b_n] \to \mathbb{R}$, where $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers and $a_i < b_i$ for $i = 1, 2, \ldots, n$, there is a well-defined multiple integral

$$\int_{x_n=a_n}^{b_n} \cdots \int_{x_2=a_2}^{b_2} \int_{x_1=a_1}^{b_1} f(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n,$$

in which, at each stage of evaluation, the integrand is a continuous function of its arguments. To evaluate this integral, one integrates first with respect to x_1, then with respect to x_2, and so on, finally integrating with respect to x_n.

In fact, if the function f is continuous, the order of evaluation of the integrals with respect to the individual variables does not affect the value of the multiple integral. We prove this first for continuous functions of two variables.

Theorem 7.2 *Let $f\colon [a, b] \times [c, e] \to \mathbb{R}$ be a continuous real-valued function on the closed rectangle $[a, b] \times [c, e]$. Then*

$$\int_c^e \left(\int_a^b f(x, y)\, dx \right) dy = \int_a^b \left(\int_c^e f(x, y)\, dy \right) dx.$$

Proof Let $f\colon [a, b] \times [c, e] \to \mathbb{R}$ is continuous, and is therefore uniformly continuous on $[a, b] \times [c, e]$ (see Theorem 4.22). Let some positive real number ε be given. It follows from the uniform continuity of the function f that there exists some positive real number δ with the property that

$$|f(x, y) - f(u, v)| < \varepsilon$$

for all $x, u \in [a, b]$ and $y, v \in [c, e]$ satisfying $|x - u| < \delta$ and $|y - v| < \delta$. Let P be a partition of $[a, b]$, and let Q be a partition of $[c, e]$, where

$$P = \{u_0, u_1, \ldots, u_p\}, \quad Q = \{v_0, v_1, \ldots, v_q\},$$

$$a = u_0 < u_1 < \cdots < u_p = b, \quad c = v_0 < v_1 < \cdots < v_q = e,$$

$u_j - u_{j-1} < \delta$ for $j = 1, 2, \ldots, p$ and $v_k - v_{k-1} < \delta$ for $k = 1, 2, \ldots, q$. Then

$$|f(x, y) - f(u_j, v_k)| < \varepsilon$$

whenever $u_{j-1} \leq x \leq u_j$ for some integer j between 1 and p and $v_{k-1} \leq y \leq v_k$ for some integer k between 1 and q.

Now

$$\int_c^e \left(\int_a^b f(x,y)\,dx \right)\,dy = \sum_{k=1}^{q} \sum_{j=1}^{p} \int_{v_{k-1}}^{v_k} \left(\int_{u_{j-1}}^{u_j} f(x,y)\,dx \right)\,dy.$$

(This follows from straightforward applications of Proposition 6.10 and Proposition 6.4.) Moreover

$$\int_{u_{j-1}}^{u_j} f(x,y)\,dx \leq \big(f(u_j,v_k) + \varepsilon)(u_j - u_{j-1})$$

for all $y \in [v_{k-1}, v_k]$, and therefore

$$\int_{v_{k-1}}^{v_k} \left(\int_{u_{j-1}}^{u_j} f(x,y)\,dx \right)\,dy \leq \Big(f(u_j,v_k) + \varepsilon\Big)(v_k - v_{k-1})(u_j - u_{j-1})$$

for all integers j between 1 and p and integers k between 1 and q. It follows that

$$\begin{aligned} \int_c^e \left(\int_a^b f(x,y)\,dx \right)\,dy &\leq \sum_{k=1}^{q} \sum_{j=1}^{p} \Big(f(u_j,v_k) + \varepsilon\Big)(v_k - v_{k-1})(u_j - u_{j-1}) \\ &= S + \varepsilon(b-a)(e-c), \end{aligned}$$

where

$$S = \sum_{k=1}^{q} \sum_{j=1}^{p} f(u_j,v_k)(v_k - v_{k-1})(u_j - u_{j-1}).$$

Similarly

$$\begin{aligned} \int_c^e \left(\int_a^b f(x,y)\,dx \right)\,dy &\geq \sum_{k=1}^{q} \sum_{j=1}^{p} \Big(f(u_j,v_k) - \varepsilon\Big)(v_k - v_{k-1})(u_j - u_{j-1}) \\ &= S - \varepsilon(b-a)(e-c). \end{aligned}$$

Thus

$$\left| \int_c^e \left(\int_a^b f(x,y)\,dx \right)\,dy - S \right| \leq \varepsilon(b-a)(e-c).$$

On interchanging the roles of the variables x and y, we conclude similarly that

$$\left| \int_a^b \left(\int_c^e f(x,y)\,dy \right)\,dx - S \right| \leq \varepsilon(b-a)(e-c).$$

It follows that

$$\left| \int_c^e \left(\int_a^b f(x,y)\,dx \right) dy - \int_a^b \left(\int_c^e f(x,y)\,dy \right) dx \right| \quad \leq \quad 2\varepsilon(b-a)(e-c).$$

Moreover the inequality just obtained must hold for every positive real number ε, no matter how small the value of ε. It follows that

$$\int_c^e \left(\int_a^b f(x,y)\,dx \right) dy = \int_a^b \left(\int_c^e f(x,y)\,dy \right) dx,$$

as required. ∎

Now let us consider a multiple integral involving a continuous function of three real variables. Let

$$f\colon [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \to \mathbb{R}$$

be a continuous real-valued function, where a_1, a_2, a_3, b_1, b_2 and b_3 are real numbers satisfying $a_1 < b_1$, $a_2 < b_2$ and $a_3 < b_3$. It follows from Theorem 7.2 that

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x_1, x_2, x_3)\,dx_2\,dx_1 = \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2, x_3)\,dx_1\,dx_2$$

for all real numbers x_3 satisfying $a_3 < x_3 < b_3$. It follows that

$$\int_{a_3}^{b_3} \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x_1, x_2, x_3)\,dx_2\,dx_1\,dx_3 \quad = \quad \int_{a_3}^{b_3} \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2, x_3)\,dx_1\,dx_2\,dx_3.$$

Also it follows from Proposition 7.1 that the function sending (x_2, x_3) to

$$\int_{a_1}^{b_1} f(x_1, x_2, x_3)\,dx_1$$

for all $(x_2, x_3) \in [a_2, b_2] \times [a_3, b_3]$ is a continuous function of (x_2, x_3). It then follows from Theorem 7.2 that

$$\int_{a_2}^{b_2} \int_{a_3}^{b_3} \int_{a_1}^{b_1} f(x_1, x_2, x_3)\,dx_1\,dx_3\,dx_2 \quad = \quad \int_{a_3}^{b_3} \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2, x_3)\,dx_1\,dx_2\,dx_3.$$

Repeated applications of these results establish that the value of the repeated integral with respect to the real variables x_1, x_2 and x_3 is independent of the order in which the successive integrations are performed.

Corresponding results hold for integration of continuous real-valued functions of four or more real variables. In general, if the integrand is a continuous real-valued function of n real variables, and if this function is integrated over a product of n closed bounded intervals, by repeated integration, then the value of the integral is independent of the order in which the integrals are performed.

7.2 A Counterexample involving an Unbounded Function

Example Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined such that

$$f(x,y) = \begin{cases} \dfrac{4xy(x^2-y^2)}{(x^2+y^2)^3} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Set $u = x^2 + y^2$. Then

$$f(x,y) = \frac{2x(2x^2-u)}{u^3}\frac{\partial u}{\partial y},$$

and therefore, when $x \neq 0$,

$$\begin{aligned} \int_{y=0}^{1} f(x,y)\,dy &= \int_{u=x^2}^{x^2+1} \left(\frac{4x^3}{u^3} - \frac{2x}{u^2}\right) du \\ &= \left[-\frac{2x^3}{u^2} + \frac{2x}{u}\right]_{u=x^2}^{x^2+1} \\ &= -\frac{2x^3}{(x^2+1)^2} + \frac{2x}{x^2+1} \\ &= \frac{2x}{(x^2+1)^2} \end{aligned}$$

It follows that

$$\begin{aligned} \int_{x=0}^{1}\left(\int_{y=0}^{1} f(x,y)\,dy\right) dx &= \int_{x=0}^{1} \frac{2x}{(x^2+1)^2}\,dx \\ &= \left[-\frac{1}{x^2+1}\right]_0^1 = \frac{1}{2}. \end{aligned}$$

Now $f(y,x) = -f(x,y)$ for all x and y. Interchanging x and y in the above evaluation, we find that

$$\begin{aligned} \int_{y=0}^{1}\left(\int_{x=0}^{1} f(x,y)\,dx\right) dy &= \int_{x=0}^{1}\left(\int_{y=0}^{1} f(y,x)\,dy\right) dx \\ &= -\int_{x=0}^{1}\left(\int_{y=0}^{1} f(x,y)\,dy\right) dx \\ &= -\frac{1}{2}. \end{aligned}$$

Thus

$$\int_{x=0}^{1} \left(\int_{y=0}^{1} f(x,y)\,dy \right) dx \neq \int_{y=0}^{1} \left(\int_{x=0}^{1} f(x,y)\,dx \right) dy.$$

when

$$f(x,y) = \frac{4xy(x^2 - y^2)}{(x^2 + y^2)^3}$$

for all $(x,y) \in \mathbb{R}^2$ distinct from $(0,0)$. Note that, in this case $f(2t,t) \to +\infty$ as $t \to 0^+$, and $f(t,2t) \to -\infty$ as $t \to 0^-$. Thus the function f is not continuous at $(0,0)$ and does not remain bounded as $(x,y) \to (0,0)$.