Course MAU23203: Michaelmas Term 2019.
Assignment 2 Solutions.

1. *This question concerns differentiation of a function of two real variables from first principles. Accordingly you may use freely any results concerning limits of functions stated and proved in Section 4 of the module notes, but you should not appeal to any lemma, proposition, theorem or corollary included in Section 9 of the notes.*

   (a) *Let p and q be real numbers. Then there exist real numbers r and s and a real-valued function $u\colon \mathbb{R}^2 \to \mathbb{R}$ of two real variables, where the constants r and s and the function u depend on and are uniquely determined by the values of p and q for which the following two properties both hold:*

   $$x^3 - 3xy^2 = p^3 - 3pq^2 + r(x-p) - s(y-q) + u(x,y)$$

   for all real numbers x and y, and also

   $$\lim_{(x,y)\to(p,q)} \frac{1}{\sqrt{(x-p)^2 + (y-q)^2}} u(x,y) = 0.$$

   Determine the constants r and s and the function u, where those constants and that function are expressed in terms of the values of the constants p and q.

$$\begin{aligned}
x^3 - 3xy^2 - p^3 + 3pq^2 &= (x-p)(x^2 + xp + p^2) \\
&\qquad - 3(x-p)y^2 - 3p(y^2 - q^2) \\
&= 3p^2(x-p) + (x-p)(x^2 + xp - 2p^2) \\
&\qquad - 3q^2(x-p) - 3(x-p)(y^2 - q^2) \\
&\qquad - 3p(y^2 - q^2) \\
&= 3(p^2 - q^2)(x-p) + (x-p)^2(x + 2p) \\
&\qquad - 3(x-p)(y-q)(y+q) \\
&\qquad - 6pq(y-q) - 3p(y-q)^2 \\
&= r(x-p) - s(x-q) + u(x,y)
\end{aligned}$$

where

$$r = 3(p^2 - q^2), \quad s = 2pq$$

and

$$\begin{aligned} u(x,y) &= (x-p)^2(x+2p) - 3(x-p)(y-q)(y+q) - 3p(y-q)^2 \\ &= 3p(x-p)^2 - 6q(x-p)(y-q) - 3p(y-q)^2 \\ &\qquad + (x-p)^3 - 3(x-p)(y-q)^2. \end{aligned}$$

Moreover

$$\begin{aligned} \lim_{(x,y)\to(p,q)} \frac{1}{\sqrt{(x-p)^2+(y-q)^2}}\,(x-p)^2 &= 0, \\ \lim_{(x,y)\to(p,q)} \frac{1}{\sqrt{(x-p)^2+(y-q)^2}}\,(x-p)(y-q) &= 0, \\ \lim_{(x,y)\to(p,q)} \frac{1}{\sqrt{(x-p)^2+(y-q)^2}}\,(y-q)^2 &= 0, \end{aligned}$$

and therefore

$$\lim_{(x,y)\to(p,q)} \frac{1}{\sqrt{(x-p)^2+(y-q)^2}} u(x,y) = 0.$$

For completeness, expanding the expression for $u(x,y)$, we find that

$$u(x,y) = x^3 - 3xy^2 - 3(p^2-q^2)x + 6pqy + 2p^3 - 6pq^2.$$

ALTERNATIVELY

The question is worded so as to allow you to take as given that

$$\lim_{(x,y)\to(p,q)} \frac{1}{\sqrt{(x-p)^2+(y-q)^2}} u(x,y) = 0.$$

Considering limits as $x \to p$ along the line $y = q$, and as $y \to q$ along the line $x = p$, one can find values for r and s, and thereby complete the question.

ALTERNATIVELY

Set $f(x,y) = x^3 - 3xy^2$. Then

$$\begin{aligned} f(p+h, q+k) &= p^3 + 3p^2h + 3ph^2 + h^3 - 3pq^2 - 3q^2h \\ &\qquad - 6pqk - 6qhk - 3pk^2 - 3hk^2 \\ &= p^3 - 3pq^2 + 3(p^2 - q^2)h - 6pqk \\ &\qquad + 3ph^2 - 6qhk - 3pk^2 + h^3 - 3hk^2 \end{aligned}$$

Now $h = x - p$ and $k = y - q$. It follows that $f(x,y)$ is expressible in the required form with $r = 3(p^2 - q^2)$, $s = 6pq$ and

$$\begin{aligned} u(x,y) &= 3p(x-p)^2 - 6q(x-p)(y-q) - 3p(y-q)^2 \\ &\qquad + (x-p)^3 - 3(x-p)(y-q)^2. \end{aligned}$$

One may then show that

$$\lim_{(x,y)\to(p,q)} \frac{1}{\sqrt{(x-p)^2 + (y-q)^2}} u(x,y) = 0,$$

using the method of the first solution set out above.

(b) *By making a straightforward substitution in the result proved in (a), or otherwise, determine the unique real-valued function* $v\colon \mathbb{R}^2 \to \mathbb{R}$ *for which the following two properties both hold:*

$$3x^2y - y^3 = 3p^2q - q^3 + s(x-p) + r(y-q) + v(x,y)$$

for all real numbers x *and* y*, and also*

$$\lim_{(x,y)\to(p,q)} \frac{1}{\sqrt{(x-p)^2 + (y-q)^2}} v(x,y) = 0.$$

Replacing x, y, p and q by $-y$, x, $-q$ and p respectively, and accordingly replacing s by $-s$, we find that

$$3x^2y - y^3 - 3p^2q + p^3 = s(x-p) + r(y-q) + u(x,y) + v(x,y),$$

where

$$\begin{aligned} v(x,y) &= -3q(x-p)^2 + 6p(x-p)(y-q) + 3q(y-q)^2 \\ &\qquad - 3(x-p)^2(y-q) - (y-q)^3. \end{aligned}$$

(c) *Let* $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ *be the function from* $\mathbb{R}^2$ *to itself defined such that*

$$\varphi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^3 - 3xy^2 \\ 3x^2y - y^3 \end{pmatrix}$$

Applying the results obtained in parts (a) and (b) of this question, determine, in terms of the values of p *and* q*, the* 2×2 *matrix that represents the derivative of the function* φ *at a point* (p, q) *of* $\mathbb{R}^2$.

It follows from previous parts of the question that

$$\varphi\begin{pmatrix} x \\ y \end{pmatrix} = \varphi\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} r & -s \\ s & r \end{pmatrix}\begin{pmatrix} x - p \\ y - q \end{pmatrix} + \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}.$$

Moreover

$$\lim_{(x,y)\to(p,q)} \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It follows from the definition of differntiability that the map φ is differentiable at (p, q), and its derivative at (p, q) is represented by the matrix

$$\begin{pmatrix} r & -s \\ s & r \end{pmatrix},$$

where

$$r = 3(p^2 - q^2), \quad s = 2pq.$$