Course MAU23203: Michaelmas Term 2019. Assignment 1 Solutions.

1. *Throughout this question, let* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *be the function defined such that*

$$f(x,y) = \begin{cases} \dfrac{2x^3y^2}{x^6+y^4} & \textit{if } (x,y) \neq (0,0); \\ (0,0) & \textit{if } (x,y) = (0,0). \end{cases}$$

(a) *Let* u *be a positive real number. Determine the maximum and minimum values of the function* $f(x,y)$ *on the line* $x = u$*, and determine also the values of* y *at which the function* $f(x,y)$ *attains those maximum and minimum values. (In other words, considering* $f(u,y)$ *as a function of the real variable* y *alone, determine the maximum and minimum values achieved by this function, and determine the values of* y *where those maximum and minimum values are achieved.)*

Let u be a fixed positive number. Then

$$0 \le (u^3 - y^2)^2 = u^6 + y^4 - 2u^3y^2$$

for all real numbers y. It follows that

$$0 \le 2u^3y^2 \le u^6 + y^4$$

for all real numbers y. Thus $0 \le f(u,y) \le 1$ for all real numbers y. Moroever $f(u,y) = 1$ if and only if $y = \pm\sqrt{u^3}$ and $f(u,y) = 0$ if and only if $y = 0$. Thus the function f restricted to the line $x = u$ achieves its minimum value 0 at $y = 0$ and its maximum value 1 at $y = \pm\sqrt{u^3}$.

ALTERNATIVELY

Setting $x = u$, where u is a fixed positive number, we find that

$$\frac{d}{du}\left(\frac{2u^3y^2}{u^6+y^4}\right) = \frac{4u^3y(u^6+y^4) - 8u^3y^5}{(u^6+y^4)} = \frac{4u^3y(u^6-y^4)}{(u^6+y^4)}.$$

Therefore $f(u,y)$ increases as y increases from 0 to $\sqrt{u^3}$, and decreases as y increases past $\sqrt{u^3}$. Also $f(u,y) \ge 0$ and $f(u,-y) = f(u,y)$ for all real numbers y. It follows that, as a function of y, $f(u,y)$ achieves a minimum value of 0 at $y = 0$ and a maximum value of 1 at $y = \pm\sqrt{u^3}$.

(b) *Now let u be a negative real number. Determine the maximum and minimum values of the function $f(x, y)$ on the line $x = u$, and determine also the values of y at which the function $f(x, y)$ attains those maximum and minimum values. [The answers for this part of the question can easily be deduced from those for the preceding part (a).]*
Note that $f(u, y) = -f(-u, y)$ when $u < 0$. Therefore, on restricting the function f to the line $y = u$, where $u < 0$, the maximum value 0 of $f(u, y)$ is achieved at $y = 0$, and the minimum value -1 is achieved at $y = \pm\sqrt{-u^3}$.

(c) *Let (u, v) be a point of $\mathbb{R}^2$ distinct from $(0, 0)$. Considering separately the cases when $v \neq 0$ and when $v = 0$, prove that*

$$\lim_{t \to 0} f(tu, tv) = 0.$$

Note that if $v \neq 0$ and $t \neq 0$ then

$$f(tu, tv) = \frac{2t^5u^3v^2}{t^4(t^2u^6 + v^4)} = \frac{2tu^3v^2}{t^2u^6 + v^4}$$

It follows that if $v \neq 0$ then

$$f(tu, tv) = \frac{2tu^3v^2}{t^2u^6 + v^4}$$

for all real numbers t (including $t = 0$). Moreover $f(tu, tv)$ is a continuous function of t. It follows that

$$\lim_{t \to 0} f(tu, tv) = f(0, 0) = 0$$

in the case where $v \neq 0$. The same is true when $v = 0$, because, in that case, $f(tu, 0) = 0$ for all real numbers t.

(d) *Determine whether or not it is the case that*

$$\lim_{(x,y) \to (0,0)} f(x, y) = 0,$$

providing a rigorous justification for your answer.
It is not that case that $\lim_{(x,y) \to (0,0)} f(x, y) = 0$. If this were the case then the function f would be continuous at $(0, 0)$, and therefore the composition function $f \circ \gamma$ would be continuous at 0, where $\gamma[0, +\infty) \to \mathbb{R}^2$ is

defined so that $\gamma(t) = (t, \sqrt{t^3})$ for all non-negative real numbers t. But $f(\gamma(0)) - 0$ and $f(\gamma(t)) = 1$ for all positive real numbers t. Therefore the function $f \circ \gamma$ is not continuous at 0. It follows that the function f is not continuous at $(0, 00$ and therefore $\lim_{(x,y)\to(0,0)} f(x, y)$ does not equal $(0, 0)$. (Indeed this limit does not exist.)

ALTERNATIVELY

Let $\varepsilon = \frac{1}{2}$. If it were the case that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$, then there would exist some positive real number δ such that $|f(x, y)| < \varepsilon = \frac{1}{2}$ whenever $|(x, y)| < \delta$. In other words $-\frac{1}{2} < f(x, y) < \frac{1}{2}$ whenever $\sqrt{x^2 + y^2} < \delta$. But, given $\delta > 0$, let $u = \min(\frac{1}{2}\delta, 1)$. Then $\sqrt{u^3} \leq u \leq \frac{1}{2}\delta$. Thus if $x = u$ and $y = \sqrt{u^3}$ then $\sqrt{x^2 + y^2} < \delta$ but $f(x, y) = 1$. Therefore the formal definition of limits is not satisfied by the function f at $(0, 0)$.

[Caution: in answering the above, particularly part (d), you should either make direct use of the ε—δ definition of limits or else you should correctly make use of the lemmas, propositions, theorems and corollaries included in the module notes. You should not rely on "obvious" results concerning limits that "intuitively" ought "clearly" to be true, but, through some oversight, seem to have been omitted from the notes: it may well happen that "obvious" results based on assumptions about how limits "ought" to behave turn out to be false in general, and lead you to incorrect conclusions.]

2. *Throughout this question, let c be a real numbera satisfying $c > 1$, and let $f(x) = \log x$ for all real numbers x satisfying $1 \leq x \leq c$ (where* log *denotes the natural logarithm function).*

   (a) *Let some positive integer m be given, and let P_m denote the partition of the interval $[1, c]$ with division points at $c^{\frac{j}{m}}$ for $j = 0, 1, 2, \ldots, m$. Show that*

$$\begin{aligned} U(P_m, f) &= c \log c - \frac{c-1}{r_m - 1} \log r_m, \\ L(P_m, f) &= c \log c - \frac{r_m(c-1)}{r_m - 1} \log r_m, \end{aligned}$$

   where $r_m = c^{\frac{1}{m}}$.

For some given m, let $u_j = c^{\frac{j}{m}} = r_m^j$ for $j = 1, 2, \ldots, m$, and let M_j and m_j denote the maximum and minimum values of the natural logarithm function on the interval $[u_{j-1}, u_j]$ for $j = 1, 2, \ldots, m$. The logarithm function is increasing, and therefore

$$\begin{aligned} M_j &= \log u_j = j \log r_m \\ m_j &= \log u_{j-1} = (j-1) \log r_m \end{aligned}$$

for $j = 1, 2, \ldots, m$. The upper and lower sums $U(P_m, f)$ and $L(P_m, f)$ are given by the formulae

$$U(P_m, f) = \sum_{j=1}^{m} M_j (u_j - u_{j-1}), \quad L(P_m, f) = \sum_{j=1}^{m} m_j (u_j - u_{j-1}).$$

Now $u_j - u_{j-1} = (r_m - 1) r_m^{j-1}$ for $j = 1, 2, \ldots, n$. Note also that $r_m^m = c$ and $m \log r_m = \log c$. Using the identity given on the assignmnet sheet, it follows that

$$\begin{aligned} U(P_m, f) &= \sum_{j=1}^{m} j r^{j-1} (r_m - 1) \log r_m \\ &= \frac{m r_m^{m+1} - (m+1) r_m^m + 1}{(r_m - 1)^2} \times (r_m - 1) \log r_m \\ &= \frac{(m(r_m - 1)c + 1 - c) \log r_m}{(r_m - 1)} \\ &= c \log c - \frac{c - 1}{r_m - 1} \log r_m. \end{aligned}$$

Now $m_j = M_j - \log r_m$. It follows that

$$\begin{aligned} L(P_m, f) &= U(P_m, f) - \log r_m \times \sum_{j-1}^{m} (u_j - j_{j-1}) \\ &= U(P_m, f) - (u_m - u_1) \log r_m \\ &= U(P_m, f) - (c - 1) \log r_m \\ &= c \log c - (c - 1) \left(\frac{1}{r_m - 1} + 1 \right) \log r_m \\ &= c \log c - \frac{r_m (c - 1)}{r_m - 1} \log r_m. \end{aligned}$$

ALTERNATIVELY

Using the expression for $U(P_m, f)$, resulting from applying the definition of the upper sum to the logarithm function on the interval $[1, c]$, we find that

$$\begin{aligned}
U(P_m, f) &= \sum_{j=1}^{m} j r_m^{j-1}(r_m - 1)\log r_m \\
&= \sum_{j=1}^{m} (j r_m^j - j r_m^{j-1})\log r_m \\
&= \left(m r_m^m + \sum_{j=0}^{m-1} j r_m^j - \sum_{j=1}^{m} j r_m^{j-1} \right) \log r_m \\
&= \left(mc + \sum_{j=0}^{m-1} j r_m^j - \sum_{j=0}^{m-1} (j+1) r_m^j \right) \log r_m \\
&= m r_m^m \log r_m - \sum_{j=0}^{m-1} r_m^j \, \log r_m \\
&= r_m^m \log r_m^m - \frac{r_m^m - 1}{r_m - 1} \log r_m \\
&= c \log c - \frac{c-1}{r_m - 1} \log r_m
\end{aligned}$$

(where we have made use of the identity $r_m^m = c$). Given the expression for $U(P_m, f)$, an expression for $L(P_m, f)$ may be found in a similar fashion, or by applying the method employed in the other solution presented above.

(b) *For each positive integer m, let P_m be the partition of the interval $[0, c]$ defined as described in (a). Prove that*

$$\lim_{m \to +\infty} U(P_m, f) = c \log c + 1 - c$$

and

$$\lim_{m \to +\infty} L(P_m, f) = c \log c + 1 - c$$

Note that

$$\lim_{r \to 1} \frac{\log r}{r - 1} = \left. \frac{d(\log r)}{dr} \right|_{r=1} = 1.$$

Now $r_m \to 1$ as $m \to +\infty$. It follows that

$$\lim_{m \to +\infty} \frac{\log r_m}{r_m - 1} = 1.$$

It follows that

$$\begin{aligned}
\lim_{m\to+\infty} U(P_m, f) &= c\log c - (c-1) \lim_{m\to+\infty} \frac{\log r_m}{r_m - 1} \\
&= c\log c - (c-1) \\
\lim_{m\to+\infty} U(P_m, f) &= c\log c - (c-1) \left(\lim_{m\to+\infty} r_m\right) \left(\lim_{m\to+\infty} \frac{\log r_m}{r_m - 1}\right) \\
&= c\log c - (c-1)
\end{aligned}$$

(c) *Use the results of (b) to prove (from first principles, and without any appeal to the Fundamental Theorem of Calculus) that the natural logarithm function is Riemann-integrable on the interval* $[1, c]$ *and that*

$$\int_1^c \log x \, dx = c\log c + 1 - c.$$

It follows from the basic definitions that

$$L(P_m f) \leq \mathcal{L}\int_1^c \log x \, dx \leq \mathcal{U}\int_1^c \log x \, dx \leq U(P_m, f)$$

for all positive integers m. Taking limits as $m \to +\infty$, we conclude that

$$\lim_{m\to+\infty} L(P_m f) \leq \mathcal{L}\int_1^c \log x \, dx \leq \mathcal{U}\int_1^c \log x \, dx \leq \lim_{m\to+\infty} U(P_m, f).$$

But

$$\lim_{m\to+\infty} L(P_m f) = \lim_{m\to+\infty} U(P_m f) = c\log c - (c-1).$$

It follows that

$$\mathcal{L}\int_1^c \log x \, dx = \mathcal{U}\int_1^c \log x \, dx = c\log c - (c-1).$$

The upper and lower Riemann integrals are equal, therefore the natural logarithm function is Riemann-integrable on the interval $[1, c]$, and its integral has the expected value.

[Note that

$$\sum_{j=1}^{m} j r^{j-1} = \frac{m r^{m+1} - (m+1) r^m + 1}{(r-1)^2}.$$

This identity follows on differentiating the standard identity $\sum_{j=0}^{m} r^j = \frac{r^{m+1}-1}{r-1}$ *with respect to* r. *Note also that*

$$\lim_{r\to 1} \frac{\log r}{r-1} = \frac{d}{dr}(\log r)\Big|_{r=1} = 1.]$$