Course MAU23203: Michaelmas Term 2019.

Assignment 1.

To be handed in by Tuesday 5th November, 2019.

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Please complete the cover sheet on the back of this page and attach it to the front of your completed assignment script, in particular signing the declaration with regard to plagiarism. Please make sure also that you include both name and student number on work handed in.

Solutions to problems should be expressed in appropriately concise and correct logical language. Attempted solutions that are incoherent, unclear or logically confused will not gain substantial credit.

Module MAU23203—Analysis in Several Real Variables,
Michaelmas Term 2019.
Assignment I.

Name (please print): . . . .

Student number: . . . .

Date submitted: . . . .

I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar for the current year, found at

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Signed:

. . . .

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1. Throughout this question, let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined such that

$$f(x,y) = \begin{cases} \dfrac{2x^3y^2}{x^6+y^4} & \text{if } (x,y) \neq (0,0); \\ (0,0) & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Let u be a positive real number. Determine the maximum and minimum values of the function $f(x,y)$ on the line $x = u$, and determine also the values of y at which the function $f(x,y)$ attains those maximum and minimum values. (In other words, considering $f(u,y)$ as a function of the real variable y alone, determine the maximum and minimum values achieved by this function, and determine the values of y where those maximum and minimum values are achieved.)

(b) Now let u be a negative real number. Determine the maximum and minimum values of the function $f(x,y)$ on the line $x = u$, and determine also the values of y at which the function $f(x,y)$ attains those maximum and minimum values. [The answers for this part of the question can easily be deduced from those for the preceding part (a).]

(c) Let (u,v) be a point of $\mathbb{R}^2$ distinct from $(0,0)$. Considering separately the cases when $v \neq 0$ and when $v = 0$, prove that

$$\lim_{t\to 0} f(tu,tv) = 0.$$

(d) Determine whether or not it is the case that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0,$$

providing a rigorous justification for your answer.

[Caution: in answering the above, particularly part (d), you should either make direct use of the ε—δ definition of limits or else you should correctly make use of the lemmas, propositions, theorems and corollaries included in the module notes. You should not rely on "obvious" results concerning limits that "intuitively" ought "clearly" to be true, but, through some oversight, seem to have been omitted from the notes: it may well happen that "obvious" results based on assumptions about how limits "ought" to behave turn out to be false in general, and lead you to incorrect conclusions.]

2. Throughout this question, let c be a real numbera satisfying $c > 1$, and let $f(x) = \log x$ for all real numbers x satisfying $1 \leq x \leq c$ (where log denotes the natural logarithm function).

(a) Let some positive integer m be given, and let P_m denote the partition of the interval $[1, c]$ with division points at $c^{\frac{j}{m}}$ for $j = 0, 1, 2, \ldots, m$. Show that

$$\begin{aligned} U(P_m, f) &= c \log c - \frac{c-1}{r_m - 1} \log r_m, \\ L(P_m, f) &= c \log c - \frac{r_m(c-1)}{r_m - 1} \log r_m, \end{aligned}$$

where $r_m = c^{\frac{1}{m}}$.

(b) For each positive integer m, let P_m be the partition of the interval $[0, c]$ defined as described in (a). Prove that

$$\lim_{m \to +\infty} U(P_m, f) = c \log c + 1 - c$$

and

$$\lim_{m \to +\infty} L(P_m, f) = c \log c + 1 - c$$

(c) Use the results of (b) to prove (from first principles, and without any appeal to the Fundamental Theorem of Calculus) that the natural logarithm function is Riemann-integrable on the interval $[1, c]$ and that

$$\int_1^c \log x \, dx = c \log c + 1 - c.$$

[Note that

$$\sum_{j=1}^{m} j r^{j-1} = \frac{m r^{m+1} - (m+1) r^m + 1}{(r-1)^2}.$$

This identity follows on differentiating the standard identity $\sum_{j=1}^{m} r^j = \frac{r^{m+1} - 1}{r - 1}$ with respect to r. Note also that

$$\lim_{r \to 1} \frac{\log r}{r - 1} = \frac{d}{dr}(\log r)\bigg|_{r=1} = 1.]$$