MAU22200, Hilary Term 2020.
Problems I: Interchanging limits and integrals

1. For each positive real number u let Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \neq 0 \text{ and } x \text{ is rational;} \\ x & \text{if } x = 0 \text{ or } x \text{ is irrational.} \end{cases}$$

Is the function f Lebesgue-integrable on the interval $[0,1]$? If so, what is the value of $\int_0^1 f(x)\,dx$?

2. For each positive real number u let $f_u\colon \mathbb{R} \to \mathbb{R}$ be the function of a real variable defined so that

$$f_u(x) = \frac{u^6}{u^8 + (x-u)^4}$$

for all real numbers x. We examine the behaviour of the integrals $\int_0^1 f_u(x)\,dx$. You may take it for granted that, for continuous functions such as the functions f_u, when integrated over bounded intervals, the values of the integrals with respect to Lebesgue measure, as determined in the theory of the Lebesgue integral, coincide with the values that result from applying the usual rules of differential and integral calculus.

(a) Considering separately the cases when $x = 0$ and when $x \neq 0$, show that $\lim\limits_{u \to 0^+} f_u(x) = 0$ for all real numbers x. (In other words show that, for each real x, the limit of $f_u(x)$ as u tends to zero from above is equal to zero.)
[There is no need to use a ε–δ argument. Standard theorems concerning limits of sums, differences, products, quotients and/or compositions of functions can be applied.]

(b) For each positive real number u, determine real numbers $p(u)$ and $q(u)$ such that

$$\int_0^1 f_u(x)\,dx = \int_{p(u)}^{q(u)} \frac{1}{1+t^4}\,dt.$$

[The value of the integral on the right hand side can be represented by an expression involving the natural logarithm and inverse tangent functions. You are neither required nor expected to carry though that calculation, merely find the appropriate values of $p(u)$ and $q(u)$ for given positive u.]

(c) Is it the case that

$$\lim_{u \to 0+} \int_0^1 f_u(x)\, dx = 0?$$

[Briefly justify your answer.]

3. In this question let $f\colon [0, +\infty) \to \mathbb{R}$ be a non-negative Lebesgue-measurable function of a real variable, defined for all non-negative values of that variable, with the property that

$$\int_0^{+\infty} f(x)\, dx = 1.$$

(a) Briefly explain how it follows from Lebesgue's Dominated Convergence Theorem that

$$\lim_{j \to +\infty} \int_j^{+\infty} f(x)\, dx = 0$$

(the limit being taken as j tends to infinity through positive integer values). [No reference should be made to Levi's Monotone Convergence Theorem.]

(b) Briefly explain how it follows from Levi's Monotone Convergence Theorem that

$$\lim_{j \to +\infty} \int_j^{+\infty} f(x)\, dx = 0$$

(the limit being taken as j tends to infinity through positive integer values). [No reference should be made to Lebesgue's Dominated Convergence Theorem.]

4. Let $f_1, f_2, f_3, \ldots$ be an infinite sequence of non-negative Lebesgue-measurable functions of a real variable, define for all real numbers x. Suppose that $\int_{-\infty}^{+\infty} f_j(x)\, dx < +\infty$ for all j, that $f_j(x) \geq f_{j+1}(x)$ for all real numbers x and all positive integers j and that $\lim\limits_{j \to +\infty} f_j(x) = 0$ for all real numbers x. Prove that there cannot exist any real number K with the property that

$$\int_{-\infty}^{+\infty} e^{-f_j(x)}\, dx \leq K$$

for all positive integers j.