MAU22200, Hilary Term 2020.
Problems I: Interchanging limits and integrals
Worked Solutions to Problems

1. *For each positive real number u let Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined as follows:*

$$f(x) = \begin{cases} \dfrac{1}{x} & \textit{if } x \neq 0 \textit{ and } x \textit{ is rational;} \\ x & \textit{if } x = 0 \textit{ or } x \textit{ is irrational.} \end{cases}$$

Is the function f Lebesgue-integrable on the interval $[0,1]$? If so, what is the value of $\int_0^1 f(x)\,dx$?

The function f is Lebesgue-integrable on the interval $[0,1]$. Indeed the set of rational numbers contained in $[0,1]$ is a set of measure zero. Therefore the function f is equal almost everywhere to the identity function g, where $g(x) = x$ for all $x \in [0,1]$. Moreover it follows from this that

$$\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx = \int_0^1 x\,dx = \tfrac{1}{2}.$$

2. *For each positive real number u let $f_u\colon \mathbb{R} \to \mathbb{R}$ be the function of a real variable defined so that*

$$f_u(x) = \frac{u^6}{u^8 + (x-u)^4}$$

for all real numbers x. We examine the behaviour of the integrals $\int_0^1 f_u(x)\,dx$. You may take it for granted that, for continuous functions such as the functions f_u, when integrated over bounded intervals, the values of the integrals with respect to Lebesgue measure, as determined in the theory of the Lebesgue integral, coincide with the values that result from applying the usual rules of differential and integral calculus.

(a) *Considering separately the cases when $x = 0$ and when $x \neq 0$, show that $\lim_{u \to 0^+} f_u(x) = 0$ for all real numbers x. (In other words show that, for each real x, the limit of $f_u(x)$ as u tends to zero from above is equal to zero.)*

[There is no need to use a ε–δ argument. Standard theorems concerning limits of sums, differences, products, quotients and/or compositions of functions can be applied.]

First we note that

$$\lim_{u\to 0^+} f_u(0) = \lim_{u\to 0^+} \frac{u^6}{u^8+u^4} = \lim_{u\to 0^+} \frac{u^2}{u^4+1} = \frac{\displaystyle\lim_{u\to 0^+} u^2}{\displaystyle\lim_{u\to 0^+} (u^4+1)} = \frac{0}{1} = 0.$$

Thus the required result holds in the case when $x = 0$.
Now suppose that $x \neq 0$. Then

$$\lim_{u\to 0^+} f_u(x) = \frac{\displaystyle\lim_{u\to 0^+} u^6}{\displaystyle\lim_{u\to 0^+} \left(u^8 + (x-u)^4\right)} = \frac{0}{x^4} = 0.$$

Thus $\displaystyle\lim_{u\to 0^+} f_u(x) = 0$ for all real numbers x.

(b) *For each positive real number u, determine real numbers $p(u)$ and $q(u)$ such that*

$$\int_0^1 f_u(x)\,dx = \int_{p(u)}^{q(u)} \frac{1}{1+t^4}\,dt.$$

[The value of the integral on the right hand side can be represented by an expression involving the natural logarithm and inverse tangent functions. You are neither required nor expected to carry though that calculation, merely find the appropriate values of $p(u)$ and $q(u)$ for given positive u.]

Note that, for all real numbers x, and for all positive real numbers u,

$$f_u(x) = \frac{u^6}{u^8+(x-u)^4} = \frac{1}{u^2\left(1+\left(\dfrac{x}{u^2}-\dfrac{1}{u}\right)^4\right)}$$

Substituting $x = u^2t + u$, and applying the rule for integration by substitution, we find that

$$\int_0^1 f_u(x)\,dx = \int_{p(u)}^{q(u)} \frac{1}{1+t^4}\,dt,$$

where

$$p(u) = -\frac{1}{u} \quad \text{and} \quad q(u) = \frac{1}{u^2} - \frac{1}{u}.$$

(c) *Is it the case that*

$$\lim_{u\to 0^+} \int_0^1 f_u(x)\,dx = 0?$$

[Briefly justify your answer.]

It is not the case that the limit of the integrals is equal to zero. The function sending each real number t to $1/(1+t^4)$ is integrable on the real line. Let

$$A = \int_{-\infty}^{\infty} \frac{1}{1+t^4}\,dt = 2\int_0^{+\infty} \frac{1}{1+t^4}\,dt = 2\lim_{R\to+\infty}\int_0^R \frac{1}{1+t^4}\,dt.$$

Then $A > 0$. (In fact, $A = \pi/\sqrt{2}$, but this does not need to be proved here.) Moreover, given any positive real number ε, there exists some real number K such that

$$A - \varepsilon < \int_{-s}^{s} \frac{1}{1+t^4}\,dt < A$$

whenever $s > K$. Now, given K, there exists some positive real number δ such that $p(u) < -K$ and $q(u) > K$ whenever $0 < u < \delta$, because $\lim_{u\to 0^+} p(u) = -\infty$ and $\lim_{u\to 0^+} q(u) = +\infty$. It follows that

$$A - \varepsilon < \int_0^1 f_u(x)\,dx < A$$

whenever $0 < u < \delta$, and therefore

$$\lim_{u\to 0^+} \int_0^1 f_u(x)\,dx = A.$$

Note in particular that

$$\lim_{u\to 0^+} \int_0^1 f_u(x)\,dx \neq \int_0^1 \left(\lim_{u\to 0^+} f_u(x)\right)\,dx$$

in this example!

3. *In this question let* $f\colon [0,+\infty) \to \mathbb{R}$ *be a non-negative Lebesgue-measurable function of a real variable, defined for all non-negative values of that variable, with the property that*

$$\int_0^{+\infty} f(x)\,dx = 1.$$

(a) *Briefly explain how it follows from Lebesgue's Dominated Convergence Theorem that*

$$\lim_{j\to+\infty} \int_j^{+\infty} f(x)\,dx = 0$$

(the limit being taken as j tends to infinity through positive integer values). [No reference should be made to Levi's Monotone Convergence Theorem.]

Let functions $f_1, f_2, f_3, \ldots$ be defined on $[0, +\infty)$ so that

$$f_j(x) = \begin{cases} f(x) & \text{if } 0 \le x < j; \\ 0 & \text{if } x \ge j. \end{cases}$$

Then the functions f_j and $f - f_j$ are all measurable, and $0 \le f(x) - f_j(x) \le f(x)$ for all non-negative real numbers x and for all positive integers j. Moreover $\lim\limits_{j\to+\infty}(f(x) - f_j(x)) = 0$ for all non-negative real numbers x. It follows from Lebesgue's Dominated Convergence Theorem that

$$\begin{aligned} \lim_{j\to+\infty} \int_j^{+\infty} f(x)\,dx &= \lim_{j\to+\infty} \int_0^{+\infty} (f(x) - f_j(x))\,dx \\ &= \int_0^{+\infty} \lim_{j\to+\infty} (f(x) - f_j(x)\,dx = 0, \end{aligned}$$

as required.

(b) *Briefly explain how it follows from Levi's Monotone Convergence Theorem that*

$$\lim_{j\to+\infty} \int_j^{+\infty} f(x)\,dx = 0$$

(the limit being taken as j tends to infinity through positive integer values). [No reference should be made to Lebesgue's Dominated Convergence Theorem.]

Let functions $f_1, f_2, f_3, \ldots$ be defined on $[0, +\infty)$ so that

$$f_j(x) = \begin{cases} f(x) & \text{if } 0 \le x < j; \\ 0 & \text{if } x \ge j. \end{cases}$$

Then the functions f_j are all measurable, and for each non-negative real number x, the infinite sequence $f_1(x), f_2(x), f_3(x), \ldots$ converges to $f(x)$. It follows from Levi's Monotone Convergence Theorem that

$$\lim_{j\to+\infty} \int_0^{+\infty} f_j(x)\,dx = \int_0^{+\infty} f(x)\,dx,$$

and therefore

$$\begin{aligned}\lim_{j\to+\infty}\int_j^{+\infty} f(x)\,dx &= \lim_{j\to+\infty}\int_0^{+\infty}(f(x)-f_j(x))\,dx\\ &= \int_0^{+\infty} f(x)\,dx - \lim_{j\to+\infty}\int_0^{+\infty} f_j(x))\,dx = 0,\end{aligned}$$

as required.

4. *Let $f_1, f_2, f_3, \ldots$ be an infinite sequence of non-negative Lebesgue-measurable functions of a real variable, define for all real numbers x. Suppose that $\int_{-\infty}^{+\infty} f_j(x)\,dx < +\infty$ for all j, that $f_j(x) \geq f_{j+1}(x)$ for all real numbers x and all positive integers j and that $\lim_{j\to+\infty} f_j(x) = 0$ for all real numbers x. Prove that there cannot exist any real number K with the property that*

$$\int_{-\infty}^{+\infty} e^{-f_j(x)}\,dx \leq K$$

for all positive integers j.

Let real-valued functions $g_1, g_2, g_3, \ldots$ be defined on $\mathbb{R}$ so that $g_j(x) = e^{-f_j(x)}$ for all real numbers x. Then, for each real number x, the infinite sequence $g_1(x), g_2(x), g_3(x) \ldots$ is non-decreasing, $0 < g_j(x) \leq 1$ for all j, and $\lim_{j\to+\infty} g_j(x) = 1$. It follows from Levi's Monotone Convergence Theorem that

$$\begin{aligned}\lim_{j\to+\infty}\int_{-\infty}^{+\infty} e^{-f_j(x)}\,dx &= \lim_{j\to+\infty}\int_{-\infty}^{+\infty} g_j(x)\,dx\\ &= \int_{-\infty}^{+\infty}\lim_{j\to+\infty} g_j(x)\,dx = +\infty.\end{aligned}$$

Therefore, given any positive real number K, there exists some positive integer N such that

$$\int_{-\infty}^{+\infty} e^{-f_j(x)}\,dx > K$$

whenever $j \geq N$. The result follows.

Alternative Proof

Let K be a positive constant, and let a Lebesgue-measurable set E be chosen in $\mathbb{R}$ for which $K < \mu(E) < +\infty$, where $\mu(E)$ denotes the Lebesgue measure of E. (This measurable set E could, for example, be a bounded interval of length exceeeding K.) Also let real-valued functions $g_1, g_2, g_3, \ldots$ be defined on $\mathbb{R}$ so that $g_j(x) = e^{-f_j(x)}$ for all real numbers x. It follows from Egorov's Theorem that the infinite sequence $g_1, g_2, g_3, \ldots$ of real-valued functions converges almost uniformly on E, and therefore there exists a subset F of E such that $\mu(F) < \mu(E) - K$ and the infinite sequence $g_1, g_2, g_3, \ldots$ of measurable real-valued functions converges uniformly on $E \setminus F$. Now $\mu(E \setminus F) \geq \mu(E) - \mu(F) > K$. Choose some positive real number ε small enough to ensure that $(1-\varepsilon)\mu(E \setminus F) > K$. Then there exists some positive integer N such that $g_j(x) > 1 - \varepsilon$ whenever $x \in E \setminus F$ and $j \geq N$, because the infinite sequence $g_1, g_2, g_3, \ldots$ of real-valued functions converges uniformly on $E \setminus F$. But then

$$\int_{-\infty}^{+\infty} e^{-f_j(x)}\,dx = \int_{\mathbb{R}} g_j\,d\mu \geq \int_{E\setminus F} g_j\,d\mu \geq (1-\varepsilon)\mu(E \setminus F) > K$$

whenever $j \geq N$. The result follows.