Module MA22200: Advanced Analysis (Semester 2) Hilary Term 2020 Part III (Section 8)

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Contents

8 The Lebesgue Integral

8.1 Measurable Functions

Definition Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let $f\colon X \to [-\infty, +\infty]$ be a function on X with values in the set $[-\infty, +\infty]$ of extended real numbers. The function f is said to be *measurable* with respect to the σ-algebra $\mathcal{A}$ if $\{x \in X : f(x) < c\} \in \mathcal{A}$ for all real numbers c.

Definition Let $(X, \mathcal{A}, \mu)$ be a measure space. A function $f\colon X \to [-\infty, +\infty]$ defined on X is said to be *measurable* if it is measurable with respect to the σ-algebra $\mathcal{A}$ of measurable subsets of X.

It follows from these definitions that a function $f\colon X \to [-\infty, +\infty]$ defined on a measure space $(X, \mathcal{A}, \mu)$ is measurable if and only if $\{x \in X : f(x) < c\}$ is a measurable set for all real numbers c.

Definition Let $(X, \mathcal{A}, \mu)$ be a measure space, and let E be a measurable subset of X. A function $f\colon E \to [-\infty, +\infty]$ defined on E is said to be *measurable on E* if, for all real numbers c,

$$\{x \in E : f(x) < c\}$$

is a measurable subset of X (i.e., if and only if this set belongs to the σ-algebra $\mathcal{A}$ of measurable subsets of X).

Proposition 8.1 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, let $f\colon X \to [-\infty, +\infty]$ be a function on X, with values in the set $[-\infty, +\infty]$ of extended real numbers, which is measurable with respect to the σ-algebra $\mathcal{A}$, and let a, b and c be real numbers, where $a \leq b$. Then the following sets also belong to the σ-algebra $\mathcal{A}$:*

(i) $\{x \in X : f(x) \geq c\}$;

(ii) $\{x \in X : f(x) \leq c\}$;

(iii) $\{x \in X : f(x) > c\}$;

(iv) $\{x \in X : a \leq f(x) \leq b\}$;

(v) $\{x \in X : a < f(x) < b\}$;

(vi) $\{x \in X : a \leq f(x) < b\}$;

(vii) $\{x \in X : a < f(x) \leq b\}$;

(viii) $\{x \in X : f(x) = c\}$;

(ix) $\{x \in X : f(x) = -\infty\}$;

(x) $\{x \in X : f(x) = +\infty\}$;

(xi) $\{x \in X : f(x) < +\infty\}$;

(xii) $\{x \in X : f(x) > -\infty\}$;

(xiii) $\{x \in X : f(x) \in \mathbb{R}\}$.

Proof The set $\{x \in X : f(x) \geq c\}$ is the complement of a set $\{x \in X : f(x) < c\}$ belonging to the σ-algebra $\mathcal{A}$, and must therefore itself belong to this σ-algebra. This proves (i).

The set $\{x \in X : f(x) \leq c\}$ may be represented as a countable intersection

$$\bigcap_{n=1}^{+\infty} \left\{ x \in X : f(x) < c + \frac{1}{n} \right\}$$

of sets that are of the form $\{x \in X : f(x) < c + n^{-1}\}$ for some natural number n. These sets belong to the σ-algebra $\mathcal{A}$, and any countable intersection of sets belonging to $\mathcal{A}$ must itself belong to this σ-algebra. Therefore $\{x \in X : f(x) \leq c\}$ belongs to the σ-algebra. This proves (ii).

The set $\{x \in X : f(x) > c\}$ is the complement of a set $\{x \in X : f(x) \leq c\}$ which belongs to the σ-algebra $\mathcal{A}$, and must therefore itself belong to $\mathcal{A}$. This proves (iii).

The set $\{x \in X : a \leq f(x) \leq b\}$ is the intersection of sets $\{x \in X : f(x) \geq a\}$ and $\{x \in X : f(x) \leq b\}$ that belong to the σ-algebra $\mathcal{A}$. It follows that $\{x \in X : a \leq f(x) \leq b\}$ must itself belong to $\mathcal{A}$. Similarly $\{x \in X : a < f(x) < b\}$ is the intersection of sets $\{x \in X : f(x) > a\}$ and $\{x \in X : f(x) < b\}$, $\{x \in X : a \leq f(x) < b\}$ is the intersection of sets $\{x \in X : f(x) \geq a\}$ and $\{x \in X : f(x) < b\}$, and $\{x \in X : a < f(x) \leq b\}$ is the intersection of sets $\{x \in X : f(x) > a\}$ and $\{x \in X : f(x) \leq b\}$, and therefore $\{x \in X : a < f(x) < b\}$, $\{x \in X : a \leq f(x) < b\}$ and $\{x \in X : a < f(x) \leq b\}$ belong to $\mathcal{A}$. This proves (iv), (v), (vi) and (vii). Moreover (viii) is a special case of (iv).

The set $\{x \in X : f(x) = -\infty\}$ may be represented as a countable intersection

$$\bigcap_{n=1}^{+\infty} \{x \in X : f(x) < -n\}$$

of sets belonging to $\mathcal{A}$, and must therefore itself belong to $\mathcal{A}$. This proves (ix).

Similarly the set $\{x \in X : f(x) = +\infty\}$ may be represented as a countable intersection

$$\bigcap_{n=1}^{+\infty} \{x \in X : f(x) \geq n\}$$

of sets belonging to $\mathcal{A}$, and must therefore itself belong to $\mathcal{A}$. This proves (x).

The set $\{x \in X : f(x) < +\infty\}$ is the complement of the set specified in (x), and must therefore belong to $\mathcal{A}$. Similarly the set $\{x \in X : f(x) > -\infty\}$ is the complement of the set specified in (ix), and must therefore belong to $\mathcal{A}$. This proves (xi) and (xii).

Finally we note that $\{x \in X : f(x) \in \mathbb{R}\}$ is the intersection of the sets $\{x \in X : f(x) < +\infty\}$ and $\{x \in X : f(x) > -\infty\}$ specified in (xi) and (xii), and must therefore belong to $\mathcal{A}$, as required. ∎

Corollary 8.2 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, let $f\colon X \to [-\infty, +\infty]$ be a function on X, with values in the set $[-\infty, +\infty]$ of extended real numbers, which is measurable with respect to the σ-algebra $\mathcal{A}$, and let m be a real number. Then mf is measurable with respect to $\mathcal{A}$.*

Proof The result is immediate when $m = 0$. Let c be a real number. If $m > 0$ then

$$\{x \in X : mf(x) < c\} = \{x \in X : f(x) < c/m\},$$

and if $m < 0$ then

$$\{x \in X : mf(x) < c\} = \{x \in X : f(x) > c/m\}.$$

It then follows immediately from Proposition 8.1 and the definition of measurable functions that $\{x \in X : mf(x) < c\} \in \mathcal{A}$. Therefore mf is measurable, as required. ∎

Definition A subset V of the extended real line $[-\infty, +\infty]$ is said to be *open* if and only if it satisfies the following conditions:

- given any real number p belonging to V, there exists some positive real number δ for which $\{t \in \mathbb{R} : p - \delta < t < p + \delta\} \subset V$;
- in cases where $+\infty \in V$ there exists some real number L large enough to ensure that $\{t \in \mathbb{R} : t > L\} \subset V$;
- in cases where $-\infty \in V$ there exists some real number L large enough to ensure that $\{t \in \mathbb{R} : t < -L\} \subset V$.

The empty set $\emptyset$ is open in $[-\infty, +\infty]$, and $[-\infty, +\infty]$ is open in itself. Any union of open subsets of $[-\infty, +\infty]$ is itself open in $[-\infty, +\infty]$, and any finite intersection of open subsets of $[-\infty, +\infty]$ is itself open in $[-\infty, +\infty]$.

Lemma 8.3 *Any open set in the extended real line $[-\infty, +\infty]$ is the union of open intervals that are of the forms (p, q), $(p, +\infty]$, $[-\infty, q)$ with rational endpoints p and q.*

Proof Let V be open in $[-\infty, +\infty]$, and let $v \in V$, where $-\infty < v < +\infty$. Then there exists some positive real number δ such that $(v - \delta, v + \delta) \subset V$. Let rational numbers p and q be chosen such that $v - \delta < p < v < q < v + \delta$. Then $v \in (p, q)$.

If $+\infty \in V$ then some rational number p can be chosen to ensure that $(p, +\infty] \subset V$, and if $-\infty \in V$ then some rational number q can be chosen to ensure that $[-\infty, q) \in V$. The result follows. ∎

Proposition 8.4 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f \colon X \to [-\infty, +\infty]$ be a measurable function on X, and let V be an open set in the extended real line $[-\infty, +\infty]$. Then $f^{-1}(V)$ is a measurable set.*

Proof Let $\mathcal{C}$ be the collection consisting of all open intervals of the form (p, q), $(p, +\infty]$ and $[-\infty, q)$ contained in V for which p and q are rational numbers. Then the collection $\mathcal{C}$ is countable, and $V = \bigcup_{J \in \mathcal{C}} J$. The preimage of a union of subsets of $[-\infty, +\infty]$ is the union of the preimages of those sets, and therefore $f^{-1}(V) = \bigcup_{J \in \mathcal{C}} f^{-1}(J)$. Now it follows from applications of Proposition 8.1 that $f^{-1}(J) \in \mathcal{A}$ for all $J \in \mathcal{C}$. Thus the preimages $f^{-1}(J)$ of all the intervals in the countable collection $\mathcal{C}$ are measurable sets, and therefore $f^{-1}(V) \in \mathcal{A}$, as required. ∎

Proposition 8.5 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f \colon X \to [-\infty, +\infty]$ be a measurable function on X, and let B be a Borel set in the extended real line $[-\infty, +\infty]$. Then $f^{-1}(B)$ is a measurable set.*

Proof Let $\mathcal{G}$ be the collection consisting of all subsets G of the extended real line $[-\infty, +\infty]$ for which $f^{-1}(G) \in \mathcal{A}$. If $G \in \mathcal{G}$ then $f^{-1}([-\infty, +\infty] \setminus G) = X \setminus f^{-1}(G) \in \mathcal{A}$, because the complement of every member of the σ-algebra $\mathcal{A}$ must itself belong to $\mathcal{A}$. It then follows from the specification of $\mathcal{G}$ that $[-\infty, +\infty] \setminus G \in \mathcal{G}$. Thus the complement, in the extended real line, of every member of the collection $\mathcal{G}$ must itself belong to $\mathcal{G}$.

Now let $(G_i :\in I)$ be a countable collection of subsets of $[-\infty, +\infty]$ that all belong to $\mathcal{G}$. Then

$$f^{-1}\left(\bigcup_{i \in I} G_i\right) = \bigcup_{i \in I} f^{-1}(G_i) \in \mathcal{A},$$

because every countable union of sets belonging to $\mathcal{A}$ must itself belong to $\mathcal{A}$. It follows that $\mathcal{G}$ is a σ-algebra of subsets of the extended real line $[-\infty, +\infty]$. Also every open subset of $[-\infty, +\infty]$ belongs to $\mathcal{G}$ (Proposition 8.4). It follows that $\mathcal{G}$ contains the σ-algebra generated by the open subsets of $[-\infty, +\infty]$. The latter σ-algebra is the σ-algebra of Borel sets in $[-\infty, +\infty]$. The result follows. ∎

Proposition 8.6 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, let $f: X \to [-\infty, +\infty]$ and $g: X \to [-\infty, +\infty]$ be functions on X, with values in the set $[-\infty, +\infty]$ of extended real numbers, which are measurable with respect to the σ-algebra $\mathcal{A}$. Then, given any real number c, the set*

$$\{x \in X : f(x) + g(x) \text{ is defined and } f(x) + g(x) < c\}$$

is measurable with respect to the σ-algebra $\mathcal{A}$.

Proof Let u and v be elements of the extended real number system $[-\infty, +\infty]$ for which $u+v$ is defined and satisfies $u+v < c$. Then $u < +\infty$ and $v < +\infty$. We show that there exists a rational number q such that $u < q$ and $v < c-q$. Now if $u = -\infty$ it suffices to choose q so that $q < c-v$. If $v = -\infty$ it suffices to choose q such that $q > u$. If u and v are real numbers then $u < c-v$, and therefore we may choose a rational number q that satisfies the inequalities. $u < q < c-v$. But then $u < q$ and $v < c-q$. This completes the case-by-case analysis that establishes that, given any elements u and v of the extended real number system for which $u+v$ is defined and satisfies $u < c$, there exists some rational number q such that $u < q$ and $v < c - q$.

The result just established ensures that

$$\{x \in X : f(x) + g(x) \text{ is defined and } f(x) + g(x) < c\} = \bigcup_{q \in \mathbb{Q}} E_q,$$

where

$$E_q = \{x \in X : f(x) < q \text{ and } g(x) < c - q\}.$$

for each rational number q. Now, for each rational number q, the sets

$$\{x \in X : f(x) < q\} \quad \text{and} \quad \{x \in X : g(x) < c - q\}$$

are measurable with respect to $\mathcal{A}$, because the functions f and g are measurable. It follows that, for each rational number q, the set E_q, being the intersection of two measurable sets, must itself be measurable with respect to $\mathcal{A}$. It then follows that the set

$$\{x \in X : f(x) + g(x) \text{ is defined and } f(x) + g(x) < c\}$$

is a countable union of measurable sets, and therefore is itself measurable with respect to $\mathcal{A}$. ∎

Corollary 8.7 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, let $f\colon X \to [-\infty, +\infty]$ and $g\colon X \to [-\infty, +\infty]$ be functions on X, with values in the set $[-\infty, +\infty]$ of extended real numbers, which are measurable with respect to the σ-algebra $\mathcal{A}$. Suppose that $f(x) + g(x)$ is defined for all $x \in X$. Then $f + g$ is measurable with respect to $\mathcal{A}$.*

Proof Proposition 8.6 ensures that, for all real numbers c, the set $\{x \in X : f(x) + g(x) < c\}$ is measurable with respect to $\mathcal{A}$. It then follows from the definition of measurable functions that $f + g$ is measurable with respect to $\mathcal{A}$, as required. ∎

Proposition 8.8 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, let $f\colon X \to [-\infty, +\infty]$ and $g\colon X \to [-\infty, +\infty]$ be functions on X, with values in the set $[-\infty, +\infty]$ of extended real numbers, which are measurable with respect to the σ-algebra $\mathcal{A}$. Then $f \cdot g$ is measurable with respect to $\mathcal{A}$, where $(f \cdot g)(x) = f(x)g(x)$ for all $x \in X$.*

Proof First we prove the result in the special case where the functions f and g are real-valued, so that $-\infty < f(x) < +\infty$ and $-\infty < g(x) < +\infty$ for all $x \in X$. In that case

$$f(x)g(x) = \tfrac{1}{2}\Bigl((f(x) + g(x))^2 - f(x)^2 - g(x)^2\Bigr)$$

for all $x \in X$. Now

$$\{x \in X : f(x)^2 < c\} = \{x \in X : -\sqrt{c} < f(x) < \sqrt{c}\}.$$

for all positive real numbers c, and

$$\{x \in X : f(x)^2 < c\} = \emptyset$$

for all non-positive real numbers c. It follows (on applying the results of Proposition 8.1) that the function $x \mapsto f(x)^2$ is measurable. Similarly the functions $x \mapsto g(x)^2$ and $x \mapsto (f(x) + g(x))^2$ are measurable. Sums and scalar multiples of measurable functions are measurable (see Corollary 8.2 and Proposition 8.6). It follows that, if the functions f and g are measurable and real-valued then the function $f \cdot g$ is measurable.

Now suppose that there is some $x \in X$ for which either $f(x)$ or $g(x)$ is equal to $+\infty$ or $-\infty$. In that case let

$$Z = \{x \in X : f(x) = \pm\infty \text{ or } g(x) = \pm\infty\},$$

and define

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for all } x \in X \setminus Z; \\ 0 & \text{for all } x \in Z. \end{cases}$$

and

$$\tilde{g}(x) = \begin{cases} g(x) & \text{for all } x \in X \setminus Z; \\ 0 & \text{for all } x \in Z. \end{cases}$$

Then the functions $\tilde{f}$ and $\tilde{g}$ are measurable and real-valued on X, and therefore $\tilde{f} \cdot \tilde{g}$ is measurable on X.

Now the functions $\tilde{f}$ and $\tilde{g}$ agree with the functions f and g on the set $X \setminus Z$. It follows that, for all real numbers c, the set

$$\{x \in X \setminus Z : f(x)g(x) < c\}$$

is measurable. Also $f(x)g(x) = 0$ if and only if either $f(x) = 0$ or $g(x) = 0$. It follows that

$$\{x \in X : f(x)g(x) = 0\}$$

is measurable.

The definition of products in the extended real number system involving $+\infty$ and $-\infty$ ensure that the possible values for $f(x)g(x)$ on Z are $+\infty$, 0 and $-\infty$. Also $f(x)g(x) = +\infty$ in exactly four cases: $f(x) = +\infty$ and $g(x) > 0$; $f(x) = -\infty$ and $g(x) < 0$; $g(x) = +\infty$ and $f(x) > 0$; $g(x) = -\infty$ and $f(x) < 0$. It follows easily from this that

$$\{x \in X : f(x)g(x) = +\infty\}$$

is measurable. Similarly the set

$$\{x \in X : f(x)g(x) = -\infty\}$$

is measurable. These results are sufficient to establish that

$$\{x \in X : f(x)g(x) < c\}$$

is a measurable set for all real numbers c and therefore the function $f \cdot g$ is measurable on X, as required. ∎

Lemma 8.9 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let $f_1, f_2, \ldots, f_m$ be functions on X with values in the set $[-\infty, +\infty]$ of extended real numbers. Suppose that each of the functions $f_1, f_2, \ldots, f_m$ is measurable with respect to the σ-algebra $\mathcal{A}$. Then so are*

$$\max(f_1, f_2, \ldots, f_m) \quad \textit{and} \quad \min(f_1, f_2, \ldots, f_m).$$

Proof Let c be a real number. Then

$$\{x \in X : \max(f_1, f_2, \ldots, f_m) < c\} = \bigcap_{i=1}^{m} \{x \in X : f_i(x) < c\}$$

and

$$\{x \in X : \min(f_1, f_2, \ldots, f_m) < c\} = \bigcup_{i=1}^{m} \{x \in X : f_i(x) < c\}.$$

It follows that $\{x \in X : \max(f_1, f_2, \ldots, f_m) < c\}$ is a finite intersection of sets belonging to $\mathcal{A}$, and must therefore itself belong to $\mathcal{A}$. Similarly $\{x \in X : \min(f_1, f_2, \ldots, f_m) < c\}$ is a finite union of sets belonging to $\mathcal{A}$, and must therefore itself belong to $\mathcal{A}$. The result follows. ∎

Proposition 8.10 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let $f_1, f_2, f_3, \ldots$ be an infinite sequence of functions on X with values in the set $[-\infty, +\infty]$ of extended real numbers. Suppose that each of the functions $f_1, f_2, f_3, \ldots$ is measurable with respect to the σ-algebra $\mathcal{A}$. Then so are g and h, where*

$$g(x) = \inf\{f_i(x) : i \in \mathbb{N}\}, \quad h(x) = \sup\{f_i(x) : i \in \mathbb{N}\}$$

for all $x \in X$.

Proof Let c be a real number, and let x be a point of X. Then $g(x) < c$ if and only if there exists some natural number j for which $f_j(x) < c$, and $h(x) < c$ if and only if there exists some natural number k such that $f_j(x) < c - k^{-1}$ for all natural numbers j. Therefore

$$\{x \in X : g(x) < c\} = \bigcup_{j=1}^{+\infty} \{x \in X : f_j(x) < c\}$$

and

$$\{x \in X : h(x) < c\} = \bigcup_{k=1}^{+\infty} \bigcap_{j=1}^{+\infty} \left\{x \in X : f_j(x) < c - \frac{1}{k}\right\}$$

The measurability of the function f ensures that $\{x \in X : f_j(x) < c\}$ is measurable for all real numbers c and positive integers j. Also countable unions and countable intersections of measurable sets are measurable, because the collection $\mathcal{A}$ of measurable sets in X is a σ-algebra. Therefore the functions g and h are measurable, as required. ∎

Let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable functions on a measure space $(X, \mathcal{A}, \mu)$ taking values in the extended real line $[-\infty, \infty]$. We can construct from this infinite sequence a non-decreasing sequence of functions $g_1, g_2, g_3, \ldots$ and a non-increasing sequence of functions $h_1, h_2, h_3, \ldots$, where

$$g_j(x) = \inf\{f_k(x) : k \geq j\} \quad \text{and} \quad h_j(x) = \sup\{f_k(x) : k \geq j\}$$

for all positive integers j. It follows from Proposition 8.10 that the functions $g_1, g_2, g_3, \ldots$ and $h_1, h_2, h_3, \ldots$ are all measurable.

For all $x \in X$, the lower limit $f_*(x)$ and upper limit $f^*(x)$ of the infinite sequence $f_1(x), f_2(x), f_3(x), \ldots$ are defined such that

$$f_*(x) = \liminf_{j \to +\infty} f_j(x) = \lim_{j \to +\infty} g_j(x) = \sup_{j \to +\infty} g_j(x)$$

and

$$f^*(x) = \limsup_{j \to +\infty} f_j(x) = \lim_{j \to +\infty} h_j(x) = \inf_{j \to +\infty} h_j(x).$$

where the measurable functions $g_1, g_2, g_3, \ldots$ and $h_1, h_2, h_3, \ldots$ are defined in the manner described immediately above. It then follows, on applying Proposition 8.10, that the lower limit function f_* and the upper limit function f^* are both measurable. We formally state this result in the following corollary.

Corollary 8.11 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let $f_1, f_2, f_3, \ldots$ be an infinite sequence of functions on X with values in the set $[-\infty, +\infty]$ of extended real numbers. Suppose that each of the functions $f_1, f_2, f_3, \ldots$ is measurable with respect to the σ-algebra $\mathcal{A}$. Then so are f^* and f_*, where*

$$f^*(x) = \limsup_{j \to +\infty} f_j(x), \quad f_*(x) = \liminf_{j \to +\infty} f_j(x)$$

for all $x \in X$.

Corollary 8.12 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let $f_1, f_2, f_3, \ldots$ be an infinite sequence of functions on X with values in the set $[-\infty, +\infty]$ of extended real numbers. Suppose that each of the functions $f_1, f_2, f_3, \ldots$ is measurable with respect to the σ-algebra $\mathcal{A}$. Let*

$$X_0 = \{x \in X : \lim_{j \to +\infty} f_j(x) \text{ is defined}\}$$

Then $X_0 \in \mathcal{A}$. Moreover if $f(x) = \lim_{j \to +\infty} f_j(x)$ for all $x \in X_0$, then f is a measurable function on X_0.

Proof Note that

$$X_0 = \{x \in X : \limsup_{j \to +\infty} f_j(x) - \liminf_{j \to +\infty} f_j(x) = 0\}.$$

It follows from Proposition 8.1 that $X_0 \in \mathcal{A}$. Moreover the function f coincides with the measurable functions f^* on X_0, where $f^*(x) = \limsup_{j \to +\infty} f_j(x)$, and must therefore be a measurable function on X_0, as required. ∎

We see therefore that if $(X, \mathcal{A}, \mu)$ is a measure space then the limit of any convergent sequence of measurable functions on X must itself be measurable.

8.2 Integrable Simple Functions

Definition Let X be a set, and let E be a subset of X. The *characteristic function* of E is defined to be the function $\chi_E \colon X \to \mathbb{R}$ defined so that

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E; \\ 0 & \text{if } x \notin E. \end{cases}$$

Let E be a subset of X, where $(X, \mathcal{A}, \mu)$ is a measure space. It follows directly from the relevant definitions that the subset E is measurable if and only if its characteristic function χ_E is a measurable function on X.

Definition Let $(X, \mathcal{A}, \mu)$ be a measure space. A real-valued function $f \colon X \to \mathbb{R}$ on X is said to be an *integrable simple function* if there exist real numbers $c_1, c_2, \ldots, c_m$ and measurable subsets $E_1, E_2, \ldots, E_m$ of X, where $\mu(E_j) < +\infty$ for $j = 1, 2, \ldots, m$, such that

$$f(x) = \sum_{j=1}^{m} c_j \chi_{E_j}(x)$$

for all $x \in X$, where χ_{E_j} denotes the characteristic function of E_j for $j = 1, 2, \ldots, m$.

It follows directly from the definition of integrable simple functions that any real linear combination of integrable simple functions is itself an integrable simple functions, and thus the integrable simple functions on a measure space thus constitute a real vector space.

Lemma 8.13 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $E_1, E_2, \ldots, E_m$ be a finite collection of measurable subsets of X. Then there exists a finite list $G_1, G_2, \ldots, G_r$ of pairwise disjoint measurable subsets of X such that $\bigcup_{i=1}^{r} G_i = \bigcup_{j=1}^{m} E_j$ and, for each integer j between 1 and m, E_j is the disjoint union of those sets G_i for which $G_i \subset E_j$.*

Proof For each subset S of $\{1, 2, \ldots, m\}$ let F_S be the set consisting of all elements x of X that satisfy $x \in E_j$ for all $j \in S$ and $x \notin E_j$ for all $j \in \{1, 2, \ldots, m\} \setminus S$. Then

$$F_S = \left(\bigcap\nolimits_{j \in S} E_j\right) \cap \left(\bigcap\nolimits_{j \notin S} E_j^c\right) = \left(\bigcap\nolimits_{j \in S} E_j\right) \setminus \left(\bigcup\nolimits_{j \notin S} E_j\right),$$

where $E_j^c = X \setminus E_j$ for $j = 1, 2, \ldots, m$. Any finite intersection of measurable sets is measurable. It follows that each set F_S is measurable.

Let $r = 2^m - 1$, and let $S_0, S_1, \ldots S_r$ be a listing of the subsets of $\{1, 2, \ldots, m\}$ with $S_0 = \emptyset$ in which every subset of $\{1, 2, \ldots, m\}$ occurs exactly once, let

$$G_0 = F_{S_0} = E_1^c \cap E_2^c \cap \cdots \cap E_m^c = X \setminus \bigcup_{j=1}^{m} E_j,$$

and let $G_k = F_{S_k}$ for $k = 1, 2, \ldots, r$. Then, given any element x of X, there exists exactly one subset S of $\{1, 2, \ldots, m\}$ for which $x \in F_S$. This subset S consists of those, and only those, integers j between 1 and m for which $x \in E_j$. Thus, given any element x of $X \setminus G_0$, there exists exactly one integer i between 1 and r for which $x \in G_i$. It follows that the sets $G_1, G_2, \ldots, G_r$ are pairwise disjoint, and their union is the complement of the set G_0. But $X \setminus G_0 = \bigcup\limits_{j=1}^{m} E_j$. We conclude therefore that the sets $G_1, G_2, \ldots, G_r$ are pairwise disjoint and $\bigcup\limits_{i=1}^{r} G_i = \bigcup\limits_{j=1}^{m} E_j$.

Let i and j be integers, where $1 \leq i \leq r$ and $1 \leq j \leq m$. If $j \in S_i$ then $G_i \subset E_j$; and if $j \notin S_i$ then $G_i \cap E_j = \emptyset$. But every element of $\bigcup\limits_{j=1}^{m} E_j$. belongs to exactly one of the sets $G_1, G_2, \ldots, G_r$. It follows that E_j is the union of those sets G_i for which $j \in S_i$, and therefore E_j is the union of those sets G_i for which $G_i \subset E_j$, as required. ∎

Proposition 8.14 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f\colon X \to \mathbb{R}$ be an integrable simple function on X, and let $f = \sum_{j=1}^{m} c_j \chi_{E_j}$, where, for each integer j between 1 and m, c_j is a real number and χ_{E_j} is the characteristic function of a measurable set E_j for which $\mu(E_j) < +\infty$. Let the non-zero values taken on by the function f be $v_1, v_2, \ldots, v_n$, where no real numbers occurs more than once in this list, and let $F_k = \{x \in X : f(x) = v_k\}$ for $k = 1, 2, \ldots, n$. Then the sets $F_1, F_2, \ldots, F_n$ are measurable and pairwise disjoint, $\mu(F_k) < +\infty$ for $k = 1, 2, \ldots, n$, and*

$$\sum_{j=1}^{m} c_j \mu(E_j) = \sum_{k=1}^{n} v_k \mu(F_k).$$

Proof It follows from Lemma 8.13 that there exists a finite list $G_1, G_2, \ldots, G_r$ of pairwise disjoint measurable subsets of X such that $\bigcup\limits_{i=1}^{r} G_i = \bigcup\limits_{i=1}^{m} E_j$ and, for each integer j between 1 and m, E_j is the disjoint union of those sets G_i for which $G_i \subset E_j$. Let J be the set consisting of those ordered pairs (i, j) of integers for which $1 \leq i \leq r$, $0 \leq j \leq m$ and $G_i \subset E_j$. The additivity of the measure μ ensures that the measure $\mu(E_j)$ of E_j is the sum of the measures $\mu(G_i)$ of those sets G_i in the list $G_1, G_2, \ldots, G_r$ for which $G_i \subset E_j$. It follows that

$$\sum_{j=1}^{n} c_j \mu(E_j) = \sum_{(i,j)\in J} c_j \mu(G_i) = \sum_{i=1}^{r} w_i \mu(G_i),$$

where w_i is the sum of those real numbers c_j for which $G_i \subset E_j$.

Let i be an integer between 1 and n, and let $x \in G_i$. Then $f(x)$ is the sum of those c_j for which $G_i \subset E_j$, and therefore $f(x) = w_i$. Thus the function f takes the value w_i throughout the set G_i. It follows that that, for each integer i between 1 and r, either the function f is zero throughout G_i or else there exists exactly one integer k between 1 and n for which $w_i = v_k$. Therefore, for each integer k between 1 and n, the set F_k is the disjoint union of those sets G_i for which $w_i = v_k$. It follows that each set F_k is measurable, and $\mu(F_k)$ is the sum of the measures $\mu(G_i)$ of the sets G_i for which $w_i = v_k$. It then follows that

$$\sum_{j=1}^{n} c_j \mu(E_j) = \sum_{i=1}^{r} w_i \mu(G_i) = \sum_{k=1}^{n} v_k \mu(F_k),$$

as required. ∎

Definition Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f\colon X \to \mathbb{R}$ be an integrable simple function on X. The *integral* $\int_X f\,d\mu$ of the function f on X is defined so that

$$\int_X f\,d\mu = \sum_{k=1}^{n} v_k \mu(F_k),$$

where $v_1, v_2, \ldots, v_n$ are distinct and are the non-zero values taken on by the function f, and where

$$F_k = \{x \in X : f(x) = v_k\}$$

for $k = 1, 2, \ldots, n$.

Corollary 8.15 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f\colon X \to \mathbb{R}$ be an integrable simple function on X, and let $f = \sum_{j=1}^{m} c_j \chi_{E_j}$, where, for each integer j between 1 and m, c_j is a real number and χ_{E_j} is the characteristic function of a measurable set E_j for which $\mu(E_j) < +\infty$. Then*

$$\int_X f\,d\mu = \sum_{j=1}^{m} c_j \mu(E_j).$$

Proof The result follows immediately on combining the definition of the integral $\int_X f \, d\mu$ with the result of Proposition 8.14. ∎

Proposition 8.16 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be integrable simple functions on X, and let c be a real number. Then*

$$\int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu$$

and

$$\int_X cf \, d\mu = c \int_X f \, d\mu.$$

Proof The integrable simple simple functions f and g can both be represented as linear combinations of characteristic functions of measurable sets. The results therefore follow immediately on applying the result of Corollary 8.15. ∎

Corollary 8.17 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be integrable simple functions on X. Suppose that $f(x) \leq g(x)$ for all $x \in X$. Then*

$$\int_X f \, d\mu \leq \int_X g \, d\mu.$$

Proof The function $g - f$ is a non-negative integrable simple function. The definition of the integral of such functions therefore ensures that $\int_X (g - f) \, d\mu \geq 0$. But it follows from Proposition 8.16 that

$$\int_X (g - f) \, d\mu = \int_X g \, d\mu - \int_X f \, d\mu.$$

The result follows. ∎

Definition Let $(X, \mathcal{A}, \mu)$ be a measure space, let E be a measurable subset of X, and let $f: X \to [0, +\infty)$ be a non-negative integrable simple function on X. The integral $\int_E f \, d\mu$ of s over E is defined by the formula

$$\int_E f \, d\mu = \int_X f \cdot \chi_E \, d\mu,$$

where $f \cdot \chi_E$ denotes the product of the function f and the characteristic function χ_E of the set E.

Proposition 8.18 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $s: X \to [0, +\infty)$ be a non-negative integrable simple function on X, and let $\nu(E) = \int_E s \, d\mu$ for all measurable sets E. Then ν is a measure defined on the σ-algebra $\mathcal{A}$ of measurable subsets of X.*

Proof The function s is a non-negative integrable simple function on X, and therefore there exist non-negative real numbers $c_1, c_2, \ldots, c_m$ and measurable sets $F_1, F_2, \ldots, F_m$ such that $s(x) = \sum_{j=1}^{m} c_j \chi_{F_j}(x)$ for all $x \in X$. Let E be a measurable set in X. Then $s(x)\chi_E(x) = \sum_{j=1}^{m} c_j \chi_{F_j \cap E}(x)$ for all $x \in X$, and therefore

$$\nu(E) = \int_E s\,d\mu = \int_X s \cdot \chi_E\,d\mu = \sum_{j=1}^{m} c_j \mu(F_j \cap E).$$

Let $\mathcal{E}$ be a countable collection of pairwise disjoint measurable sets. It follows from the countable additivity of the measure μ that

$$\begin{aligned} \nu\left(\bigcup_{E \in \mathcal{E}} E\right) &= \sum_{j=1}^{m} c_j \mu\left(\bigcup_{E \in \mathcal{E}} (F_j \cap E)\right) \\ &= \sum_{j=1}^{m} c_j \sum_{E \in \mathcal{E}} \mu(F_j \cap E) \\ &= \sum_{E \in \mathcal{E}} \nu(E). \end{aligned}$$

Thus the function ν is countably additive, and is therefore a measure defined on $\mathcal{A}$, as required. ∎

Corollary 8.19 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $s\colon X \to [0, +\infty)$ be a non-negative integrable simple function on X, and let $E_1, E_2, E_3, \ldots$ be an infinite sequence of measurable subsets of X, where $E_j \subset E_{j+1}$ for all positive integers j. Then*

$$\lim_{j \to +\infty} \int_{E_j} f\,d\mu = \int_E f\,d\mu,$$

where $E = \bigcup_{j=1}^{+\infty} E_j$.

Proof Let $\nu(F) = \int_F s\,d\mu$ for all measurable sets F. Then ν is a measure on X. It follows that

$$\nu\left(\bigcup_{j=1}^{+\infty} E_j\right) = \lim_{j \to +\infty} \nu(E_j)$$

(Lemma 7.25). The result follows. ∎

8.3 Integrals of Non-Negative Measurable Functions

We shall extend the definition of the integral to non-negative measurable functions that are not necessarily simple. In developing the properties of this integral, we shall need the result that a non-negative measurable function on a measure set is the limit of a non-decreasing sequence of integrable simple functions. We now proceed to prove this result.

Proposition 8.20 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f\colon X \to [0, +\infty]$ be a non-negative measurable function on X. Then there exists an infinite sequence $s_1, s_2, s_3, \ldots$ of non-negative integrable simple functions with the following properties:*

(i) $0 \leq s_j(x) \leq s_{j+1}(x)$ *for all* $j \in \mathbb{N}$ *and* $x \in X$*;*

(ii) $\lim\limits_{j \to +\infty} s_j(x) = f(x)$ *for all* $x \in X$*.*

Proof For each positive integer j let

$$F_j = \{x \in X : f(x) \geq 2^j\},$$

and for each integer k satisfying $1 \leq k \leq 4^j$, let

$$E_{j,k} = \left\{ x \in X : \frac{k-1}{2^j} \leq f(x) < \frac{k}{2^j} \right\}.$$

Then the sets F_j and $E_{j,k}$ are measurable sets. Let

$$s_j(x) = 2^j \chi_{F_j}(x) + \sum_{k=1}^{4^j} \frac{k-1}{2^j} \chi_{E_{j,k}}(x)$$

for all $j \in \mathbb{N}$ and $x \in X$. Then s_j is a integrable simple function on X which takes the value $2^{-j}(k-1)$ when $2^{-j}(k-1) \leq f(x) < 2^{-j}k$ for some integer k between 1 and 4^j, and takes the value 2^j when $f(x) \geq 2^j$. One can readily verify that $0 \leq s_j(x) \leq s_{j+1}(x)$ for all $j \in \mathbb{N}$ and $x \in X$. If $f(x) \leq 2^j$ then $0 \leq f(x) - s_j(x) < 2^{-j}$. It follows that if $f(x) < +\infty$ then $\lim\limits_{j \to +\infty} s_j(x) = f(x)$. If $f(x) = +\infty$ then $s_j(x) = 2^j$ for all positive integers j, and therefore $\lim\limits_{j \to +\infty} s_j(x) = f(x)$ in this case as well. The result is thus established. ∎

Definition Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f\colon X \to [0, +\infty]$ be a measurable function on X taking values in the set $[0, +\infty]$ of non-negative extended real numbers. The *integral* $\int_X f \, d\mu$ of f over X is defined to be the supremum of the integrals $\int_X s \, d\mu$ as s ranges over all non-negative integrable simple functions on X that satisfy $s(x) \leq f(x)$ for all $x \in X$.

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f: X \to [0, +\infty]$ be a measurable function on X taking values in the set $[0, +\infty]$ of non-negative extended real numbers. It follows from the above definition that $\int_X f \, d\mu = C$ for some non-negative extended real number C if and only if the following two conditions are satisfied:

(i) $\int_X s \, d\mu \leq C$ for all non-negative integrable simple functions s on X that satisfy $s(x) \leq f(x)$ for all $x \in X$.

(ii) given any non-negative real number c satisfying $c < C$, there exists some non-negative integrable simple function s on X such that $s(x) \leq f(x)$ for all $x \in X$ and $\int_X s \, d\mu > c$.

It follows directly from Corollary 8.17 that the definition of the integral for non-negative measurable functions is consistent with that previously given for integrable simple functions.

Lemma 8.21 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f: X \to [0, +\infty]$ and $g: X \to [0, +\infty]$ be measurable functions on X with values in the set $[0, +\infty]$ of non-negative extended real numbers. Suppose that $f(x) \leq g(x)$ for all $x \in X$. Then*

$$\int_X f \, d\mu \leq \int_X g \, d\mu$$

Proof This follows immediately from the definition of the integral, since any non-negative integrable simple function s on X satisfying $s(x) \leq f(x)$ for all $x \in X$ will also satisfy $s(x) \leq g(x)$ for all $x \in X$. ∎

8.4 Levi's Monotone Convergence Theorem

We now prove an important theorem which states that the integral of the limit of a non-decreasing sequence of non-negative measurable functions is equal to the limit of the integrals of those functions. A number of other important results follow as consequences of this basic theorem.

Theorem 8.22 (Levi's Monotone Convergence Theorem) *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable functions on X with values in the set $[0, +\infty]$ of non-negative extended real numbers, and let $f: X \to [0, +\infty]$ be defined such that $f(x) = \lim\limits_{j \to +\infty} f_j(x)$ for all $x \in X$. Suppose that $0 \leq f_j(x) \leq f_{j+1}(x)$ for all $j \in \mathbb{N}$ and $x \in X$. Then*

$$\lim_{j \to +\infty} \int_X f_j \, d\mu = \int_X f \, d\mu$$

Proof It follows from Corollary 8.12 that the limit function f is measurable. Moreover $\int_X f_j \, d\mu \leq \int_X f \, d\mu$, and therefore $\lim\limits_{j \to +\infty} \int_X f_j \, d\mu \leq \int_X f \, d\mu$.

Let s be a non-negative integrable simple function on X which satisfies $s(x) \leq f(x)$ for all $x \in X$, and let c be a real number satisfying $0 < c < 1$. If $f(x) > 0$ then $f(x) > cs(x)$ and therefore there exists some positive integer j such that $f_j(x) \geq cs(x)$. If $f(x) = 0$ then $s(x) = 0$, and therefore $f_j(x) \geq cs(x)$ for all positive integers j. It follows that $\bigcup\limits_{j=1}^{+\infty} E_j = X$, where

$$E_j = \{x \in X : f_j(x) \geq cs(x)\}.$$

Now

$$c \int_{E_j} s \, d\mu \leq \int_{E_j} f_j \, d\mu \leq \int_X f_j \, d\mu \leq \lim_{j \to +\infty} \int_X f_j \, d\mu.$$

Also $E_j \subset E_{j+1}$ for all positive integers j. It therefore follows from Corollary 8.19 that

$$c \int_X s \, d\mu = \lim_{j \to +\infty} c \int_{E_j} s \, d\mu \leq \lim_{j \to +\infty} \int_X f_j \, d\mu.$$

Moreover this inequality holds for all real numbers c satisfying $0 < c < 1$, and therefore

$$\int_X s \, d\mu \leq \lim_{j \to +\infty} \int_X f_j \, d\mu.$$

This inequality holds for all non-negative integrable simple functions s satisfying $s(x) \leq f(x)$ for all $x \in X$. It now follows from the definition of the integral of a measurable function that $\int_X f \, d\mu \leq \lim\limits_{j \to +\infty} \int_X f_j \, d\mu$, and therefore $\int_X f \, d\mu = \lim\limits_{j \to +\infty} \int_X f_j \, d\mu$, as required. ∎

Proposition 8.23 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f: X \to [0, +\infty]$ and $g: X \to [0, +\infty]$ be non-negative measurable functions on X. Then*

$$\int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu.$$

Proof It follows from Proposition 8.20 that there exist infinite sequences $s_1, s_2, s_3, \ldots$ and $t_1, t_2, t_3, \ldots$ of non-negative integrable simple functions such that $0 \leq s_j(x) \leq s_{j+1}(x)$ and $0 \leq t_j(x) \leq t_{j+1}(x)$ for all $j \in \mathbb{N}$ and $x \in X$, $\lim\limits_{j \to +\infty} s_j(x) = f(x)$ and $\lim\limits_{j \to +\infty} t_j(x) = g(x)$. Then

$$\lim_{j \to +\infty} (s_j(x) + t_j(x)) = f(x) + g(x).$$

It therefore follows from Proposition 8.16 and Levi's Monotone Convergence Theorem (Theorem 8.22) that

$$\begin{aligned}\int_X (f+g)\,d\mu &= \lim_{j\to+\infty} \int_X (s_j+t_j)\,d\mu \\ &= \lim_{j\to+\infty} \left(\int_X s_j\,d\mu + \int_X t_j\,d\mu\right) \\ &= \lim_{j\to+\infty} \int_X s_j\,d\mu + \lim_{j\to+\infty} \int_X t_j\,d\mu \\ &= \int_X f\,d\mu + \int_X g\,d\mu,\end{aligned}$$

as required. ∎

Proposition 8.24 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f_1, f_2, f_3, \ldots$ be an infinite sequence of non-negative measurable functions on X. Then*

$$\int_X \left(\sum_{j=1}^{+\infty} f_j\right)\,d\mu = \sum_{j=1}^{+\infty} \int_X f_j\,d\mu.$$

Proof It follows from Proposition 8.23 that

$$\int_X \left(\sum_{j=1}^{N} f_j\right)\,d\mu = \sum_{j=1}^{N} \int_X f_j\,d\mu$$

for all positive integers N. It then follows from Levi's Monotone Convergence Theorem (Theorem 8.22) that

$$\int_X \left(\sum_{j=1}^{+\infty} f_j\right)\,d\mu = \lim_{N\to+\infty} \int_X \left(\sum_{j=1}^{N} f_j\right)\,d\mu = \sum_{j=1}^{+\infty} \int_X f_j\,d\mu,$$

as required. ∎

8.5 Fatou's Lemma

Lemma 8.25 (Fatou's Lemma) *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of non-negative measurable functions on X. Then*

$$\int_X \left(\liminf_{j\to+\infty} f_j(x)\right)\,d\mu \le \liminf_{j\to+\infty} \int_X f_j\,d\mu.$$

Proof Let $g_j(x) = \inf\{f_k(x) : k \geq j\}$ for all $j \in \mathbb{N}$ and $x \in X$. Then the functions $g_1, g_2, g_3, \ldots$ are measurable (Proposition 8.10), and $\lim\limits_{j\to+\infty} g_j(x) = f_*(x)$ for all $x \in X$, where $f_*(x) = \liminf\limits_{j\to+\infty} f_j(x)$ for all $x \in X$. Also $0 \leq g_j(x) \leq g_{j+1}(x)$ for all $j \in \mathbb{N}$ and $x \in X$. It follows from Levi's Monotone Convergence Theorem (Theorem 8.22) that

$$\int_X f_*\,d\mu = \lim_{j\to+\infty} \int_X g_j\,d\mu.$$

Now $g_j(x) \leq f_k(x)$ for all $x \in X$ and for all positive integers j and k satisfying $j \leq k$. It follows that

$$\int_X g_j\,d\mu \leq \int_X f_k\,d\mu \quad \text{whenever } j \leq k,$$

and therefore

$$\int_X g_j\,d\mu \leq \inf\left\{\int_X f_k\,d\mu : k \geq j\right\}.$$

It follows that

$$\begin{aligned}\int_X f_*\,d\mu = \lim_{j\to+\infty} \int_X g_j\,d\mu &\leq \lim_{j\to+\infty} \inf\left\{\int_X f_k\,d\mu : k \geq j\right\} \\ &= \liminf_{j\to+\infty} \int_X f_j\,d\mu,\end{aligned}$$

as required. ∎

8.6 Integration of Functions with Positive and Negative Values

Definition Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f\colon X \to [-\infty, +\infty]$ be a measurable function on X. The function f is said to be *integrable* if $\int_X |f|\,dx < +\infty$.

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f\colon X \to [-\infty, +\infty]$ be a measurable function on X. Then f gives rise to non-negative measurable functions f_+ and f_- on X, where $f_+(x) = \max(f(x), 0)$ and $f_-(x) = \max(-f(x), 0)$ for all $x \in X$. Moreover $f(x) = f_+(x) - f_-(x)$ and $|f(x)| = f_+(x) + f_-(x)$ for all $x \in X$. Now $\int_X f_+\,d\mu \leq \int_X |f|\,d\mu$, $\int_X f_-\,d\mu \leq \int_X |f|\,d\mu$ and $\int_X |f|\,d\mu = \int_X f_+\,d\mu + \int_X f_- d\mu$. It follows that $\int_X |f|\,d\mu < +\infty$ if and only if $\int_X f_+\,d\mu < +\infty$ and $\int_X f_-\,d\mu < +\infty$.

Definition Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f\colon X \to [-\infty, +\infty]$ be an integrable function on X. The *integral* $\int_X f \, d\mu$ of f on X is defined by the identity

$$\int_X f \, d\mu = \int_X f_+ \, d\mu - \int_X f_- \, d\mu,$$

where $f_+(x) = \max(f(x), 0)$ and $f_-(x) = \max(-f(x), 0)$ for all $x \in X$.

Lemma 8.26 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f\colon X \to [-\infty, +\infty]$ be an integrable function on X, and let $u\colon X \to [0, +\infty]$ and $v\colon X \to [0, +\infty]$ be non-negative integrable functions on X such that $f(x) = u(x) - v(x)$ for all $x \in X$. Then*

$$\int_X f \, d\mu = \int_X u \, d\mu - \int_X v \, d\mu.$$

Proof Let $f_+(x) = \max(f(x), 0)$ and $f_-(x) = \max(-f(x), 0)$ for all $x \in X$. Then $f(x) = f_+(x) - f_-(x) = u(x) - v(x)$ for all $x \in X$, and therefore $f_+(x) + v(x) = f_-(x) + u(x)$ for all $x \in X$. It follows from Proposition 8.23 that

$$\int_X f_+ \, d\mu + \int_X v \, d\mu = \int_X f_- \, d\mu + \int_X u \, d\mu.$$

But then

$$\int_X f \, d\mu = \int_X f_+ \, d\mu - \int_X f_- \, d\mu = \int_X u \, d\mu - \int_X v \, d\mu,$$

as required. ∎

Lemma 8.27 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f\colon X \to [-\infty, +\infty]$ and $g\colon X \to [-\infty, +\infty]$ be integrable functions on X. Then*

$$\int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu,$$

and

$$\int_X cf \, d\mu = c \int_X f \, d\mu$$

for all real numbers c.

Proof Let

$$f_+(x) = \max(f(x), 0), \quad f_-(x) = \max(-f(x), 0),$$

$$g_+(x) = \max(f(x), 0), \quad g_-(x) = \max(-f(x), 0),$$

$$u(x) = f_+(x) + g_+(x) \quad \text{and} \quad v(x) = f_-(x) + g_-(x)$$

for all $x \in X$. Then the functions u and v are integrable, and $f(x) + g(x) = u(x) - v(x)$ for all $x \in X$. It follows from Lemma 8.26 that

$$\begin{aligned}
\int_X (f+g)\,d\mu &= \int_X u\,d\mu - \int_X v\,d\mu \\
&= \int_X f_+\,d\mu + \int_X g_+\,d\mu - \int_X f_-\,d\mu - \int_X g_-\,d\mu \\
&= \int_X f\,d\mu + \int_X g\,d\mu.
\end{aligned}$$

The identity $\int_X cf\,d\mu = c\int_X f\,d\mu$ follows directly from the definition of the integral, on considering separately the cases when $c > 0$, $c = 0$ and $c < 0$. ∎

8.7 Lebesgue's Dominated Convergence Theorem

Theorem 8.28 (Lebesgue's Dominated Convergence Theorem) *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X, and let f be a measurable real-valued function on X, where $f(x) = \lim_{j \to +\infty} f_j(x)$ for all $x \in X$. Suppose that there exists a non-negative integrable function $g\colon X \to [0, +\infty]$ such that $|f_j(x)| \leq g(x)$ for all $j \in \mathbb{N}$ and $x \in X$. Then the function f is integrable, and*

$$\lim_{j \to +\infty} \int_X f_j\,d\mu = \int_X f\,d\mu.$$

Proof The conditions of the theorem ensure that $g(x) + f_j(x) \geq 0$ and $g(x) - f_j(x) \geq 0$ for all $x \in X$. Also

$$\limsup_{j \to +\infty} f_j(x) = \liminf_{j \to +\infty} f_j(x) = f(x)$$

for all $x \in X$ (see Proposition 5.2). It therefore follows from Fatou's Lemma (Lemma 8.25) that

$$\begin{aligned}
\int_X g(x)\,d\mu + \int_X f(x)\,d\mu &= \int_X (g(x) + f(x))\,d\mu \\
&= \int_X \left(\liminf_{j \to +\infty} (g(x) + f_j(x)) \right) d\mu \\
&\leq \liminf_{j \to +\infty} \int_X (g(x) + f_j(x))\,d\mu \\
&= \int_X g(x)\,d\mu + \liminf_{j \to +\infty} \int_X f_j(x)\,d\mu
\end{aligned}$$

and

$$\begin{aligned}
\int_X g(x)\,d\mu - \int_X f(x)\,d\mu &= \int_X (g(x) - f(x))\,d\mu \\
&= \int_X \left(\liminf_{j\to+\infty} (g(x) - f_j(x))\right)\,d\mu \\
&\leq \liminf_{j\to+\infty} \int_X (g(x) - f_j(x))\,d\mu \\
&= \int_X g(x)\,d\mu - \limsup_{j\to+\infty} \int_X f_j(x)\,d\mu,
\end{aligned}$$

and therefore

$$\int_X f(x)\,d\mu \;\leq\; \liminf_{j\to+\infty} \int_X f_j(x)\,d\mu \leq \limsup_{j\to+\infty} \int_X f_j(x)\,d\mu \leq \int_X f(x)\,d\mu.$$

Now the extreme left hand side and extreme right hand side of this chain of inequalities are of course identical. Therefore

$$\int_X f(x)\,d\mu = \liminf_{j\to+\infty} \int_X f_j(x)\,d\mu = \limsup_{j\to+\infty} \int_X f_j(x)\,d\mu.$$

It follows follows from this (on applying Proposition 5.2) that the sequence of integrals $\int_X f_j(x)\,d\mu$ for $j = 1, 2, 3, \ldots$ is convergent, and

$$\lim_{j\to+\infty} \int_X f_j(x)\,d\mu = \int_X f(x)\,d\mu,$$

as required. ∎

8.8 Basic Results concerning Integrable Functions

Lemma 8.29 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f\colon X \to [0, +\infty]$ be a non-negative integrable function on X, let c be a real number, where $c > 0$. Then*

$$\mu(\{x \in X : f(x) \geq c\}) \leq \frac{1}{c} \int_X f\,d\mu.$$

Proof Let $E_c = \{x \in X : f(x) \geq c\}$, let χ_{E_c} be the characteristic function of the set E_c, and let $s\colon X \to [0, +\infty)$ be the integrable simple function defined such that $s(x) = c\chi_{E_c}(x)$ for all $x \in X$. Then $f(x) \geq 0$ for all $x \in X$, $s(x) = 0$ whenever $f(x) < c$, and $s(x) = c$ whenever $f(x) \geq c$. It follows that $s(x) \leq f(x)$ for all $x \in X$. The definitions of the integrals $\int_X s \, d\mu$ and $\int_X f \, d\mu$ then ensure that

$$c \, \mu(\{x \in X : f(x) \geq c\}) = c \, \mu(E_c) = \int_X s \, d\mu \leq \int_X f \, d\mu.$$

The result follows. ∎

Proposition 8.30 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f \colon X \to [0, +\infty]$ be a non-negative integrable function on X. Suppose that $\int_X f \, d\mu = 0$. Then*

$$\mu(\{x \in X : f(x) \neq 0\}) = 0.$$

Proof The function f is non-negative. Thus if $x \in X$ and $f(x) \neq 0$ then $f(x) > 0$, and therefore there exists some positive integer j for which $f(x) > 1/j$. It follows that

$$\{x \in X : f(x) \neq 0\} = \bigcup_{j=1}^{+\infty} F_j,$$

where $F_j = \{x \in X : f(x) > 1/j\}$ for all positive integers j. Now $\int_X f \, d\mu = 0$ by assumption. It follows from Lemma 8.29 that $\mu(F_j) = 0$ for all positive integers j. Now $\mu\left(\bigcup_{j=1}^{+\infty} F_j\right) \leq \sum_{j=1}^{+\infty} \mu(F_j)$. It follows that

$$\mu(\{x \in X : f(x) \neq 0\}) = \mu\left(\bigcup_{j=1}^{+\infty} F_j\right) = 0,$$

as required. ∎

Corollary 8.31 *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f \colon X \to [-\infty, +\infty]$ and $g \colon X \to [-\infty, +\infty]$ be integrable functions on X. Suppose that $\int_X |f - g| \, d\mu = 0$. Then*

$$\mu(\{x \in X : f(x) \neq g(x)\}) = 0.$$

Proof The result follows immediately on applying Proposition 8.30 to the non-negative integrable function that sends x to $|f(x) - g(x)|$ for all $x \in X$. ∎

8.9 Properties that hold Almost Everywhere

Let $(X, \mathcal{A}, \mu)$ be a measure space, and, for each $x \in X$, let $P(x)$ be some property that may or may not hold at x. If the set of elements x of X for which the property holds has measure zero then we say that the property $P(x)$ holds *almost nowhere* on X. If the set of elements x of X for which the property does not hold has measure zero then we say that the property $P(x)$ holds *almost everywhere* on X.

The result of Corollary 8.31 may be stated as follows. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f\colon X \to [-\infty, +\infty]$ and $g\colon X \to [-\infty, +\infty]$ be integrable functions on X. Suppose that $\int_X |f - g| \, d\mu = 0$. Then the functions f and g are equal almost everywhere on X.

Lemma 8.32 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let Let f, g and h be integrable functions on X. Suppose that the functions f and g are equal almost everywhere on X and also that the functions g and h are equal almost everywhere on X. Then the functions f and h are equal almost everywhere on X.*

Proof Note that if $f(x) \neq h(x)$ then either $f(x) \neq g(x)$ or $g(x) \neq h(x)$. It follows that

$$\begin{aligned} &\{x \in X : f(x) \neq h(x)\} \\ &\qquad \subset \quad \{x \in X : f(x) \neq g(x)\} \cup \{x \in X : g(x) \neq h(x)\}, \end{aligned}$$

and therefore

$$\begin{aligned} &\mu(\{x \in X : f(x) \neq h(x)\}) \\ &\qquad \leq \quad \mu(\{x \in X : f(x) \neq g(x)\}) + \mu(\{x \in X : g(x) \neq h(x)\}). \end{aligned}$$

But

$$\mu(\{x \in X : f(x) \neq g(x)\}) = 0$$

and

$$\mu(\{x \in X : g(x) \neq h(x)\}) = 0.$$

It follows that $\mu(\{x \in X : f(x) \neq h(x)\} = 0$, and thus the functions f and h are equal almost everywhere, as required. ∎