Module MAU22200: Advanced Analysis (Semester 2) Hilary Term 2020 Part II (Section 7)

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7 Measure Spaces

7.1 Blocks

Definition We define a *block* in \mathbb{R}^n to be a subset of \mathbb{R}^n that is a Cartesian product of subsets of \mathbb{R} that are bounded intervals or singleton sets.

Let B be a block in \mathbb{R}^n . Then there exist bounded intervals or singleton sets I_1, I_2, \ldots, I_n in \mathbb{R} such that $B = I_1 \times I_2 \times \cdots \times I_n$. Let a_i and b_i denote the endpoints of the interval I_i or singleton set for $i = 1, 2, \ldots, n$, where $a_i \leq b_i$. Then the interval I_i must coincide with one of the intervals (a_i, b_i) , $(a_i, b_i], [a_i, b_i)$ and $[a_i, b_i]$ determined by its endpoints, where

$$(a_i, b_i) = \{ x \in \mathbb{R} : a_i < x < b_i \}, \quad (a_i, b_i] = \{ x \in \mathbb{R} : a_i < x \le b_i \}$$
$$[a_i, b_i) = \{ x \in \mathbb{R} : a_i \le x < b_i \}, \quad [a_i, b_i] = \{ x \in \mathbb{R} : a_i \le x \le b_i \}.$$

Definition Let *B* be a block in \mathbb{R}^n , and let $B = I_1 \times I_2 \times \cdots \times I_n$, where, for each integer *i* between 1 and *n*, either I_i is an interval of strictly positive length or else I_i is a singleton set consisting of a single real number. We define the *dimension* of the block *B* to be the number of values of *i* for which the subset I_i of \mathbb{R} is an interval of positive length.

Thus a k-dimensional block B in \mathbb{R}^n is the Cartesian product of k bounded intervals of strictly positive length and n - k singleton sets.

The following two results, characterizing open and closed blocks in *n*dimensional Euclidean space \mathbb{R}^n follow directly on applying the definition of open and closed sets in \mathbb{R}^n .

Lemma 7.1 A block in \mathbb{R}^n is open in \mathbb{R}^n if it is the Cartesian product of n bounded open intervals.

Lemma 7.2 A block in \mathbb{R}^n is closed in \mathbb{R}^n if it is the Cartesian product of bounded closed intervals and singleton sets.

Note that a closed one-dimensional block in \mathbb{R} is a closed bounded interval, and a closed one-dimensional block in \mathbb{R}^n is a closed bounded line segment parallel to one of the coordinate axes. A closed two-dimensional block in \mathbb{R}^n is a closed rectangle with each side parallel to some coordinate axis.

Definition Let B be a block in \mathbb{R}^n , and let

$$B = I_1 \times I_2 \times \dots \times I_n$$

where I_i is a bounded interval or singleton set in \mathbb{R} for i = 1, 2, ..., n. The (*n*-dimensional) content m(B) is defined so that

$$m(B) = \prod_{i=1}^{n} (b_i - a_i),$$

where a_i and b_i are the lower and upper endpoints respectively of the interval or singleton set I_i for i = 1, 2, ..., n. (Thus, for each integer *i* between 1 and $n, a_i = \inf I_i, b_i = \sup I_i, b_i > a_i$ in cases where I_i is an interval of positive length, and $b_i = a_i$ in cases where I_i is a singleton set.)

Proposition 7.3 Let B be a block in n-dimensional Euclidean space \mathbb{R}^n , and let B_1, B_2, \ldots, B_s be a finite collection of blocks in \mathbb{R}^n . Suppose that the blocks B_1, B_2, \ldots, B_s are pairwise disjoint and $B = \bigcup_{k=1}^s B_k$. Then $m(B) = \sum_{k=1}^s m(B_k)$.

Proof The statement of this proposition is an immediate consequence, or particular case, of Corollary 6.24.

Proposition 7.4 Let B_1, B_2, \ldots, B_s be a finite list whose members are blocks in \mathbb{R}^n . Then there exists a finite list D_1, D_2, \ldots, D_q of blocks in \mathbb{R}^n such that the blocks D_1, D_2, \ldots, D_q are pairwise disjoint and such that, for k = $1, 2, \ldots, s$, the block B_k is the union of those blocks in the list D_1, D_2, \ldots, D_q that are contained in B_k . Moreover the content $m(B_k)$ is equal to the sum of the contents $m(D_j)$ of those blocks D_j in the list D_1, D_2, \ldots, D_q for which $D_j \subset B_k$.

Proof The collection of subsets of \mathbb{R} consisting of the empty set, the singleton sets that are of the form $\{c\}$ for some real number c, and the bounded intervals is a semiring of subsets of \mathbb{R} . (Lemma 6.5). Applying Proposition 6.14 we deduce that those subsets of \mathbb{R}^n that are blocks constitute a semiring of subsets of \mathbb{R}^n . Indeed the definition of blocks ensures that each block in \mathbb{R}^n is a Cartesian product of subsets of \mathbb{R} that are singleton sets or bounded intervals. The result concerning the existence of the finite list D_1, D_2, \ldots, D_q of blocks therefore follows from Proposition 6.8.

Proposition 7.5 Let B be a block in n-dimensional Euclidean space \mathbb{R}^n , and let B_1, B_2, \ldots, B_s be a finite collection of blocks in \mathbb{R}^n . Suppose that $B \subset \bigcup_{k=1}^s B_k$. Then $m(B) \leq \sum_{k=1}^s m(B_k)$. **Proof** The collection of subsets of \mathbb{R} consisting of the empty set, the singleton sets that are of the form $\{c\}$ for some real number c, and the bounded intervals is a semiring of subsets of \mathbb{R} . The required result therefore follows immediately on applying Proposition 6.19.

Proposition 7.6 Let B be a block in n-dimensional Euclidean space \mathbb{R}^n , and let B_1, B_2, \ldots, B_s be a finite collection of blocks in \mathbb{R}^n . Suppose that the blocks B_1, B_2, \ldots, B_s are pairwise disjoint and are contained in B. Then $\sum_{k=1}^{s} m(B_k) \leq m(B)$.

Proof The collection of subsets of \mathbb{R} consisting of the empty set, the singleton sets that are of the form $\{c\}$ for some real number c, and the bounded intervals is a semiring of subsets of \mathbb{R} . The required result therefore follows immediately on applying Proposition 6.20.

Lemma 7.7 Let B be an block in \mathbb{R}^n , and let ε be any positive real number. Then there exist a closed block F and and open block V such that $F \subset B \subset V$, $m(F) > m(B) - \varepsilon$ and $m(V) < m(B) + \varepsilon$.

Proof Suppose that $B = I_1 \times I_2 \times \cdots \times I_n$, where I_1, I_2, \ldots, I_n are bounded intervals. Now

$$\lim_{h \to 0} \prod_{i=1}^{n} (m(I_i) + h) = \prod_{i=1}^{n} m(I_i) = m(B).$$

It follows that, given any positive real number ε , we can choose the positive real number δ small enough to ensure that

$$\prod_{i=1}^{n} (m(I_i) - \delta) > m(B) - \varepsilon, \quad \prod_{i=1}^{n} (m(I_i) + \delta) < m(B) + \varepsilon.$$

Let $F = J_1 \times J_2 \times \cdots \times J_n$ and $V = K_1 \times K_2 \times \cdots \times K_n$, where J_1, J_2, \ldots, J_n are closed bounded intervals chosen such that $J_i \subset I_i$ and $m(J_i) > m(I_i) - \delta$ for $i = 1, 2, \ldots, n$, and K_1, K_2, \ldots, K_n are open bounded intervals chosen such that $I_i \subset K_i$ and $m(K_i) < m(I_i) + \delta$ for $i = 1, 2, \ldots, n$. Then F is a closed block, V is an open block, $F \subset B \subset V$, $m(F) > m(B) - \varepsilon$ and $m(V) < m(B) + \varepsilon$, as required.

Any closed *n*-dimensional block F is a compact subset of \mathbb{R}^n . This means that, given any collection \mathcal{V} of open sets in \mathbb{R}^n that covers F (so that each point of F belongs to at least one of the open sets in the collection), there exists some finite collection V_1, V_2, \ldots, V_s of open sets belonging to the collection \mathcal{V} such that

$$F \subset V_1 \cup V_2 \cup \cdots \cup V_s.$$

We shall use this property of closed blocks in order to generalize Proposition 7.5 to countable infinite unions of blocks.

Proposition 7.8 Let A be a block in n-dimensional Euclidean space \mathbb{R}^n , and let C be a countable collection of blocks in \mathbb{R}^n . Suppose that $A \subset \bigcup_{B \in \mathcal{C}} B$. Then $m(A) \leq \sum_{B \in \mathcal{C}} m(B)$.

Proof There is nothing to prove if $\sum_{B \in \mathcal{C}} m(B) = +\infty$. We may therefore restrict our attention to the case where $\sum_{B \in \mathcal{C}} m(B) < +\infty$. Moreover the result is an immediate consequence of Proposition 7.5 if the collection \mathcal{C} is finite. It therefore only remains to prove the result in the case where the collection \mathcal{C} is infinite, but countable. In that case there exists an infinite sequence B_1, B_2, B_3, \ldots of blocks with the property that each block in the collection \mathcal{C} occurs exactly once in the sequence. Let some positive real number ε be given. It follows from Lemma 7.7 that there exists a closed block F such that $F \subset A$ and $m(F) \ge m(A) - \varepsilon$. Also, for each $k \in \mathbb{N}$, there exists an open block V_k such that $B_k \subset V_k$ and $m(V_k) < m(B_k) + 2^{-k}\varepsilon$. Then $F \subset \bigcup_{k=1}^{+\infty} V_k$, and thus $\{V_1, V_2, V_3, \ldots\}$ is a collection of open sets in \mathbb{R}^n which covers the closed bounded set F. It follows from the compactness of F that there exists a finite collection k_1, k_2, \ldots, k_s of positive integers such that $F \subset V_{k_1} \cup V_{k_2} \cup \cdots \cup V_{k_s}$. It then follows from Proposition 7.5 that

$$m(F) \le m(V_{k_1}) + m(V_{k_2}) + \dots + m(V_{k_s}).$$

Now

$$\frac{1}{2^{k_1}} + \frac{1}{2^{k_2}} + \dots + \frac{1}{2^{k_s}} \le \sum_{k=1}^{+\infty} \frac{1}{2^k} = 1,$$

and therefore

$$m(F) \leq m(V_{k_1}) + m(V_{k_2}) + \dots + m(V_{k_s})$$

$$\leq m(B_{k_1}) + m(B_{k_2}) + \dots + m(B_{k_s}) + \varepsilon$$

$$\leq \sum_{k=1}^{+\infty} m(B_k) + \varepsilon.$$

Also $m(A) < m(F) + \varepsilon$. It follows that

$$m(A) \le \sum_{k=1}^{+\infty} m(B_k) + 2\varepsilon.$$

Moreover this inequality holds no matter how small the value of the positive real number ε . It follows that

$$m(A) \le \sum_{k=1}^{+\infty} m(B_k),$$

as required.

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7.2Lebesgue Outer Measure

We say that a collection \mathcal{C} of *n*-dimensional blocks *covers* a subset E of \mathbb{R}^n if $E \subset \bigcup_{B \in \mathcal{C}} B$, (where $\bigcup_{B \in \mathcal{C}} B$ denotes the union of all the blocks belonging to the collection \mathcal{C}). Given any subset E of \mathbb{R}^n , we shall denote by $\mathbf{CCB}_n(E)$ the set of all countable collections of n-dimensional blocks that cover the set E.

Definition Let E be a subset of \mathbb{R}^n . We define the Lebesgue outer measure $\mu^*(E)$ of E to be the infimum, or greatest lower bound, of the quantities $\sum_{B \in \mathcal{C}} m(B)$, where this infimum is taken over all countable collections \mathcal{C} of n-dimensional blocks that cover the set E. Thus

$$\mu^*(E) = \inf \left\{ \sum m(B) : \mathcal{C} \in \mathbf{CCB}_n(E) \right\}.$$

The Lebesgue outer measure
$$\mu^*(E)$$
 of a subset E of \mathbb{R}^n is thus the greatest extended real number l with the property that $l \leq \sum_{B \in \mathcal{C}} m(B)$ for any countable collection \mathcal{C} of *n*-dimensional blocks that covers the set E . In particular, $\mu^*(E) = +\infty$ if and only if $\sum_{B \in \mathcal{C}} m(B) = +\infty$ for every countable

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collection \mathcal{C} of *n*-dimensional blocks that covers the set E.

Note that $\mu^*(E) \ge 0$ for all subsets E of \mathbb{R}^n .

Lemma 7.9 Let E be a block in \mathbb{R}^n . Then $\mu^*(E) = m(E)$, where m(E) is the content of the block E.

Proof It follows from Proposition 7.8 that $m(E) \leq \sum_{B \in \mathcal{C}} m(B)$ for any countable collection of *n*-dimensional blocks that covers the block *E*. Therefore $m(E) \leq \mu^*(E)$. But the collection $\{E\}$ consisting of the single block *E* is itself a countable collection of blocks covering *E*, and therefore $\mu^*(E) \leq m(E)$. It follows that $\mu^*(E) = m(E)$, as required.

Lemma 7.10 Let E and F be subsets of \mathbb{R}^n . Suppose that $E \subset F$. Then $\mu^*(E) \leq \mu^*(F)$.

Proof Any countable collection of *n*-dimensional blocks that covers the set F will also cover the set E, and therefore $\mathbf{CCB}_n(F) \subset \mathbf{CCB}_n(E)$. It follows that

$$\mu^{*}(F) = \inf \left\{ \sum_{B \in \mathcal{C}} m(B) : \mathcal{C} \in \mathbf{CCB}_{n}(F) \right\}$$

$$\geq \inf \left\{ \sum_{B \in \mathcal{C}} m(B) : \mathcal{C} \in \mathbf{CCB}_{n}(E) \right\} = \mu^{*}(E),$$

as required.

Proposition 7.11 Let \mathcal{E} be a countable collection of subsets of \mathbb{R}^n . Then

$$\mu^*\left(\bigcup_{E\in\mathcal{E}}E\right)\leq\sum_{E\in\mathcal{E}}\mu^*(E).$$

Proof Let $K = \mathbb{N}$ in the case where the countable collection \mathcal{E} is infinite, and let $K = \{1, 2, ..., m\}$ in the case where the collection \mathcal{E} is finite and has m elements. Then there exists a bijective function $\varphi: K \to \mathcal{E}$. We define $E_k = \varphi(k)$ for all $k \in K$. Then $\mathcal{E} = \{E_k : k \in K\}$, and any subset of \mathbb{R}^n belonging to the collection \mathcal{E} is of the form E_k for exactly one element k of the indexing set K.

Let some positive real number ε be given. Then corresponding to each element k of K there exists a countable collection \mathcal{C}_k of n-dimensional blocks covering the set E_k for which

$$\sum_{B \in \mathcal{C}_k} m(B) < \mu^*(E_k) + \frac{\varepsilon}{2^k}.$$

Let $\mathcal{C} = \bigcup_{k \in K} \mathcal{C}_k$. Then \mathcal{C} is a collection of *n*-dimensional blocks that covers the union $\bigcup_{E \in \mathcal{E}} E$ of all the sets in the collection \mathcal{E} . Moreover every block belonging to the collection \mathcal{C} belongs to at least one of the collections \mathcal{C}_k , and therefore belongs to exactly one of the collections \mathcal{D}_k , where $\mathcal{D}_k = \mathcal{C}_k \setminus \bigcup_{i < k} \mathcal{C}_j$. It follows that

$$\mu^* \left(\bigcup_{E \in \mathcal{E}} E \right) \leq \sum_{B \in \mathcal{C}} m(B) = \sum_{k \in K} \sum_{B \in \mathcal{D}_k} m(B)$$

$$\leq \sum_{k \in K} \sum_{B \in \mathcal{C}_k} m(B) \leq \sum_{k \in K} \left(\mu^*(E_k) + \frac{\varepsilon}{2^k} \right)$$

$$\leq \sum_{k \in K} \mu^*(E_k) + \varepsilon$$

Thus $\mu^* \left(\bigcup_{E \in \mathcal{E}} E\right) \leq \sum_{k \in K} \mu^*(E_k) + \varepsilon$, no matter how small the value of ε . It follows that $\mu^* \left(\bigcup_{E \in \mathcal{E}} E\right) \leq \sum_{k \in K} \mu^*(E_k)$, as required.

Proposition 7.12 Let B be a block in \mathbb{R}^n . Then

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \setminus B)$$

for all subsets A of \mathbb{R}^n .

Proof First we deal with the case when $\mu^*(A) = +\infty$, and this case either $\mu^*(A \cap B) = +\infty$ or else $\mu^*(A \setminus B) = +\infty$ because otherwise the subadditivity of Lebesgue outer measure (Proposition 7.11) would ensure that $\mu^*(A)$, being non-negative and less than the sum of two finite quantities, would itself be a finite quantity. The stated result is thus valid in cases where $\mu^*(A) = +\infty$.

Now suppose that $\mu^*(A) < +\infty$. Let some positive real number ε be given. It then follows from the definition of Lebesgue outer measure that there exists a collection $(C_i : i \in I)$ of blocks indexed by a countable set I for which

$$\sum_{i \in I} m(C_i) < \mu^*(A) + \varepsilon.$$

Then, for each $i \in I$, Proposition 7.4 guarantees the existence of a finite list $D_{i,1}, D_{i,2}, \ldots D_{i,q(i)}$ of blocks satisfying the following conditions:

- the blocks $D_{i,1}, D_{i,2}, \ldots D_{i,q(i)}$ are pairwise disjoint;
- C_i is the union of all the blocks $D_{i,k}$ for which $1 \le k \le q(i)$;
- $C_i \cap B$ is the union of those blocks $D_{i,k}$ with $1 \leq k \leq q(i)$ for which $D_{i,k} \subset C_i \cap B$.

For each $i \in I$, let L(i) denote the set of integers between 1 and q(i) for which $D_{i,k} \not\subset C_i \cap B$. and let I_0 denote the subset of I consisting of those $i \in I$ for which L(i) is non-empty. Then

$$C_i \setminus B \subset \bigcup_{k \in L(i)} D_{i,k}$$

for all $i \in I_0$, and

$$A \setminus B \subset \bigcup_{i \in I_0} (C_i \setminus B),$$

and therefore

$$A \setminus B \subset \bigcup_{i \in I_0} \bigcup_{k \in L(i)} D_{i,k}$$

It then follows from the definition of Lebesgue outer measure that

$$\mu^*(A \setminus B) \le \sum_{i \in I_0} \sum_{k \in L(i)} m(D_{i,k}),$$

where $m(D_{i,k})$ denotes the content of the block $D_{i,k}$ for all $i \in I$ and for all integers k in the range $1 \leq k \leq q(i)$. But, for each $i \in I_0$, the content $m(C_i)$ of the block C_i is equal to the sum of the contents $m(D_{i,k})$ of the blocks $D_{i,k}$ for all integer values of k satisfying $1 \leq k \leq q(i)$ (see Proposition 7.3), whilst the content $m(C_i \cap B)$ of the block $C_i \cap B$ is equal to the sum of the contents $m(D_{i,k})$ of those blocks $D_{i,k}$ with $1 \leq k \leq q(i)$ for which $D_{i,k} \subset C_i \cap B$. It follows that, for all $i \in I_0$,

$$\sum_{k \in L(i)} m(D_{i,k}) = m(C_i) - m(C_i \cap B).$$

Also $m(C_i) = m(C_i \cap B)$ for all $i \in I \setminus I_0$. It follows that

$$\mu^*(A \setminus B) \leq \sum_{i \in I_0} \sum_{k \in L(i)} m(D_{i,k})$$

=
$$\sum_{i \in I_0} (m(C_i) - m(C_i \cap B))$$

=
$$\sum_{i \in I} (m(C_i) - m(C_i \cap B)).$$

The definition of definition of Lebesgue outer measure also ensures that

$$\mu^*(A \cap B) \le \sum_{i \in I} m(C_i \cap B).$$

Adding these two inequalities, we find that

$$\mu^*(A \cap B) + \mu^*(A \setminus B) \le \sum_{i \in I} \mu(C_i) < \mu^*(A) + \varepsilon.$$

We have now shown that

$$\mu^*(A \cap B) + \mu^*(A \setminus B) < \mu^*(A) + \varepsilon$$

for all strictly positive numbers ε . It follows that

$$\mu^*(A \cap B) + \mu^*(A \setminus B) \le \mu^*(A).$$

The reverse inequality

$$\mu^*(A) \le \mu^*(A \cap B) + \mu^*(A \setminus B),$$

is a consequence of Proposition 7.11. It follows that

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \setminus B),$$

as required.

7.3 Outer Measures

Definition Let X be a set, and let $\mathcal{P}(X)$ be the collection of all subsets of X. An outer measure $\lambda: \mathcal{P}(X) \to [0, +\infty]$ on X is a function, mapping subsets of X to non-negative extended real numbers, which has the following properties:

- (i) $\lambda(\emptyset) = 0;$
- (ii) $\lambda(E) \leq \lambda(F)$ for all subsets E and F of X that satisfy $E \subset F$;
- (iii) $\lambda \left(\bigcup_{E \in \mathcal{E}} E\right) \leq \sum_{E \in \mathcal{E}} \lambda(E)$ for any countable collection \mathcal{E} of subsets of X.

Lebesgue outer measure is an outer measure on the set \mathbb{R}^n . (This follows directly from the definition of Lebesgue outer measure, and from Lemma 7.10 and Proposition 7.11.)

We shall prove that any outer measure on a set X determines a collection of subsets of X with particular properties. The subsets belonging to this collection are known as *measurable sets*. Any countable union or intersection of measurable sets is itself a measurable set. Also any difference of measurable sets is itself a measurable set. We shall also prove that if C is any countable collection of pairwise disjoint measurable sets then $\lambda \left(\bigcup_{E \in \mathcal{E}} E\right) = \sum_{E \in \mathcal{E}} \lambda(E).$

These results are fundamental to the branch of mathematics known as *measure theory*. Moreover the existence of such collections of measurable sets underlies the powerful and very general theory of integration introduced into mathematics by Lebesgue.

Definition Let λ be an outer measure on a set X. A subset E of X is said to be λ -measurable if $\lambda(A) = \lambda(A \cap E) + \lambda(A \setminus E)$ for all subsets A of X.

The above definition of measurable sets may seem at first somewhat strange and unmotivated. Nevertheless it serves to characterize a collection of subsets of X with convenient properties, as we shall see.

Proposition 7.13 Let λ be an outer measure on a set X. Then the empty set \emptyset and the whole set X are λ -measurable. Moreover the complement $X \setminus E$ of E, and the union $E \cup F$, intersection $E \cap F$ and difference $E \setminus F$ of E and F are λ -measurable for all λ -measurable subsets E and F of X.

Proof It follows directly from the definition of λ -measurability that \emptyset and X are λ -measurable.

For each subset E of X, let us denote the complement $X \setminus E$ of E in X by E^c . Then $A \setminus E = A \cap E^c$ for all subsets A and E of X, and thus a subset E of X is λ -measurable if and only if

$$\lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c)$$

for all subsets A of X. Now $(E^c)^c = E$. It follows that if a subset E of X is λ -measurable, then so is E^c . Thus $X \setminus E$ is λ -measurable for all measurable subsets E of X.

Let E and F be λ -measurable subsets of X, and let A be an arbitrary subset of X. Then

$$\lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c).$$

Also

$$\lambda(A \cap E) = \lambda(A \cap E \cap F) + \lambda(A \cap E \cap F^c)$$

and

$$\lambda(A \cap E^c) = \lambda(A \cap E^c \cap F) + \lambda(A \cap E^c \cap F^c).$$

It follows that

$$\lambda(A) = \lambda(A \cap E \cap F) + \lambda(A \cap E \cap F^c) + \lambda(A \cap E^c \cap F) + \lambda(A \cap E^c \cap F^c).$$

Now, replacing A by $A \cap (E \cup F)$, we find that

$$\begin{split} \lambda(A \cap (E \cup F)) &= \lambda(A \cap (E \cup F) \cap E \cap F) \\ &+ \lambda(A \cap (E \cup F) \cap E \cap F^c) \\ &+ \lambda(A \cap (E \cup F) \cap E^c \cap F) \\ &+ \lambda(A \cap (E \cup F) \cap E^c \cap F^c). \end{split}$$

But

$$\begin{split} A \cap (E \cup F) \cap E \cap F &= A \cap E \cap F, \\ A \cap (E \cup F) \cap E \cap F^c &= A \cap E \cap F^c, \\ A \cap (E \cup F) \cap E^c \cap F &= A \cap E^c \cap F, \\ A \cap (E \cup F) \cap E^c \cap F^c &= \emptyset. \end{split}$$

It follows therefore that

$$\lambda(A \cap (E \cup F)) = \lambda(A \cap E \cap F) + \lambda(A \cap E \cap F^c) + \lambda(A \cap E \cap F^c) + \lambda(A \cap E^c \cap F).$$

Also $A \cap (E \cup F)^c = A \cap E^c \cap E^c$. It follows that

$$\lambda(A) = \lambda(A \cap (E \cup F)) + \lambda(A \cap (E \cup F)^c),$$

for all subsets A of X, and thus the subset $E \cup F$ of X is λ -measurable.

Also if E and F are λ -measurable subsets of X then so are E^c and F^c , and therefore $E^c \cup F^c$ is a λ -measurable subset of X. But $E^c \cup F^c = (E \cap F)^c$. It follows that $E \cap F$ is λ -measurable for all λ -measurable subsets E and Fof X. Moreover $E \setminus F = E \cap F^c$, and therefore $E \setminus F$ is λ -measurable for all λ -measurable subsets E and F of X. This completes the proof.

It follows from the above proposition that any finite union or intersection of measurable sets is measurable.

We say that the sets in some collection are *pairwise disjoint* if the intersection of any two distinct sets belonging to this collection is the empty set.

Lemma 7.14 Let λ be an outer measure on a set X, let A be a subset of X, and let E_1, E_2, \ldots, E_m be pairwise disjoint λ -measurable sets. Then

$$\lambda\left(A\cap\bigcup_{k=1}^{m}E_{k}\right)=\sum_{k=1}^{m}\lambda(A\cap E_{k}).$$

Proof There is nothing to prove if m = 1. Suppose that m > 1. It follows from the definition of measurable sets that

$$\lambda \left(A \cap \bigcup_{k=1}^{m} E_k \right) = \lambda \left(\left(A \cap \bigcup_{k=1}^{m} E_k \right) \setminus E_m \right) + \lambda \left(\left(A \cap \bigcup_{k=1}^{m} E_k \right) \cap E_m \right).$$

But $\left(A \cap \bigcup_{k=1}^{m} E_k\right) \setminus E_m = A \cap \bigcup_{k=1}^{m-1} E_k$ and $\left(A \cap \bigcup_{k=1}^{m} E_k\right) \cap E_m = A \cap E_m$, because the sets E_1, E_2, \ldots, E_m are pairwise disjoint. Therefore

$$\lambda\left(A\cap\bigcup_{k=1}^{m}E_{k}\right)=\lambda\left(A\cap\bigcup_{k=1}^{m-1}E_{k}\right)+\lambda(A\cap E_{m}).$$

The required result therefore follows by induction on m.

Proposition 7.15 Let λ be an outer measure on a set X. Then the union of any countable collection of λ -measurable subsets of X is λ -measurable.

Proof The union of any two λ -measurable sets is λ -measurable (Proposition 7.13). It follows from this that the union of any finite collection of λ -measurable sets is λ -measurable.

Now let E_1, E_2, E_3, \ldots be an infinite sequence of pairwise disjoint λ -measurable subsets of X. We shall prove that the union of these sets is λ -measurable. Let A be a subset of X. Now $\bigcup_{k=1}^{m} E_k$ is a λ -measurable set for each positive integer m, because any finite union of λ -measurable sets is λ -measurable, and therefore

$$\lambda(A) = \lambda \left(A \cap \bigcup_{k=1}^{m} E_k \right) + \lambda \left(A \setminus \bigcup_{k=1}^{m} E_k \right)$$

for all positive integers m. Moreover it follows from Lemma 7.14 that

$$\lambda\left(A\cap\bigcup_{k=1}^{m}E_{k}\right)=\sum_{k=1}^{m}\lambda(A\cap E_{k}).$$

Also

$$A \setminus \bigcup_{k=1}^{+\infty} E_k \subset A \setminus \bigcup_{k=1}^m E_k,$$

and therefore

$$\lambda\left(A\setminus\bigcup_{k=1}^{m}E_{k}\right)\geq\lambda\left(A\setminus\bigcup_{k=1}^{+\infty}E_{k}\right)$$

It follows that

$$\lambda(A) \ge \sum_{k=1}^{m} \lambda(A \cap E_k) + \lambda \left(A \setminus \bigcup_{k=1}^{+\infty} E_k \right),$$

and therefore

$$\lambda(A) \geq \lim_{m \to +\infty} \sum_{k=1}^{m} \lambda(A \cap E_k) + \lambda \left(A \setminus \bigcup_{k=1}^{+\infty} E_k \right)$$
$$= \sum_{k=1}^{+\infty} \lambda(A \cap E_k) + \lambda \left(A \setminus \bigcup_{k=1}^{+\infty} E_k \right).$$

However it follows from the definition of outer measures that

$$\lambda\left(A\cap\bigcup_{k=1}^{+\infty}E_k\right)=\lambda\left(\bigcup_{k=1}^{+\infty}(A\cap E_k)\right)\leq\sum_{k=1}^{+\infty}\lambda(A\cap E_k).$$

Therefore

$$\lambda(A) \ge \lambda \left(A \cap \bigcup_{k=1}^{+\infty} E_k \right) + \lambda \left(A \setminus \bigcup_{k=1}^{+\infty} E_k \right)$$

But the set A is the union of the sets $A \cap \bigcup_{k=1}^{+\infty} E_k$ and $A \setminus \bigcup_{k=1}^{+\infty} E_k$, and therefore

$$\lambda(A) \leq \lambda \left(A \cap \bigcup_{k=1}^{+\infty} E_k \right) + \lambda \left(A \setminus \bigcup_{k=1}^{+\infty} E_k \right).$$

We conclude therefore that

$$\lambda(A) = \lambda \left(A \cap \bigcup_{k=1}^{+\infty} E_k \right) + \lambda \left(A \setminus \bigcup_{k=1}^{+\infty} E_k \right)$$

for all subsets A of X. We conclude from this that the union of any pairwise disjoint sequence of λ -measurable subsets of X. is itself λ -measurable.

Now let E_1, E_2, E_3, \ldots be a countable sequence of (not necessarily pairwise disjoint) λ -measurable sets. Then $\bigcup_{k=1}^{+\infty} E_k = \bigcup_{k=1}^{+\infty} F_k$, where $F_1 = E_1$, and $F_k = E_k \setminus \bigcup_{j=1}^{k-1} E_j$ for all integers k satisfying k > 1. Now we have proved that any finite union of λ -measurable sets is λ -measurable, and any difference of λ -measurable sets is λ -measurable. It follows that the sets F_1, F_2, F_3, \ldots are

all λ -measurable. These sets are also pairwise disjoint. We conclude that the union of the sets F_1, F_2, F_3, \ldots is λ -measurable, and therefore the union of the sets E_1, E_2, E_3, \ldots is λ -measurable.

We have now shown that the union of any finite collection of λ -measurable sets is λ -measurable, and the union of any infinite sequence of λ -measurable sets is λ -measurable. We conclude that the union of any countable collection of λ -measurable sets is λ -measurable, as required.

Corollary 7.16 Let λ be an outer measure on a set X. Then the intersection of any countable collection of λ -measurable subsets of X is λ -measurable.

Proof Let \mathcal{C} be a countable collection of λ -measurable subsets of X. Then $X \setminus \bigcap_{E \in \mathcal{C}} E = \bigcup_{E \in \mathcal{C}} (X \setminus E)$ (i.e., the complement of the intersection of the sets in the collection is the union of the complements of those sets.) Now $X \setminus E$ is λ -measurable for every $E \in \mathcal{C}$. Therefore the complement $X \setminus \bigcap_{E \in \mathcal{C}} E$ of $\bigcap_{E \in \mathcal{C}} E$ is a union of λ -measurable sets, and is thus itself λ -measurable. It follows that intersection $\bigcap_{E \in \mathcal{C}} E$ of the sets in the collection is λ -measurable, as required.

Proposition 7.17 Let λ be an outer measure on a set X, let A be a subset of X, and let C be a countable collection of pairwise disjoint λ -measurable sets. Then

$$\lambda\left(A\cap\bigcup_{E\in\mathcal{C}}E\right)=\sum_{E\in\mathcal{C}}\lambda(A\cap E).$$

Proof It follows from Lemma 7.14 that the required identity holds for any finite collection of pairwise disjoint λ -measurable sets.

Now let E_1, E_2, E_3, \ldots be an infinite sequence of pairwise disjoint λ -measurable subsets of X. Then

$$\sum_{k=1}^{m} \lambda(A \cap E_k) = \lambda \left(A \cap \bigcup_{k=1}^{m} E_k \right) \le \lambda \left(A \cap \bigcup_{k=1}^{+\infty} E_k \right)$$

for all positive integers m. It follows that

$$\sum_{k=1}^{+\infty} \lambda(A \cap E_k) = \lim_{m \to +\infty} \sum_{k=1}^{m} \lambda(A \cap E_k) \le \lambda \left(A \cap \bigcup_{k=1}^{+\infty} E_k \right).$$

But the definition of outer measures ensures that

$$\lambda\left(A \cap \bigcup_{k=1}^{+\infty} E_k\right) = \lambda\left(\bigcup_{k=1}^{+\infty} (A \cap E_k)\right) \le \sum_{k=1}^{+\infty} \lambda(A \cap E_k)$$

We conclude therefore that $\lambda \left(A \cap \bigcup_{k=1}^{+\infty} E_k\right) = \sum_{k=1}^{+\infty} \lambda(A \cap E_k)$ for any infinite sequence E_1, E_2, E_3, \ldots of pairwise disjoint λ -measurable subsets of X. Thus the required identity holds for any countable collection of pairwise disjoint λ -measurable subsets of X, as required.

7.4 Measure Spaces

Definition Let X be a set. A collection \mathcal{A} of subsets of X is said to a σ -algebra (or sigma-algebra) of subsets of X if it has the following properties:

- (i) the empty set \emptyset is a member of \mathcal{A} ;
- (ii) the complement $X \setminus E$ of any member E of \mathcal{A} is itself a member of \mathcal{A} ;
- (iii) the union of any countable collection of members of \mathcal{A} is itself a member of \mathcal{A} .

Lemma 7.18 Let X be a set, and let \mathcal{A} be a σ -algebra of subsets of X. Then the intersection of any countable collection of members of the σ -algebra \mathcal{A} is itself a member of \mathcal{A} .

Proof Let \mathcal{C} be a countable collection of sets belonging to \mathcal{A} . Then $X \setminus E \in \mathcal{A}$ for all $E \in \mathcal{C}$, and therefore $\bigcup_{E \in \mathcal{C}} (X \setminus E) \in \mathcal{A}$. But $\bigcup_{E \in \mathcal{C}} (X \setminus E) = X \setminus \bigcap_{E \in \mathcal{C}} E$. It follows that the complement of the intersection $\bigcap_{E \in \mathcal{C}} E$ of the sets in the collection \mathcal{C} is itself a member of \mathcal{A} , and therefore the intersection $\bigcap_{E \in \mathcal{C}} E$ of those sets is a member of the σ -algebra \mathcal{A} , as required.

Let X be a set, and let \mathcal{C} be a collection of subsets of X. The collection of all subsets of X is a σ -algebra. Also the intersection of any collection of σ -algebras of subsets of X is itself a σ -algebra. Let \mathcal{A} be the intersection of all σ -algebras \mathcal{B} of subsets of X that have the property that $\mathcal{C} \subset \mathcal{B}$. Then \mathcal{A} is a σ -algebra, and $\mathcal{C} \subset \mathcal{A}$. Moreover if \mathcal{B} is a σ -algebra of subsets of X, and if $\mathcal{C} \subset \mathcal{B}$ then $\mathcal{A} \subset \mathcal{B}$. The σ -algebra \mathcal{A} may therefore be regarded as the smallest σ -algebra of subsets of X for which $\mathcal{C} \subset \mathcal{A}$. We shall refer to this σ -algebra \mathcal{A} as the σ -algebra of subsets of X generated by \mathcal{C} . We see therefore that any collection of subsets of a set X generates a σ -algebra of subsets of X which is the smallest σ -algebra of subsets of X that contains the given collection of subsets. **Definition** Let X be a set, and let \mathcal{A} be a σ -algebra of subsets of X. A *measure* on \mathcal{A} is a function $\mu: \mathcal{A} \to [0, +\infty]$, taking values in the set $[0, +\infty]$ of non-negative extended real numbers, which has the property that

$$\mu\left(\bigcup_{E\in\mathcal{C}}E\right) = \sum_{E\in\mathcal{C}}\mu(E)$$

for any countable collection C of pairwise disjoint sets belonging to the σ -algebra A.

Definition A measure space (X, \mathcal{A}, μ) consists of a set X, a σ -algebra \mathcal{A} of subsets of X, and a measure $\mu: \mathcal{A} \to [0, +\infty]$ defined on this σ -algebra \mathcal{A} . A subset E of a measure space (X, \mathcal{A}, μ) is said to be measurable (or μ -measurable) if it belongs to the σ -algebra \mathcal{A} .

Theorem 7.19 Let λ be an outer measure on a set X. Then the collection \mathcal{A}_{λ} of all λ -measurable subsets of X is a σ -algebra. The members of this σ -algebra are those subsets E of X with the property that $\lambda(A) = \lambda(A \cap E) + \lambda(A \setminus E)$ for any subset A of X. Moreover the restriction of the outer measure λ to the λ -measurable sets defines a measure μ on the σ -algebra \mathcal{A}_{λ} . Thus (X, \mathcal{A}, μ) is a measure space.

Proof Immediate from Propositions 7.13, 7.15 and 7.17.

Definition A measure space (X, \mathcal{A}, μ) is said to be *complete* if, given any measurable subset E of X satisfying $\mu(E) = 0$, and given any subset F of E, the subset F is also measurable. The measure μ on \mathcal{A} is then said to be *complete*.

Lemma 7.20 Let λ be an outer measure on a set X, let \mathcal{A} be the σ -algebra consisting of the λ -measurable subsets of X, and let μ be the measure on \mathcal{A} obtained by restricting the outer measure λ to the members of \mathcal{A} . Then (X, \mathcal{A}, μ) is a complete measure space.

Proof Let *E* be a measurable set in *X* satisfying $\mu(E) = 0$, let *F* be a subset of *E*, and let *A* be a subset of *X*. Then $A \cap F \subset A \cap E$ and $A \setminus E \subset A \setminus F \subset A$, and therefore $0 \leq \lambda(A \cap F) \leq \lambda(A \cap E)$ and $\lambda(A \setminus E) \leq \lambda(A \setminus F) \leq \lambda(A)$. Now it follows from the definition of measurable sets in *X* that $\lambda(A) =$ $\lambda(A \cap E) + \lambda(A \setminus E)$. Moreover $0 \leq \lambda(A \cap E) \leq \lambda(E) = \mu(E) = 0$. It follows that $\lambda(A \cap E) = 0$ and $\lambda(A \setminus E) = \lambda(A)$. The inequalities above then ensure that $\lambda(A \cap F) = 0$ and $\lambda(A \setminus F) = \lambda(A)$. But then $\lambda(A) = \lambda(A \cap F) + \lambda(A \setminus F)$, and thus *F* is λ -measurable, as required.

7.5 Lebesgue Measure on Euclidean Spaces

We are now in a position to give the definition of *Lebesgue measure* on *n*dimensional Euclidean space \mathbb{R}^n . We have already defined an outer measure μ^* on \mathbb{R}^n known as *Lebesgue outer measure*. We defined a *block* in \mathbb{R}^n to be a subset of \mathbb{R}^n that is a Cartesian product of *n* bounded intervals. The product of the lengths of those intervals is the *content* of the block. Then, given any subset *E* of \mathbb{R}^n , we defined the *Lebesgue outer measure* $\mu^*(E)$ of the set *E* to be the infimum of the quantities $\sum_{B \in \mathcal{C}} m(B)$, where the infimum is taken over all countable collections of blocks in \mathbb{R}^n that cover the set *E*, and where m(B) denotes the content of a block *B* in such a collection. Thus

$$\sum_{B\in\mathcal{C}}m(B)\geq\mu^*(E)$$

for every countable collection \mathcal{C} of blocks in \mathbb{R}^n that covers E; and, moreover, given any positive real number ε , there exists a countable collection \mathcal{C} of blocks in \mathbb{R}^n covering E for which

$$\mu^*(E) \le \sum_{B \in \mathcal{C}} m(B) \le \mu^*(E) + \varepsilon.$$

These properties characterize the Lebesgue outer measure $\mu^*(E)$ of the set E.

We say that a subset E of \mathbb{R}^n is Lebesgue-measurable if and only if it is μ^* -measurable, where μ^* denotes Lebesgue outer measure on \mathbb{R}^n . Thus a subset E of \mathbb{R}^n is Lebesgue-measurable if and only if $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)$ for all subsets A of \mathbb{R}^n . The collection \mathcal{L}_n of all Lebesguemeasurable sets is a σ -algebra of subsets of \mathbb{R}^n , and therefore the difference of any two Lebesgue-measurable subsets of \mathbb{R}^n is Lebesgue-measurable, and any countable union or intersection of Lebesgue-measurable sets is Lebesguemeasurable. The Lebesgue measure $\mu(E)$ of a Lebesgue-measurable subset Eof \mathbb{R}^n is defined to be the Lebesgue outer measure $\mu^*(E)$ of that set. Thus Lebesgue measure μ is the restriction of Lebesgue outer measure μ^* to the σ -algebra \mathcal{L}_n of Lebesgue-measurable subsets of \mathbb{R}^n .

It follows from Lemma 7.20 that Lebesgue measure is a complete measure on \mathbb{R}^n .

Remark The Lebesgue measure $\mu(E)$ of a subset E of \mathbb{R}^2 may be regarded as the area of that set. It is not possible to assign an area to every subset of \mathbb{R}^2 in such a way that the areas assigned to such subsets have all the properties that one would expect from a well-defined notion of area. One might at first sight expect that Lebesgue outer measure would provide a natural definition of area, applicable to all subsets of the plane, that would have the properties that one would expect of a well-defined notion of area. One would expect in particular that the area of a disjoint union of two subsets of the plane would be the sum of the areas of those sets. However it is possible to construct examples of disjoint subsets E and F in the plane which interpenetrate one another to such an extent as to ensure that $\mu^*(E \cup F) < \mu^*(E) + \mu^*(F)$, where μ^* denotes Lebesgue outer measure on \mathbb{R}^2 . The σ -algebra \mathcal{L}_2 consisting of the Lebesgue-measurable subsets of the plane \mathbb{R}^2 is in fact that largest collection of subsets of the plane for which the sets in the collection have a well-defined area; the Lebesgue measure of a Lebesgue-measurable subset of the plane can be regarded as the area of that set. Similarly the σ -algebra \mathcal{L}_3 of Lebesguemeasurable subsets of three-dimensional Euclidean space \mathbb{R}^3 is the largest collection of subsets of \mathbb{R}^3 for which the sets in the collection have a welldefined volume.

Proposition 7.21 Every closed n-dimensional block in \mathbb{R}^n is Lebesgue-measurable.

Proof Proposition 7.12, ensures that closed blocks have the property that characterizes Lebesgue-measurable subsets of \mathbb{R}^n .

Proposition 7.22 Every open set in \mathbb{R}^n is Lebesgue-measurable.

Proof Let \mathcal{W} be the collection of all open blocks in \mathbb{R}^n that are Cartesian products of intervals whose endpoints are rational numbers. Now the set \mathcal{I} of all open intervals in \mathbb{R}^n whose endpoints are rational numbers is a countable set, as the function that sends such an interval to its endpoints defines an injective function from \mathcal{I} to the countable set $\mathbb{Q} \times \mathbb{Q}$. Moreover there is a bijection from the countable set \mathcal{I}^n to \mathcal{W} that sends each ordered *n*-tuple (I_1, I_2, \ldots, I_n) of open intervals to the open block $I_1 \times I_2 \times \cdots \times I_n$. It follows that the collection \mathcal{W} is countable.

Let V be an open set in \mathbb{R}^n , and let **v** be a point of V. Then there exists some positive real number δ such that $B(\mathbf{v}, \delta) \subset V$, where $B(\mathbf{v}, \delta) \subset V$ denotes the open ball of radius δ centred on **v**. Moreover there exist open blocks W belonging to \mathcal{W} for which $\mathbf{v} \in W$ and $W \subset B(\mathbf{v}, \delta)$. It follows that the open set V is the union of the countable collection

$$\{W \in \mathcal{W} : W \subset V\}$$

of open blocks. Now each open block is a Lebesgue-measurable set, and any countable union of Lebesgue-measurable sets is itself a Lebesgue-measurable set. Therefore the open set V is a Lebesgue-measurable set, as required.

Corollary 7.23 Every closed set in \mathbb{R}^n is Lebesgue-measurable.

Proof This follows immediately from Proposition 7.22, since the complement of any Lebesgue-measurable set is itself Lebesgue measurable set.

Definition A subset of \mathbb{R}^n is said to be a *Borel set* if it belongs to the σ -algebra generated by the collection of open sets in \mathbb{R}^n .

All open sets and closed sets in \mathbb{R}^n are Borel sets. The collection of all Borel sets is a σ -algebra in \mathbb{R}^n and is the smallest such σ -algebra containing all open subsets of \mathbb{R}^n .

Definition A measure defined on a σ -algebra \mathcal{A} of subsets of \mathbb{R}^n is said to be a *Borel measure* if the σ -algebra \mathcal{A} contains all the open sets in \mathbb{R}^n .

Corollary 7.24 Lebesgue measure on \mathbb{R}^n is a Borel measure, and thus every Borel set in \mathbb{R}^n is Lebesgue-measurable.

Remark The definitions of Borel sets and Borel measures generalize in the obvious fashion to arbitrary topological spaces. The collection of Borel sets in a topological space X is the σ -algebra generated by the open subsets of X. A measure defined on a σ -ring of subsets of X is said to be a Borel measure if every Borel set is measurable.

7.6 Basic Properties of Measures

Let (X, \mathcal{A}, μ) be a measure space. Then the measure μ is defined on the σ -algebra \mathcal{A} of measurable subsets of X, and takes values in the set $[0, +\infty]$, where $[0, +\infty] = [0, +\infty) \cup \{+\infty\}$. Thus $\mu(E)$ is defined for each measurable subset E of X, and is either a non-negative real number, or else has the value $+\infty$. The measure μ is by definition *countably additive*, so that

$$\mu\left(\bigcup_{E\in\mathcal{C}}E\right) = \sum_{E\in\mathcal{C}}\mu(E)$$

for every countable collection C of pairwise disjoint measurable subsets of X. In particular μ is *finitely additive*, so that if E_1, E_2, \ldots, E_r are measurable subsets of X that are pairwise disjoint, then

$$\mu(E_1 \cup E_2 \cup \dots \cup E_r) = \mu(E_1) + \mu(E_2) + \dots + \mu(E_r).$$

Also

$$\mu\left(\bigcup_{j=1}^{+\infty} E_j\right) = \sum_{j=1}^{+\infty} \mu(E_j)$$

for any infinite sequence E_1, E_2, E_3, \ldots of pairwise disjoint measurable subsets of X.

Let E and F be measurable subsets of X. Then $E = (E \cap F) \cup (E \setminus F)$, and the sets $E \cap F$ and $E \setminus F$ are measurable and disjoint. It therefore follows from the finite additivity of the measure μ that $\mu(E) = \mu(E \cap F) + \mu(E \setminus F)$. Also $E \cup F$ is the disjoint union of E and $F \setminus E$. and therefore

$$\mu(E \cup F) = \mu(E) + \mu(F \setminus E) = \mu(E \cap F) + \mu(E \setminus F) + \mu(F \setminus E).$$

It follows that

$$\mu(E \cup F) + \mu(E \cap F)$$

= $(\mu(E \cap F) + \mu(E \setminus F)) + (\mu(E \cap F) + \mu(F \setminus E))$
= $\mu(E) + \mu(F).$

Now let *E* and *F* be measurable subsets of *X* that satisfy $F \subset E$. Then $\mu(E) = \mu(F) + \mu(E \setminus F)$, and $\mu(E \setminus F) \ge 0$. It follows that $\mu(F) \le \mu(E)$. Moreover $\mu(E \setminus F) = \mu(E) - \mu(F)$, provided that $\mu(E) < +\infty$.

Lemma 7.25 Let (X, \mathcal{A}, μ) be a measure space, and let E_1, E_2, E_3, \ldots be an infinite sequence of measurable subsets of X. Suppose that $E_j \subset E_{j+1}$ for all positive integers j. Then

$$\mu\left(\bigcup_{j=1}^{+\infty} E_j\right) = \lim_{j \to +\infty} \mu(E_j).$$

Proof Let $E = \bigcup_{j=1}^{+\infty} E_j$, let $F_1 = E_1$, and let $F_j = E_j \setminus \bigcup_{k=1}^{j-1} E_k$ for all integers j satisfying j > 1. Then the sets F_1, F_2, F_3, \ldots are pairwise disjoint, the set E_j is the disjoint union of the sets F_k for which $1 \le k \le j$, and the set E is the disjoint union of all of the sets F_k . It therefore follows from the countable (and finite) additivity of the measure μ that

$$\mu(E) = \sum_{k=1}^{+\infty} \mu(F_k), \quad \mu(E_j) = \sum_{k=1}^{j} \mu(F_k).$$

But then

$$\mu(E) = \sum_{k=1}^{+\infty} \mu(F_k) = \lim_{j \to +\infty} \sum_{k=1}^{j} \mu(F_k) = \lim_{j \to +\infty} \mu(E_j),$$

as required.

Lemma 7.26 Let (X, \mathcal{A}, μ) be a measure space, and let E_1, E_2, E_3, \ldots be an infinite sequence of measurable subsets of X. Suppose that $E_{j+1} \subset E_j$ for all positive integers j, and that $\mu(E_1) < +\infty$. Then

$$\mu\left(\bigcap_{j=1}^{+\infty} E_j\right) = \lim_{j \to +\infty} \mu(E_j).$$

Proof Let $G_j = E_1 \setminus E_j$ for all positive integers j, let $E = \bigcap_{j=1}^{+\infty} E_j$, and let $G = \bigcup_{j=1}^{+\infty} G_j$. It then follows from Lemma 7.25 that $\mu(G) = \lim_{j \to +\infty} \mu(G_j)$. Now $E_j = E_1 \setminus G_j$ for all positive integers j, and $\mu(E_1) < \infty$. It follows that $\mu(E_j) = \mu(E_1) - \mu(G_j)$ for all positive integers j. Also $E = E_1 \setminus G$. Therefore

$$\mu(E) = \mu(E_1) - \mu(G) = \mu(E_1) - \lim_{j \to +\infty} \mu(G_j) = \lim_{j \to +\infty} \mu(E_j),$$

as required.

7.7 The Existence of Non-Measurable Sets

Definition For each real number u, let $\tau_u \colon \mathbb{R} \to \mathbb{R}$ be the translation mapping the set \mathbb{R} of real numbers onto itself defined so that $\tau_u(x) = x + u$ for all real numbers x. We say that an outer measure λ on \mathbb{R} is translation-invariant if $\lambda(\tau_u(E)) = \lambda(E)$ for all subsets E of \mathbb{R} and for all real numbers u.

Proposition 7.27 Let λ be a translation-invariant outer measure on the set \mathbb{R} of real numbers. Suppose that [0,1) is λ -measurable and $\lambda([0,1)) = 1$. Then there exist subsets of \mathbb{R} that are not λ -measurable.

Proof Let B = [0, 1) and, for each real number u, let $\tau_u \colon \mathbb{R} \to \mathbb{R}$ and $\rho_u \colon B \to B$ be defined such that, for all $x \in B$, $\tau_u(x) = x + u$ and $\rho_u(x)$ is the unique element of B for which $x + u - \rho_u(x)$ is an integer.

Let $u \in B$. Then

$$\rho_u(x) = \begin{cases} x+u & \text{if } x < 1-u; \\ x+u-1 & \text{if } x \ge 1-u. \end{cases}$$

Now the set B is λ -measurable. The translation-invariance of the outer measure λ then ensures that the set $\tau_{-u}(B)$ is λ -measurable. Indeed let A

be a subset of \mathbb{R} . Then

$$\lambda(A) = \lambda(\tau_u(A)) = \lambda(\tau_u(A) \cap B) + \lambda(\tau_u(A) \setminus B)$$

= $\lambda(\tau_{-u}(\tau_u(A) \cap B)) + \lambda(\tau_{-u}(\tau_u(A) \setminus B))$
= $\lambda(A \cap \tau_{-u}(B)) + \lambda(A \setminus \tau_{-u}(B)).$

Thus the set $\tau_{-u}(B)$ is λ -measurable, as claimed.

Next we show that $\lambda(\rho_u(E)) = \lambda(E)$ for all subsets E of B and for all $u \in B$. Now

$$B \cap \tau_{-u}(B) = \{x \in B : x < 1 - u\}$$

and

$$B \setminus \tau_{-u}(B) = \{ x \in B : x \ge 1 - u \}.$$

Therefore $\rho_u(x) = \tau_u(x)$ for all $x \in B \cap \tau_{-u}(B)$ and $\rho_u(x) = \tau_{u-1}(x)$ for all $x \in B \setminus \tau_{-u}(B)$. It follows that

$$\lambda(\rho_u(E) \cap B) = \lambda(\rho_u(E \cap \tau_{-u}(B))) = \lambda(\tau_u(E \cap \tau_{-u}(B)))$$
$$= \lambda(E \cap \tau_{-u}(B))$$

and

$$\lambda(\rho_u(E) \setminus B) = \lambda(\rho_u(E \setminus \tau_{-u}(B))) = \lambda(\tau_{u-1}(E \setminus \tau_{-u}(B)))$$
$$= \lambda(E \setminus \tau_{-u}(B))).$$

But

$$\lambda(\rho_u(E)) = \lambda(\rho_u(E) \cap B) + \lambda(\rho_u(E) \setminus B)$$

and

$$\lambda(E) = \lambda(E \cap \tau_{-u}(B)) + \lambda(E \setminus \tau_{-u}(B)),$$

because the sets B and $\tau_{-u}(B)$ are λ -measurable. It follows that $\lambda(\rho_u(E)) = \lambda(E)$ for all $u \in \mathbb{R}$.

Now let us define a relation \sim on the interval B, where B = [0, 1), where real numbers x and y belonging to B satisfy $x \sim y$ if and only if x - y is a rational number. Clearly $x \sim x$ for all $x \in B$, and if $x, y \in B$ satisfy $x \sim y$ then they also satisfy $y \sim x$. And if $x, y, z \in B$ satisfy $x \sim y$ and $y \sim z$ then they also satisfy $x \sim z$. Thus the relation \sim on B is reflexive, symmetric and transitive, and is therefore an equivalence relation. This equivalence relation then partitions the set B into equivalence classes: every real number in the set B belongs to a unique equivalence class; two real numbers in the set set B belong to the same equivalence class if and only if their difference is a rational number. Now the Axiom of Choice in set theory guarantees the existence of a subset E of B that contains exactly one element from each equivalence class. Then, given any real number x in the set B, there exists exactly one element z of the set E for which x - z is a rational number. If $x \ge z$ then $x = \rho_q(z)$ if and only if q = x - z. On the other hand if x < z then $x = \rho_q(z)$ if and only if q = x - z + 1. It follows that, given any real number x in the set B, there exists a unique real number z belonging to E and a unique rational number q satisfying $0 \le q < 1$ for which $x = \rho_q(z)$. We conclude from this that the set B is the union of the sets $\rho_q(E)$ as q ranges over the set T of all rational numbers q satisfying $0 \le q < 1$. Moreover the sets $\rho_q(E)$ obtained as q ranges over the countable set T are pairwise disjoint.

But $\lambda(\rho_q(E)) = \lambda(E)$ for all $q \in T$. If it were the case that $\lambda(E) = 0$, it would then follow that $\lambda(B) = 0$, because λ is an outer measure. But $\lambda(B) = 1$. It then follows that the sum $\sum_{q \in T} \lambda(\rho_q(E))$ diverges, and therefore cannot equal $\lambda(B)$, though $B = \bigcup_{q \in T} \rho_q(E)$. If the set E were λ -measurable, then all the sets $\rho_q(E)$ would be λ -measurable, and the sum of the outer measures of these pairwise-disjoint sets would be equal to $\lambda(B)$. Because this is not the case, it follows that the set E cannot be λ -measurable. The result follows.