Module MAU22200: Advanced Analysis (Semester 2) Hilary Term 2020 Part I (Sections 4, 5 and 6)

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Contents

4 Countable and Uncountable Sets

4.1 Functions between Sets

Let X and Y be sets, and let $f\colon X \to Y$ be a function from X to Y. The function f is *injective* if, given any element y of Y, there exists at most one element x of X for which $f(x) = y$. The function f is *surjective* if, given any element y of Y, there exists at least one element x of X for which $f(x) = y$. The function f is *bijective* if it is both injective and surjective. Thus the function $f\colon X \to Y$ is bijective if and only if, given any element y of Y, there exists a exactly one element x of X for which $f(x) = y$. A function $f\colon X \to Y$ is bijective if and only if it has a well-defined inverse $f^{-1}\colon Y \to X$. Injective, surjective and bijective functions may be referred to as *injections*, *surjections* and *bijections* respectively.

4.2 Countable Sets

Definition A non-empty set X is said to be *countable* if there exists an injection mapping X into the set $\mathbb{N}$ of positive integers. The empty set $\emptyset$ is also said to be countable.

Lemma 4.1 *Any subset of a countable set is countable.*

Proof Let Y be a subset of a countable set X. Then there exists an injection $f\colon X \to \mathbb{N}$ from X to the set $\mathbb{N}$ of positive integers. The restriction of this injection to the set Y gives an injection from Y to $\mathbb{N}$. ∎

Lemma 4.2 *Let X and Y be sets, and let $f\colon X \to Y$ be an injective function from X to Y. Suppose that the set Y is countable. Then the set X is countable.*

Proof The set Y is countable, and therefore there exists an injective function $g\colon Y \to \mathbb{N}$ mapping Y into the set $\mathbb{N}$ of positive integers. Then the composition function $g \circ f\colon X \to \mathbb{N}$ is injective, because the composition of any two injective functions is always itself an injective function. It follows that the set X is countable, as required. ∎

We establish a one-to-one correspondence between the set $\mathbb{N} \times \mathbb{N}$ of ordered pairs of positive integers and the set $\mathbb{N}$ of positive integers. This correspondence is implemented by a function $h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is constructed so that

$$h(1, 1) = 1,$$

$$h(2,1) = 2, \quad h(1,2) = 3,$$

$$h(3,1) = 4, \quad h(2,2) = 5, \quad h(1,3) = 6,$$

$$h(4,1) = 7, \quad h(3,2) = 8, \quad h(2,3) = 9, \quad h(1,4) = 10, \quad \text{etc.}$$

The expression for the function h should be determined so that $h(j,k) = S(j+k) + k$ for all positive integers j and k, where, for each integer m satisfying $m \geq 2$, $S(m)$ is equal to the number of ordered pairs (p,q) of positive integers satisfying $p + q < m$.

Let m be a positive integer satisfying $m \geq 3$. Then, for each integer p between 1 and $m-2$, there are $m-p-1$ positive integers q satisfying $p+q < m$. It follows that

$$S(m) = \sum_{p=1}^{m-2} (m-p-1) = \sum_{i=1}^{m-2} i = \tfrac{1}{2}(m-1)(m-2).$$

This identity also holds when $m = 2$, since $S(2) = 0$. The function $h\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ constructed to implement the one-to-one correspondence between the sets $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ therefore satisfies

$$h(j,k) = \tfrac{1}{2}(j+k-1)(j+k-2) + k$$

for all positive integers j and k. We now prove formally that this function is indeed a bijection between the sets $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.

Lemma 4.3 *Let $h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the function defined such that*

$$h(j,k) = \tfrac{1}{2}(j+k-1)(j+k-2) + k.$$

for all positive integers j and k. Then $h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection.

Proof Let n be a positive integer. Then there is a unique integer m satisfying $m \geq 2$ for which

$$\tfrac{1}{2}(m-1)(m-2) < n \leq \tfrac{1}{2}m(m-1).$$

Let $k = n - \frac{1}{2}(m-1)(m-2)$ and $j = m - k$. Then j and k are integers between 1 and $m-1$, and

$$h(j,k) = \tfrac{1}{2}(m-1)(m-2) + k = n.$$

Now let j' and k' be positive integers satisfying $h(j',k') = n$. Then

$$0 < n - \tfrac{1}{2}(j'+k'-1)(j'+k'-2) = k' \leq j'+k'-1,$$

and therefore

$$\tfrac{1}{2}(j'+k'-1)(j'+k'-2) < n \leq \tfrac{1}{2}(j'+k')(j'+k'-1).$$

It follows that $j'+k'=m$, where m is the unique integer satisfying $m \geq 2$ for which

$$\tfrac{1}{2}(m-1)(m-2) < n \leq \tfrac{1}{2}m(m-1).$$

But then

$$\tfrac{1}{2}(m-1)(m-2) + k' = n = \tfrac{1}{2}(m-1)(m-2) + k,$$

and therefore $k'=k$ and $j'=j$. Thus (j,k) is the unique ordered pair of positive integers for which $h(j,k)=n$. We have thus shown that, given any positive integer n, there exists a unique ordered pair (j,k) of positive integers for which $h(j,k)=n$. It follows that $h\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ is a bijection, as required. ∎

Lemma 4.4 *Let $h\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ be the function defined such that*

$$h(j,k) = \tfrac{1}{2}(j+k-1)(j+k-2)+k.$$

for all positive integers j and k, and let functions

$$g_n\colon \mathbb{N}^n \to \mathbb{N}$$

be constructed for $n=1,2,3\ldots$ so that $g_1(j)=j$ for all positive integers j and

$$g_n(j_1,j_2,\ldots,j_n) = h(g_{n-1}(j_1,j_2,\ldots,j_{n-1}),j_n)$$

for all $(j_1,j_2,\ldots,j_n)\in\mathbb{N}^n$ whenever $n>1$. Then each of the functions $g_n\colon \mathbb{N}^n\to\mathbb{N}$ is a bijection.

Proof The function $g_1\colon \mathbb{N}\to\mathbb{N}$ is a bijection because it is the identity function of $\mathbb{N}$. The function $g_2\colon \mathbb{N}^2\to\mathbb{N}$ coincides with the function h. It therefore follows from Lemm 4.3 that the function $g_2\colon \mathbb{N}^2\to\mathbb{N}$ is a bijection. We prove by induction on n that the function $g_n\colon \mathbb{N}^n\to\mathbb{N}$ is a bijection for all positive integers n. Suppose therefore as our induction hypothesis that n is some positive integer satisfying $n\geq 3$ and that $g_{n-1}\colon \mathbb{N}^{n-1}\to\mathbb{N}$ is a bijection. We must show that $g_n\colon \mathbb{N}^n\to\mathbb{N}$ is a bijection.

Let m be a positive integer. Then there exist uniquely-determined positive integers m' and j_n for which $h(m',j_n)=m$, because the function $h\colon \mathbb{N}^2\to\mathbb{N}$ is a bijection. There then exists a unique $(n-1)$-tuple $(j_1,j_2,\ldots,j_{n-1})$ of positive integers for which $g_{n-1}(j_1,j_2,\ldots,j_{n-1})=m'$, because $g_{n-1}\colon \mathbb{N}^{n-1}\to\mathbb{N}$ is a bijection. But then $(j_1,j_2,\ldots,j_n)$ is the unique n-tuple of positive integers for which $g_n(j_1,j_2,\ldots,j_n)=m$. We conclude therefore that $g_n\colon \mathbb{N}^n\to\mathbb{N}$ is a bijection, as required. ∎

Proposition 4.5 *Let $X_1, X_2, \ldots, X_n$ be countable sets. Then the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ of these countable sets is itself a countable set.*

Proof Let $X = X_1 \times X_2 \times \cdots X_n$. There exist injective functions $f_i: X_i \to \mathbb{N}$ from the set X_i to the set $\mathbb{N}$ of positive integers, because each set X_i is countable. Also there exists a bijection $g_n: \mathbb{N}^n \to \mathbb{N}$ from the set $\mathbb{N}^n$ of ordered n-tuples of positive integers to the set $\mathbb{N}$ of positive integers (see Lemma 4.4). Let $f: X \to \mathbb{N}$ be defined so that

$$f(x_1, x_2, \cdots, x_n) = g_n(f_1(x_1), f_2(x_2), \ldots, f_n(x_n))$$

for all $(x_1, x_2, \ldots, x_n) \in X$. We show that $f: X \to \mathbb{N}$ is injective.

Let $(x_1, x_2, \ldots, x_n)$ and $(x'_1, x'_2, \ldots, x'_n)$ be elements of the set X. Suppose that

$$f(x_1, x_2, \ldots, x_n) = f(x'_1, x'_2, \ldots, x'_n).$$

Then

$$(f_1(x_1), f_2(x_2), \ldots, f_n(x_n)) = (f_1(x'_1), f_2(x'_2), \ldots, f_n(x'_n)),$$

because the function $g_n: \mathbb{N}^2 \to \mathbb{N}$ is injective, and therefore $f_i(x_i) = f_i(x'_i)$ for $i = 1, 2, \ldots, n$. But each of the functions $f_1, f_2, \ldots, f_n$ is injective. It follows that $x_i = x'_i$ for $i = 1, 2, \ldots, n$, and thus

$$(x_1, x_2, \ldots, x_n) = (x'_1, x'_2, \ldots, x'_n).$$

This shows that the function $f: X \to \mathbb{N}$ is injective. It follows that the set X is countable, as required. ∎

Proposition 4.6 *Any countable union of countable sets is itself a countable set.*

Proof Let J be a subset of the set $\mathbb{N}$ of positive integers and, for each $j \in J$, let X_j be a countable set, and let $X = \bigcup_{j \in J} X_j$. Also, for each $j \in J$, let $g_j: X_j \to \mathbb{N}$ be an injective function from X_j to the set $\mathbb{N}$ of positive integers. (The functions g_j exist because, for all $j \in J$, the set X_j is a countable set.) For each $x \in X$ let $n(x)$ denote the smallest positive integer j in the indexing set J for which $x \in X_j$. Let $h: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection between the sets $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ (see Lemma 4.3), and let $f: X \to \mathbb{N}$ be the function defined so that

$$f(x) = h(n(x), g_{n(x)}(x))$$

for all $x \in X$.

Let x and x' be elements of X satisfying $f(x) = f(x')$. We claim that $x = x'$. Now if $f(x) = f(x')$ then $n(x) = n(x')$ and $g_{n(x)}(x) = g_{n(x')}(x')$, because the function $h: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection. Let $n = n(x)$. Then $x \in X_n$ and $x' \in X_n$, and $g_n(x) = g_n(x')$. But $g: X_n \to \mathbb{N}$ is an injective function. It follows that $x = x'$. We conclude therefore that the function $f: X \to \mathbb{N}$ is injective, and therefore the set X is countable, as required. ∎

Lemma 4.7 *The set $\mathbb{Z}$ of integers is countable.*

Proof The set $\mathbb{Z}$ is the union of the set $\mathbb{N}$ of positive integers and the set W of non-positive integers, where $W = \{n \in \mathbb{Z} : n \leq 0\}$. Let $f: W \to \mathbb{N}$ be defined such that $f(n) = 1 - n$ for all $n \in W$. Then $f: W \to \mathbb{N}$ is bijective, and therefore the set W is countable. It follows that the set $\mathbb{Z}$ of integers, being the union of the countable sets $\mathbb{N}$ and W, is itself a countable set, as required. ∎

Lemma 4.8 *The set $\mathbb{Q}$ of rational numbers is countable.*

Proof For each positive integer m, let R_m be the set consisting of all rational numbers that are of the form n/m for some positive integer n. The function mappng each $q \in R_m$ to mq is a bijection from R_m to the set $\mathbb{Z}$ of integers, and $\mathbb{Z}$ is a countable set. It follows that R_m is a countable set for each positive integer m. Now $\mathbb{Q} = \bigcup_{m=1}^{+\infty} R_m$. It follows that the set $\mathbb{Q}$ of rational numbers is a countable union of countable sets. Moreover any countable union of countable sets is itself countable (Proposition 4.6). We conclude that the set $\mathbb{Q}$ is countable, as required. ∎

Proposition 4.9 *Let $h: X \to Y$ be a surjection. Suppose that the set X is countable. Then the set Y is countable.*

Proof The set X is countable, and therefore there exists an injective function $g: X \to \mathbb{N}$ from X to the set $\mathbb{N}$ of positive integers. Given any element y of the set Y there exists at least one positive integer n with the property that $n = g(x)$ for some $x \in X$ satisfying $h(x) = y$, because the function h is surjective. For each $y \in Y$, let $f(y)$ be the smallest positive integer n with the property that $n = g(x)$ for some $x \in X$ satisfying $h(x) = y$.

Let y and y' be elements of the set Y, where $y \neq y'$. Then there exist elements x and x' of the set X for which $f(y) = g(x)$, $f(y') = g(x')$, $h(x) = y$ and $h(x') = y'$. Then $x \neq x'$, because $y \neq y'$. But then $g(x) \neq g(x')$, because the function g is injective, and therefore $f(y) \neq f(y')$. We conclude from this that the function f is injective, and therefore the set Y is countable, as required. ∎

Proposition 4.10 *A non-empty set X is countable if and only if there exists a surjective function $g: \mathbb{N} \to X$ mapping the set $\mathbb{N}$ of positive integers onto X.*

Proof Let X be a non-empty set. If there exists a surjective function $g: \mathbb{N} \to X$ mapping the set of positive integers onto X then it follows from Proposition 4.9 that the set X is countable.

Conversely suppose that X is a non-empty countable set. Then there exists an injection $f: X \to \mathbb{N}$ from X to $\mathbb{N}$. Let x_0 be some chosen element of the set X. Given a positive integer n, there exists at most one element x of the set X for which $f(x) = n$. It follows that there exists a function $g: \mathbb{N} \to X$, where $g(f(x)) = x$ for all $x \in X$ nd $g(n) = x_0$ for positive integers n that do not belong to the range $f(X)$ of the function f. This function $g: \mathbb{N} \to X$ is then a surjective function mapping the set $\mathbb{N}$ of positive integers onto the set X. The result follows. ∎

4.3 Uncountable Sets

A set that is not countable is said to be *uncountable*. Many sets occurring in mathematics are uncountable. These include the set of real numbers.

It follows directly from Lemma 4.1 that if a set X has an uncountable subset, then X must itself be uncountable.

It also follows directly from Proposition 4.9 that if $h: X \to Y$ is a surjection from a set X to a set Y, and if the set Y is uncountable, then the set X is uncountable.

Definition Let X be a set. The *power set* $\mathcal{P}(X)$ of X is the set whose elements are the subsets of the set X.

It is a straightforward exercise to prove that if a finite set X has m elements then its power set $\mathcal{P}(X)$ has 2^m elements. (This may be shown by induction on the number of elements in the finite set.) It follows that, for any finite set X, the power set $\mathcal{P}(X)$ has more elements than the set X itself, and therefore there cannot exist any surjective function from a finite set to its power set. We now show that the same is true of all sets, whether finite or infinite.

Proposition 4.11 *Let X be a set, and let $\mathcal{P}(X)$ be the power set of X. Then there cannot exist any surjective function from the set X to its power set $\mathcal{P}(X)$.*

Proof Let $f\colon X \to \mathcal{P}(X)$ be a function from a set X to its power set $\mathcal{P}(X)$, and let $B = \{x \in X : x \notin f(x)\}$. Let $x \in X$. Then $x \in B$ if and only if $x \notin f(x)$. It follows that the element x of X belongs to exactly one of the subsets B and $f(x)$ of X, and therefore $B \neq f(x)$. We conclude from this that the subset B of X is an element of the power set $\mathcal{P}(X)$ of X that does not belong to the range $f(X)$ of the function f. Thus the function f is not surjective. The result follows. ∎

Corollary 4.12 *The power set $\mathcal{P}(\mathbb{N})$ of the set $\mathbb{N}$ of positive integers is an uncountable set.*

Proof If the set $\mathcal{P}(\mathbb{N})$ were countable, there would exist a surjective function $f\colon \mathbb{N} \to \mathcal{P}(\mathbb{N})$ mapping the set $\mathbb{N}$ of positive integers onto its power set (see Proposition 4.10). But there cannot exist any surjective function mapping the set $\mathbb{N}$ onto its power set (Proposition 4.11). Therefore the set $\mathcal{P}(\mathbb{N})$ must be uncountable, as required. ∎

Proposition 4.13 *The set $\mathbb{R}$ of real numbers is uncountable.*

Proof Let the function $h\colon \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ from the power set $\mathcal{P}(\mathbb{N})$ of the set of positive integers to the set $\mathbb{R}$ of real numbers be defined so that, for all subsets B of $\mathbb{N}$,

$$h(B) = \sum_{j=1}^{+\infty} \frac{d_j}{10^j},$$

where $d_j = 1$ whenever $j \in B$ and $d_j = 0$ whenever $j \notin B$. (Thus, for example, h({2,3,5,8}) = 0.01101001.)

The function $h\colon \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ is injective. It follows that if the set $\mathbb{R}$ of real numbers were countable, then the set $\mathcal{P}(\mathbb{N})$ would also be countable (see Lemma 4.2). But the power set $\mathcal{P}(\mathbb{N})$ of the set of positive integers is uncountable (see Corollary 4.12). It follows therefore that the set $\mathbb{R}$ of real numbers is also uncountable, as required. ∎

5 Some Properties of Infinite Sequences and Series

5.1 Least Upper Bounds and Greatest Lower Bounds

Definition Let S be a set of real numbers which is bounded above. The *least upper bound*, or *supremum*, of the set S is the smallest real number that is greater than or equal to elements of the set S, and is denoted by $\sup S$.

Thus if S is a set of real numbers that is bounded above, then the least upper bound $\sup S$ of the set S is characterized by the following two properties:

- for all $x \in S$, $x \leq \sup S$;
- if u is a real number, and if, for all $x \in S$, $x \leq u$ then $\sup S \leq u$.

The *Least Upper Bound Property* of the real number system guarantees that, given any non-empty set S of real numbers that is bounded above, there exists a least upper bound $\sup S$ for the set S.

Definition Let S be a set of real numbers which is bounded below. The *greatest lower bound*, or *infimum*, of the set S is the largest real number that is less than or equal to elements of the set S, and is denoted by $\inf S$.

Thus if S is a set of real numbers that is bounded below, then the greatest lower bound $\inf S$ of the set S is characterized by the following two properties:

- for all $x \in S$, $x \geq \inf S$;
- if l is a real number, and if, for all $x \in S$, $x \geq l$ then $\inf S \geq l$.

Given any non-empty set S of real numbers that is bounded below, there exists a greatest lower bound $\inf S$ for the set S.

5.2 Monotonic Sequences

An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to be *strictly increasing* if $x_{j+1} > x_j$ for all positive integers j, *strictly decreasing* if $x_{j+1} < x_j$ for all positive integers j, *non-decreasing* if $x_{j+1} \geq x_j$ for all positive integers j, *non-increasing* if $x_{j+1} \leq x_j$ for all positive integers j. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

Theorem 5.1 *Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.*

Proof Let $x_1, x_2, x_3, \ldots$ be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Axiom that there exists a least upper bound p for the set $\{x_j : j \in \mathbb{N}\}$. We claim that the sequence converges to p.

Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j - p| < \varepsilon$ whenever $j \geq N$. Now $p - \varepsilon$ is not an upper bound for the set $\{x_j : j \in \mathbb{N}\}$ (since p is the least upper bound), and therefore there must exist some positive integer N such that $x_N > p - \varepsilon$. But then $p - \varepsilon < x_j \leq p$ whenever $j \geq N$, since the sequence is non-decreasing and bounded above by p. Thus $|x_j - p| < \varepsilon$ whenever $j \geq N$. Therefore $x_j \to p$ as $j \to +\infty$, as required.

If the sequence $x_1, x_2, x_3, \ldots$ is non-increasing and bounded below then the sequence $-x_1, -x_2, -x_3, \ldots$ is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence $x_1, x_2, x_3, \ldots$ is also convergent. ∎

5.3 Upper and Lower Limits

Let $a_1, a_2, a_3, \ldots$ be a bounded infinite sequence of real numbers, and, for each positive integer j, let

$$S_j = \{a_j, a_{j+1}, a_{j+2}, \ldots\} = \{a_k : k \geq j\}.$$

The sets $S_1, S_2, S_3, \ldots$ are all bounded. It follows that there exist well-defined infinite sequences $u_1, u_2, u_3, \ldots$ and $l_1, l_2, l_3, \ldots$ of real numbers, where $u_j = \sup S_j$ and $l_j = \inf S_j$ for all positive integers j. Now S_{j+1} is a subset of S_j for each positive integer j, and therefore $u_{j+1} \leq u_j$ and $l_{j+1} \geq l_j$ for each positive integer j. It follows that the bounded infinite sequence $(u_j : j \in \mathbb{N})$ is a non-increasing sequence, and is therefore convergent (Theorem 5.1). Similarly the bounded infinite sequence $(l_j : j \in \mathbb{N})$ is a non-decreasing sequence, and is therefore convergent. We define

$$\limsup_{j \to +\infty} a_j = \lim_{j \to +\infty} u_j = \lim_{j \to +\infty} \sup\{a_j, a_{j+1}, a_{j+2}, \ldots\}$$

and

$$\liminf_{j \to +\infty} a_j = \lim_{j \to +\infty} l_j = \lim_{j \to +\infty} \inf\{a_j, a_{j+1}, a_{j+2}, \ldots\}.$$

The quantity $\limsup_{j\to+\infty} a_j$ is referred to as the *upper limit* of the sequence $a_1, a_2, a_3, \ldots$. The quantity $\liminf_{j\to+\infty} a_j$ is referred to as the *lower limit* of the sequence $a_1, a_2, a_3, \ldots$.

Note that every bounded infinite sequence $a_1, a_2, a_3, \ldots$. of real numbers has a well-defined upper limit $\limsup_{j\to+\infty} a_j$ and a well-defined lower limit $\liminf_{j\to+\infty} a_j$.

Proposition 5.2 *A bounded infinite sequence $a_1, a_2, a_3, \ldots$. of real numbers is convergent if and only if $\liminf_{j\to+\infty} a_j = \limsup_{j\to+\infty} a_j$, in which case the limit of the sequence is equal to the common value of its upper and lower limits.*

Proof For each positive integer j, let $u_j = \sup S_j$ and $l_j = \inf S_j$, where

$$S_j = \{a_j, a_{j+1}, a_{j+2}, \ldots\} = \{a_k : k \geq j\}.$$

Then $\liminf_{j\to+\infty} a_j = \lim_{j\to+\infty} l_j$ and $\limsup_{j\to+\infty} a_j = \lim_{j\to+\infty} u_j$.

Suppose that $\liminf_{j\to+\infty} a_j = \limsup_{j\to+\infty} a_j = c$ for some real number c. Then, given any positive real number ε, there exist positive integers N_1 and N_2 such that $c - \varepsilon < l_j \leq c$ whenever $j \geq N_1$, and $c \leq u_j < c + \varepsilon$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then $a_j \in S_N$, and therefore

$$c - \varepsilon < l_N \leq a_j \leq u_N < c + \varepsilon.$$

Thus $|a_j - c| < \varepsilon$ whenever $j \geq N$. This proves that the infinite sequence $a_1, a_2, a_3, \ldots$ converges to the limit c.

Conversely let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers that converges to some value c. Let $\varepsilon > 0$ be given. Then there exists some positive integer N such that $c - \frac{1}{2}\varepsilon < a_j < c + \frac{1}{2}\varepsilon$ whenever $j \geq N$. It follows that $S_j \subset (c - \frac{1}{2}\varepsilon, c + \frac{1}{2}\varepsilon)$ whenever $j \geq N$. But then

$$c - \tfrac{1}{2}\varepsilon \leq l_j \leq u_j \leq c + \tfrac{1}{2}\varepsilon$$

whenever $j \geq N$, where $u_j = \sup S_j$ and $l_j = \inf S_j$. We see from this that, given any positive real number ε, there exists some positive integer N such that $|l_j - c| < \varepsilon$ and $|u_j - c| < \varepsilon$ whenever $j \geq N$. It follows from this that

$$\limsup_{j\to+\infty} a_j = \lim_{j\to+\infty} u_j = c \text{ and } \liminf_{j\to+\infty} a_j = \lim_{j\to+\infty} l_j = c,$$

as required. ∎

5.4 Rearrangement of Infinite Series

Example Consider the infinite series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

For each positive integer k, let p_k denote the kth partial sum of this infinite series, defined so that

$$p_k = \sum_{j=1}^{k} (-1)^{j-1} \frac{1}{j}.$$

The absolute values of the summands constitute a decreasing sequence, and accordingly examination of the form of the infinite series establishes that

$$p_1 > p_3 > p_5 > p_7 > \cdots$$

$$p_2 < p_4 < p_6 < p_8 < \cdots$$

Moreover $p_{2m} \leq p_{2m+1} \leq p_1$ and $p_{2m+1} \geq p_{2m} \geq p_2$ for all positive integers m. Thus p_1, p_3, p_5, p_7 is a decreasing sequence that is bounded below, and p_2, p_4, p_6, p_8 is an increasing sequence that is bounded above. A standard result of real analysis ensures that these bounded monotonic sequences are convergent. Moreover

$$\begin{aligned}
\lim_{m \to +\infty} p_{2m+1} &= \lim_{m \to \infty} \left(p_{2m} + \frac{1}{2m+1} \right) \\
&= \lim_{m \to \infty} p_{2m} + \lim_{m \to +\infty} \frac{1}{2m+1} \\
&= \lim_{m \to \infty} p_{2m}.
\end{aligned}$$

It then follows easily from examination of the definition of convergence that the infinite sequence $p_1, p_2, p_3, \ldots$ converges, and

$$\lim_{j \to +\infty} p_j = \lim_{m \to +\infty} p_{2m} = \lim_{m \to +\infty} p_{2m+1}.$$

Let $\alpha = \lim\limits_{j \to +\infty} p_j$. Then $p_2 < \alpha < p_1$, and thus $\frac{1}{2} < \alpha < 1$.

Now consider the infinite series

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots.$$

The individual summands are those of the infinite series previously considered, but they occur in a different order. This new infinite series is thus a *rearrangement* of the infinite series previously considered. Nevertheless the sum of this new infinite series may be represented as

$$\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\cdots$$

and therefore the sum of the new infinite series is equal to that of the infinite series

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\cdots,$$

and is therefore equal to $\frac{1}{2}\alpha$. This example demonstrates that when the terms of an infinite series are rearranged, so that they are summed together in a different order, the sum of the rearranged series is not necessarily equal to that of the original series.

The example just discussed considers the behaviour of a particular infinite series that is convergent but not *absolutely convergent*. An infinite series $\sum_{j=1}^{+\infty} a_j$ is said to be *absolutely convergent* if $\sum_{j=1}^{+\infty} |a_j|$ is convergent. The following proposition and its corollaries ensure that any absolutely convergent infinite series may be rearranged at will without affecting convergence, and without changing the value of the sum of the series. In particular an infinite series whose summands are non-negative may be rearranged without affecting the value of the sum of that infinite series.

Proposition 5.3 *Let* $\sum_{j=1}^{+\infty} a_j$ *be a convergent infinite series, where* a_j *is real and* $a_j \geq 0$ *for all positive integers* j*. Let* Q *be the subset of the real numbers consisting of the values of all sums of the form* $\sum_{j\in F} a_j$ *obtained as* F *ranges over all the non-empty finite subsets of* $\mathbb{N}$*. Then*

$$\sum_{j=1}^{+\infty} a_j = \sup Q.$$

Proof For each positive integer k, let

$$p_k = \sum_{j=1}^{k} a_j.$$

This number p_k is referred to as the kth *partial sum* of the infinite series $a_1 + a_2 + a_3 + \cdots$. The definition of the sum of this infinite series then ensures that

$$\sum_{j=1}^{+\infty} a_j = \lim_{k\to+\infty} p_k.$$

Moreover $p_1 \leq p_2 \leq p_3 \leq \cdots$, because $a_j \geq 0$ for all positive integers j, and therefore

$$\sum_{j=1}^{+\infty} a_j = \sup\{p_k : k \in \mathbb{N}\}.$$

For each non-empty finite subset F of the set $\mathbb{N}$ of positive integers, let

$$q_F = \sum_{j \in F} a_j.$$

If F and H are finite subsets of $\mathbb{N}$, and if $F \subset H$ then $q_F \leq q_H$, because the summand a_j is non-negative for all positive integers j.

Now, given any non-empty finite subset F of $\mathbb{N}$ there exists some positive integer k such that $F \subset J_k$, where $J_k = \{1, 2, \ldots, k\}$. But then

$$q_F \leq q_{J_k} = p_k \leq \sum_{j=1}^{+\infty} a_j.$$

Therefore the set Q consisting of the values of the sums q_F as F ranges over all the non-empty finite subsets F of $\mathbb{N}$ is bounded above. Moreover it is non-empty. The Least Upper Bound Principle then ensures that the set Q has a well-defined least upper bound $\sup Q$.

Let $s = \sup Q$. We have shown that $q_F \leq \sum\limits_{j=1}^{+\infty} a_j$ for each non-empty finite subset F of $\mathbb{N}$. It follows that $s \leq \sum\limits_{j=1}^{+\infty} a_j$. But $p_k \in Q$ for all positive integers k, because $p_k = q_{J_k}$, and therefore $p_k \leq s$. Taking limits as $k \to +\infty$, we find that

$$\sum_{j=1}^{+\infty} a_j = \lim_{k \to +\infty} p_k \leq s.$$

The inequalities just obtained together ensure that

$$\sum_{j=1}^{+\infty} a_j = s = \sup Q,$$

as required. ∎

A *permutation* of the set $\mathbb{N}$ of positive integers is a function $\sigma: \mathbb{N} \to \mathbb{N}$ from the set $\mathbb{N}$ to itself that is bijective. A function $\sigma: \mathbb{N} \to \mathbb{N}$ is thus a permutation if and only if it has a well-defined inverse $\sigma^{-1}: \mathbb{N} \to \mathbb{N}$. This is the case if and only if, given any positive integer k, there exists a unique positive integer j for which $k = \sigma(j)$.

Definition An infinite sequence $b_1, b_2, b_3, \ldots$ of real numbers is said to be a *rearrangement* of an infinite sequence $a_1, a_2, a_3, \ldots$ if there exists a permutation σ of the set $\mathbb{N}$ of positive integers such that $b_k = a_{\sigma(k)}$ for all positive integers k. In this situation we also say that the infinite series $\sum_{k=1}^{+\infty} b_k$ is a rearrangement of the infinite series $\sum_{j=1}^{+\infty} a_j$.

Corollary 5.4 *Let $\sum_{j=1}^{+\infty} a_j$ be a convergent infinite series, and let $\sum_{k=1}^{+\infty} b_k$ be a rearrangement of infinite series $\sum_{j=1}^{+\infty} a_j$. Suppose that $a_j \geq 0$ for all positive integers j. Then the infinite series $\sum_{k=1}^{+\infty} b_k$ is convergent, and $\sum_{k=1}^{+\infty} b_k = \sum_{j=1}^{+\infty} a_j$.*

Proof There exists a permutation $\sigma\colon \mathbb{N} \to \mathbb{N}$ of the set $\mathbb{N}$ of positive integers such that $b_k = a_{\sigma(k)}$ for all positive integers k. Let $q_F = \sum_{j \in F} a_j$ for all non-empty finite subsets F of $\mathbb{N}$, and let $r_G = \sum_{k \in G} b_k$ for all non-empty finite subsets G of $\mathbb{N}$. Then

$$q_{\sigma(G)} = \sum_{j \in \sigma(G)} a_j = \sum_{k \in G} a_{\sigma(k)} = \sum_{k \in G} b_k = r_G$$

for all non-empty finite subsets G of $\mathbb{N}$, and accordingly $q_F = r_{\sigma^{-1}(F)}$ for all non-empty finite subsets F of $\mathbb{N}$. Moroever G is a non-empty finite subset of $\mathbb{N}$ if and only if $\sigma(G)$ is a non-empty finite subset of $\mathbb{N}$. It follows that the set Q consisting of all sums of the form q_F as F ranges over the non-empty finite subsets of $\mathbb{N}$ is also the set consisting of all sums of the form r_G as G ranges over the non-empty finite subsets of $\mathbb{N}$. It follows from Proposition 5.3 that

$$\sum_{j=1}^{+\infty} a_j = \sup Q = \sum_{k=1}^{+\infty} b_k,$$

as required. ∎

It follows from Corollary 5.4 that, given any collection $(c_\alpha : \alpha \in A)$ of *non-negative* real numbers c_α indexed by the members of a countable set A, we can form the sum $\sum_{\alpha \in A} c_\alpha$. If the countable indexing set A is infinite then there exists an infinite sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ in which each element of the set A occurs exactly once. Then

$$\sum_{\alpha \in A} c_\alpha = \sum_{j=1}^{+\infty} c_{\alpha_j}.$$

The requirement that $c_\alpha \geq 0$ for all $\alpha \in A$ ensures that the value of $\sum\limits_{j=1}^{+\infty} c_{\alpha_j}$ does not depend on the choice of infinite sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ enumerating the elements of the indexing set A.

Let $c_1, c_2, c_3, \ldots$ be an infinite sequence of real numbers that are not necessarily all non-negative or all non-positive, and let $c_j^+ = \max(c_j, 0)$ and $c_j^- = \min(0, c_j)$ for all positive integers j. Then $c_j^+ \geq 0$, $c_j^- \leq 0$, $c_j = c_j^+ + c_j^-$ and $|c_j| = c_j^+ - c_j^- = c_j^+ + |c_j^-|$ for all positive integers j. Moreover, for each positive integer j, at most one of the numbers c_j^+ asnd c_j^- is non-zero. Now $0 \leq c_j^+ \leq |c_j|$ and $0 \leq |c_j^-| \leq |c_j|$ for all positive integers j. It follows from this that $\sum\limits_{j=1}^{+\infty} |c_j|$ is convergent if and only if both $\sum\limits_{j=1}^{+\infty} c_j^+$ and $\sum\limits_{j=1}^{+\infty} c_j^-$ convergent. In this case we say that the infinite series $\sum_{j=1}^{+\infty} c_j$ is *absolutely convergent*.

Corollary 5.5 *Let $\sum\limits_{j=1}^{+\infty} a_j$ be an absolutely convergent infinite series, and let $\sum\limits_{k=1}^{+\infty} b_k$ be a rearrangement of infinite series $\sum\limits_{j=1}^{+\infty} a_j$. Then the infinite series $\sum\limits_{k=1}^{+\infty} b_k$ is absolutely convergent, and $\sum\limits_{k=1}^{+\infty} b_k = \sum\limits_{j=1}^{+\infty} a_j$.*

Proof There exists a permutation $\sigma \colon \mathbb{N} \to \mathbb{N}$ of the set $\mathbb{N}$ of positive integers with the property that $b_k = a_{\sigma(k)}$ for all positive integers k. Let $a_j^+ = \max(a_j, 0)$ and $a_j^- = \min(0, a_j)$ for all positive integers j and $b_k^+ = \max(b_k, 0)$ and $b_k^- = \min(0, b_k)$ for all positive integers k. The absolute convergence of $\sum\limits_{j=1}^{\infty} a_j$ then ensures that the infinite series $\sum\limits_{j=1}^{\infty} a_j^+$ and $\sum\limits_{j=1}^{\infty} a_j^-$ both converge. It then follows from Corollary 5.4 that

$$\sum_{j=1}^{+\infty} |a_j| = \sum_{j=1}^{+\infty} a_j^+ - \sum_{j=1}^{+\infty} a_j^- = \sum_{k=1}^{+\infty} b_k^+ - \sum_{k=1}^{+\infty} b_k^- = \sum_{k=1}^{+\infty} |b_k|$$

and

$$\sum_{j=1}^{+\infty} a_j = \sum_{j=1}^{+\infty} a_j^+ + \sum_{j=1}^{+\infty} a_j^- = \sum_{k=1}^{+\infty} b_k^+ + \sum_{k=1}^{+\infty} b_k^- = \sum_{k=1}^{+\infty} b_k.$$

The result follows. ∎

5.5 The Extended Real Number System

It is sometimes convenient to make use of the *extended real line* $[-\infty, +\infty]$. This is the set $\mathbb{R} \cup \{-\infty, +\infty\}$ obtained on adjoining to the real line $\mathbb{R}$ two extra elements $+\infty$ and $-\infty$ that represent points at 'positive infinity' and 'negative infinity' respectively. We define

$$c + (+\infty) = (+\infty) + c = +\infty$$

and

$$c + (-\infty) = (-\infty) + c = -\infty$$

for all real numbers c. We also define products of non-zero real numbers with these extra elements $\pm\infty$ so that

$$\begin{array}{rcll} c \times (+\infty) & = & (+\infty) \times c = +\infty & \text{when } c > 0, \\ c \times (-\infty) & = & (-\infty) \times c = -\infty & \text{when } c > 0, \\ c \times (+\infty) & = & (+\infty) \times c = -\infty & \text{when } c < 0, \\ c \times (-\infty) & = & (-\infty) \times c = +\infty & \text{when } c < 0, \end{array}$$

We also define

$$0 \times (+\infty) = (+\infty) \times 0 = 0 \times (-\infty) = (-\infty) \times 0 = 0,$$

and

$$(+\infty) \times (+\infty) = (-\infty) \times (-\infty) = +\infty,$$

$$(+\infty) \times (-\infty) = (-\infty) \times (+\infty) = -\infty.$$

The sum of $+\infty$ and $-\infty$ is not defined. We define $-(+\infty) = -\infty$ and $-(-\infty) = +\infty$. The difference $p - q$ of two extended real numbers is then defined by the formula $p - q = p + (-q)$, unless $p = q = +\infty$ or $p = q = -\infty$, in which cases the difference of the extended real numbers p and q is not defined.

We extend the definition of inequalities to the extended real line in the obvious fashion, so that $c < +\infty$ and $c > -\infty$ for all real numbers c, and $-\infty < +\infty$.

Given any real number c, we define

$$\begin{array}{rcl} [c, +\infty] & = & [c, +\infty) \cup \{+\infty\} = \{p \in [-\infty, \infty] : p \geq c\}, \\ (c, +\infty] & = & (c, +\infty) \cup \{+\infty\} = \{p \in [-\infty, \infty] : p > c\}, \\ [-\infty, c] & = & (-\infty, c] \cup \{-\infty\} = \{p \in [-\infty, \infty] : p \leq c\}, \\ [-\infty, c) & = & (-\infty, c) \cup \{-\infty\} = \{p \in [-\infty, \infty] : p < c\}. \end{array}$$

There is an order-preserving bijective function $\varphi\colon [-\infty,+\infty] \to [-1,1]$ from the extended real line $[-\infty,+\infty]$ to the closed interval $[-1,1]$ which is defined such that $\varphi(+\infty) = 1$, $\varphi(-\infty) = -1$, and $\varphi(c) = \dfrac{c}{1+|c|}$ for all real numbers c. Let us define $\rho(p,q) = |\varphi(q) - \varphi(p)|$ for all extended real numbers p and q. Then the set $[-\infty,+\infty]$ becomes a metric space with distance function ρ. Moreover the function $\varphi\colon [-\infty,+\infty] \to [-1,1]$ is a homeomorphism from this metric space to the closed interval $[-1,1]$. It follows directly from this that $[-\infty,+\infty]$ is a compact metric space. Moreover an infinite sequence $(p_j : j \in \mathbb{N})$ of extended real numbers is convergent if and only if the corresponding sequence $(\varphi(p_j) : j \in \mathbb{N})$ of real numbers is convergent.

Given any non-empty set S of extended real numbers, we can define $\sup S$ to be the least extended real number p with the property that $s \leq p$ for all $s \in S$. If the set S does not contain the extended real number $+\infty$, and if there exists some real number B such that $s \leq B$ for all $s \in S$, then $\sup S < +\infty$; otherwise $\sup S = +\infty$. Similarly we define $\inf S$ to be the greatest extended real number p with the property that $s \geq p$ for all $s \in S$. If the set S does not contain the extended real number $-\infty$, and if there exists some real number A such that $s \geq A$ for all $s \in S$, then $\inf S > +\infty$; otherwise $\inf S = -\infty$. Moreover

$$\varphi(\sup S) = \sup \varphi(S) \text{ and } \varphi(\inf S) = \inf \varphi(S),$$

where $\varphi\colon [-\infty,+\infty] \to [-1,1]$ is the homeomorphism defined such that $\varphi(+\infty) = 1$, $\varphi(-\infty) = -1$ and $\varphi(c) = c(1+|c|)^{-1}$ for all real numbers c.

Given any sequence $(p_j : j \in \mathbb{N})$ of extended real numbers, we define the *upper limit* $\limsup_{j\to+\infty} p_j$ and the *lower limit* $\liminf_{j\to+\infty} p_j$ of the sequence so that

$$\limsup_{j\to+\infty} p_j = \lim_{j\to+\infty} \sup\{p_k : k \geq j\}$$

and

$$\liminf_{j\to+\infty} p_j = \lim_{j\to+\infty} \inf\{p_k : k \geq j\}.$$

Every sequence of extended real numbers has both an upper limit and a lower limit. Moreover an infinite sequence of extended real numbers converges to some extended real number if and only if the upper and lower limits of the sequence are equal. (These results follow easily from the corresponding results for bounded sequences of real numbers, on using the identities

$$\varphi(\limsup_{j\to+\infty} p_j) = \limsup_{j\to+\infty} \varphi(p_j), \quad \varphi(\liminf_{j\to+\infty} p_j) = \liminf_{j\to+\infty} \varphi(p_j),$$

where $\varphi\colon [-\infty,+\infty] \to [-1,1]$ is the homeomorphism defined above.)

The function that sends a pair (p, q) of extended real numbers to the extended real number $p + q$ is not defined when $p = +\infty$ and $q = -\infty$, or when $p = -\infty$ and $q = +\infty$ but is continuous elsewhere. The function that sends a pair (p, q) of extended real numbers to the extended real number pq is defined everywhere. This function is discontinuous when $p = \pm\infty$ and $q = 0$, and when $p = 0$ and $q = \pm\infty$. It is continuous for all other values of the extended real numbers p and q.

Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of extended real numbers which does not include both the values $+\infty$ and $-\infty$, and let $p_k = \sum_{j=0}^{k} a_j$ for all natural numbers k. If the infinite sequence $p_1, p_2, p_3, \ldots$ of extended real numbers converges in the extended real line $[-\infty, +\infty]$ to some extended real number p, then this value p is said to be the *sum* of the infinite series $\sum\limits_{j=1}^{+\infty} a_j$, and we write $\sum\limits_{j=1}^{+\infty} a_j = p$.

It follows easily from this definition that if $+\infty$ is one of the values of the infinite series $a_1, a_2, a_3, \ldots$, then $\sum\limits_{j=1}^{+\infty} a_j = +\infty$. Similarly if $-\infty$ is one of the values of this infinite series then then $\sum\limits_{j=1}^{+\infty} a_j = -\infty$. Suppose that the members of the sequence $a_1, a_2, a_3, \ldots$ are all real numbers. Then $\sum\limits_{j=1}^{+\infty} a_n = +\infty$ if and only if, given any real number B, there exists some real number N such that $\sum\limits_{j=1}^{k} a_n > B$ whenever $k \geq N$. Similarly $\sum\limits_{j=1}^{+\infty} a_j = -\infty$ if and only if, given any real number A, there exists some real number N such that $\sum\limits_{j=1}^{k} a_j < A$ whenever $k \geq N$.

6 Semirings and Rings of Sets

6.1 Distributive Laws in Set Theory

We explore some basic properties of intersections and products of unions of sets that are manifestations of distributive laws satisfied in basic set theory.

Proposition 6.1 *Let $A_1, A_2, \ldots, A_n$ be sets and, for each integer i between 1 and n, let $(D_{i,j} : j \in J_i)$ be a collection of subsets of A_i, indexed by some set J_i, for which $\bigcup_{j \in J_i} D_{i,j} = A_i$. Let*

$$J = J_1 \times J_2 \times \cdots \times J_n.$$

Then

$$\bigcap_{i=1}^{n} A_i = \bigcup_{(j_1, j_2, \ldots, j_n) \in J} E_{j_1, j_2, \ldots, j_n},$$

where

$$E_{j_1, j_2, \ldots, j_n} = D_{1,j_1} \cap D_{2,j_2} \cap \cdots \cap D_{n,j_n}$$

for each $(j_1, j_2, \ldots, j_n) \in J$. Moreover if, for each integer i between 1 and n, the sets $D_{i,j}$ with $j \in J_i$ are pairwise disjoint, then the sets $E_{j_1, j_2, \ldots, j_n}$ with $(j_1, j_2, \ldots, j_n) \in J$ are pairwise disjoint.

Proof Let x be an element of $E_{j_1, j_2, \ldots, j_n}$ for some element $(j_1, j_2, \ldots, j_n)$ of the indexing set J. Then $x \in A_i$ for each integer between i and n, because

$$E_{j_1, j_2, \ldots, j_n} \subset D_{i,j_i} \subset A_i,$$

and therefore $x \in \bigcap_{i=1}^{n} A_i$.

Now let x be an element of $\bigcap_{i=1}^{n} A_i$. Then $x \in A_i$ for each integer i between 1 and n and therefore, for each integer i between 1 and n there exists an element j_i of the indexing set J_i for which $x \in D_{i,j_i}$. But then $x \in E_{j_1, j_2, \ldots, j_n}$. These results establish that

$$\bigcap_{i=1}^{n} A_i = \bigcup_{(j_1, j_2, \ldots, j_n) \in J} E_{j_1, j_2, \ldots, j_n}.$$

Now suppose that, for each integer i between 1 and n, the sets $D_{i,j}$ with $j \in J_i$ are pairwise disjoint. Let $(j_1, j_2, \ldots, j_n) \in J$ and $(j'_1, j'_2, \ldots, j'_n) \in J$. Suppose that

$$(j_1, j_2, \ldots, j_n) \neq (j'_1, j'_2, \ldots, j'_n).$$

Then there is some integer i between 1 and n for which $j_i \neq j'_i$. But then

$$E_{j_1,j_2,\ldots,j_n} \subset D_{i,j_i}, \quad E_{j'_1,j'_2,\ldots,j'_n} \subset D_{i,j'_i}$$

and $D_{j_1} \cap D_{i,j'_i} = \emptyset$. It follows that

$$E_{j_1,j_2,\ldots,j_n} \cap E_{j'_1,j'_2,\ldots,j'_n} = \emptyset.$$

Thus the sets $E_{j_1,j_2,\ldots,j_n}$ with $(j_1, j_2, \ldots, j_n) \in J$ are pairwise disjoint, as required. ∎

Corollary 6.2 *Let $A_1, A_2, \ldots, A_n$ be subsets of a set X and, for each integer i between 1 and n, let $D_{i,1}, D_{i,2}, \ldots, D_{i,q(i)}$ be pairwise disjoint subsets of X satisfying $\bigcup_{j_i=1}^{q(i)} D_{i,j_i} = A_i$. Let the indexing set J consist of those n-tuples $(j_1, j_2, \ldots, j_n)$ that satisfy $1 \leq j_i \leq q(i)$ for $i = 1, 2, \ldots, n$. Then*

$$\bigcap_{i=1}^{n} A_i = \bigcup_{(j_1,j_2,\ldots,j_n)\in J} E_{j_1,j_2,\ldots,j_n}$$

where $E_{j_1,j_2,\ldots,j_n} = D_{i,1} \cap D_{i,2} \cap \cdots \cap D_{i,q(i)}$ for all $(j_1, j_2, \ldots, j_n) \in J$. Moreover the sets $E_{j_1,j_2,\ldots,j_n}$ with $(j_1, j_2, \ldots, j_n) \in J$ are pairwise disjoint.

Proposition 6.3 *Let $A_1, A_2, \ldots, A_n$ be sets and, each integer i between 1 and n, let $(D_{i,j} : j \in J_i)$ be a collection of subsets of A_i, indexed by some set J_i, for which $\bigcup_{j\in J_i} D_{i,j} = A_i$. Let*

$$J = J_1 \times J_2 \times \cdots \times J_n.$$

Then

$$A_1 \times A_2 \times \cdots \times A_n = \bigcup_{(j_1,j_2,\ldots,j_n)\in J} F_{j_1,j_2,\ldots,j_n},$$

where

$$F_{j_1,j_2,\ldots,j_n} = D_{1,j_1} \times D_{2,j_2} \times \cdots \times D_{n,j_n}$$

for each $(j_1, j_2, \ldots, j_n) \in J$. Moreover if, for each integer i between 1 and n, the sets $D_{i,j}$ with $j \in J_i$ are pairwise disjoint, then the sets $F_{j_1,j_2,\ldots,j_n}$ with $(j_1, j_2, \ldots, j_n) \in J$ are pairwise disjoint.

Proof Let $(j_1, j_2, \ldots, j_n)$ be an n-tuple of integers belonging to the indexing set J, and let $(x_1, x_2, \ldots, x_n)$ be an element of the corresponding set $F_{j_1,j_2,\ldots,j_n}$. Then $x_i \in A_i$ for each integer between i and n, and therefore

$$(x_1, x_2, \ldots, x_n) \in A_1 \times A_2 \times \cdots \times A_n.$$

Now let $(x_1, x_2, \ldots, x_n)$ be an element of $A_1 \times A_2 \times \cdots \times A_n$. Then $x_i \in A_i$ for each integer i between 1 and n and therefore, for each integer i between 1 and n there exists an element j_i of the indexing set J_i for which $x_i \in D_{i,j_i}$. But then

$$(x_1, x_2, \ldots, x_n) \in F_{j_1,j_2,\ldots,j_n}.$$

These results establish that

$$A_1 \times A_2 \times \cdots \times A_n = \bigcup_{(j_1,j_2,\ldots,j_n)\in J} F_{j_1,j_2,\ldots,j_n}.$$

Now suppose that, for each integer i between 1 and n, the sets $D_{i,j}$ with $j \in J_i$ are pairwise disjoint. Let $(j_1, j_2, \ldots, j_n) \in J$ and $(j'_1, j'_2, \ldots, j'_n) \in J$. Suppose that

$$(j_1, j_2, \ldots, j_n) \neq (j'_1, j'_2, \ldots, j'_n).$$

Then there is some integer i between 1 and n for which $j_i \neq j'_i$. Suppose there were to exist some element $(x_1, x_2, \ldots, x_n)$ of $A_1 \times A_2 \times \cdots \times A_n$ that belonged to both $F_{j_1,j_2,\ldots,j_n}$ and $F_{j'_1,j'_2,\ldots,j'_n}$. Then it would be the case that both $x_i \in D_{i,j_i}$ and $x_i \in D_{i,j'_i}$. But this is not possible, because $D_{i,j_i} \cap D_{i,j'_i} = \emptyset$. It follows that

$$F_{j_1,j_2,\ldots,j_n} \cap F_{j'_1,j'_2,\ldots,j'_n} = \emptyset.$$

Thus the sets $F_{j_1,j_2,\ldots,j_n}$ with $(j_1, j_2, \ldots, j_n) \in J$ are pairwise disjoint, as required. ∎

Corollary 6.4 *Let $A_1, A_2, \ldots, A_n$ be sets, and, for each integer i between 1 and n, let $D_{i,1}, D_{i,2}, \ldots, D_{i,q(i)}$ be pairwise disjoint subsets of A_i satisfying $\bigcup_{j_i=1}^{q(i)} D_{i,j_i} = A_i$. Let the indexing set J consist of those n-tuples $(j_1, j_2, \ldots, j_n)$ that satisfy $1 \leq j_i \leq q(i)$ for $i = 1, 2, \ldots, n$. Then*

$$A_1 \times A_2 \times \cdots \times A_n = \bigcup_{(j_1,j_2,\ldots,j_n)\in J} F_{j_1,j_2,\ldots,j_n}$$

where

$$F_{j_1,j_2,\ldots,j_n} = D_{1,j_1} \times D_{2,j_2} \times \cdots \times D_{n,j_n}$$

for all $(j_1, j_2, \ldots, j_n) \in J$. Moreover the sets $F_{j_1,j_2,\ldots,j_n}$ with $(j_1, j_2, \ldots, j_n) \in J$ are pairwise disjoint.

6.2 Semirings of Sets

Definition Let X be a set. A collection $\mathcal{S}$ of subsets of X is said to be a *semiring* of subsets of X if it satisfies the following two properties:—

1. (i) the empty set $\emptyset$ belongs to $\mathcal{S}$;
2. (ii) the intersection $A \cap B$ of any two members A and B of $\mathcal{S}$ belongs to $\mathcal{S}$.
3. (iii) The difference $A \setminus B$ of any two members of $\mathcal{S}$ can be represented as a finite union of pairwise disjoint members of $\mathcal{S}$.

A set is said to be a *singleton set* (or a *one-point set*) if it has only one element.

Lemma 6.5 *Let $\mathcal{J}$ be the collection of subsets of the real line $\mathbb{R}$ consisting of the empty set, all the singleton sets, and all bounded intervals in $\mathbb{R}$. Then $\mathcal{J}$ is a semiring of subsets of $\mathbb{R}$.*

Proof The specification of the collection $\mathcal{J}$ ensures that the empty set $\emptyset$ is a member of $\mathcal{J}$.

If A is equal to the empty set then $A \cap B$ and $A \setminus B$ are both equal to the empty set for all members B of the collection $\mathcal{J}$. It follows that $A \cap B$ and $A \setminus B$ are members of $\mathcal{J}$ in all cases where $A = \emptyset$ and B is a member of $\mathcal{J}$. Also if A is a member of the collection $\mathcal{J}$, and if $B = \emptyset$ then $A \cap B$ and $A \setminus B$, being equal to $\emptyset$ and A respectively, are members of the collection $\mathcal{J}$. If A and B are members of $\mathcal{J}$, and if A is a singleton set (so that $A = \{c\}$ for some real number c) then $A \cap B$ and $A \setminus B$ belong to $\mathcal{J}$, because each of those sets, if non-empty, is equal to the singleton set A. It suffices therefore to verify that $A \cap B$ is a member of $\mathcal{J}$, and that $A \setminus B$ is a finite union of pairwise disjoint members of $\mathcal{J}$, in those cases where A is a bounded interval and B is either a singleton set or a bounded interval.

The intersection of a bounded interval and a singleton set is either a singleton set or is equal to the empty set. Therefore the intersection of a bounded interval and a singleton set is a member of $\mathcal{J}$. The intersection of two bounded intervals is a bounded interval, a singleton set or the empty set. It follows that the intersection of any two bounded intervals is a member of $\mathcal{J}$. We conclude that $A \cap B$ is a member of $\mathcal{J}$ in all cases where A is a bounded interval and B is either a singleton set or else a bounded interval. It only remains therefore to verify that $A \setminus B$ is a member of $\mathcal{J}$ in those cases where A is a bounded interval and B is either a singleton set or a bounded interval.

Let A be a bounded interval, and let B be either a singleton set or a bounded interval. Then the complement $\mathbb{R} \setminus B$ of B is the disjoint union of two unbounded intervals L and R, where

$$\begin{aligned} L &= \{x \in \mathbb{R} : x < y \text{ for all } y \in B\}, \\ R &= \{x \in \mathbb{R} : x > y \text{ for all } y \in B\}. \end{aligned}$$

But then

$$A \setminus B = A \cap (\mathbb{R} \setminus B) = (A \cap L) \cup (A \cap R).$$

Thus $A \setminus B$ is the disjoint union of two bounded sets $A \cap L$ and $A \cap R$. Moreover each of these sets $A \cap L$ and $A \cap R$ is a bounded interval, a singleton set or is equal to the empty set. Thus either $A \setminus B$ is a member of $\mathcal{J}$ or else $A \setminus B$ is the disjoint union of two disjoint non-empty members of $\mathcal{J}$ in all cases where A is a bounded interval and B is either a singleton set or a bounded interval. This completes the proof that, for all members A and B of $\mathcal{J}$, the difference $A \setminus B$ is expressible as the union of a finite collection of pairwise disjoint members of $\mathcal{J}$. The result follows. ∎

Lemma 6.6 *Let $\mathcal{S}$ be a semiring of subsets of some given set X. Then, for all members A and B of the semiring $\mathcal{S}$, the union $A \cup B$ of A and B is expressible as a finite union of pairwise disjoint members of the semiring $\mathcal{S}$.*

Proof It follows from the definition of semirings of sets that $A \cap B$ is a member of $\mathcal{S}$ and that $A \setminus B$ and $B \setminus A$ can both be expressed as finite unions of parwise disjoint members of the semiring $\mathcal{S}$. The union $A \cup B$ of the sets A and B is the disjoint union of the sets $A \cap B$, $A \setminus B$ and $B \setminus A$. The result follows directly. ∎

Lemma 6.7 *Let $\mathcal{S}$ be a semiring of subsets of some given set X, and let $A_1, A_2, \ldots, A_k$ be subsets of X. Suppose that each of the sets $A_1, A_2, \ldots, A_k$ can be represented as a finite union of pairwise disjoint members of the semiring $\mathcal{S}$. Then the intersection $\bigcap_{i=1}^{k} A_i$ of those sets can also be so expressed.*

Proof There exist positive integers $q(1), q(2), \ldots, q(k)$ and members $B_{i,j}$ of $\mathcal{S}$ for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, q(k)$ so that, for each integer i between 1 and k, the sets $B_{i,1}, \ldots, B_{i,q(i)}$ are pairwise disjoint, and

$$A_i = \bigcup_{j=1}^{q(i)} B_{i,j}.$$

Let F be the set consisting of all k-tuples $(j_1, j_2, \ldots, j_k)$ of integers satisfying $1 \leq j_i \leq q(i)$ for $i = 1, 2, \ldots, k$, and, for each $(j_1, j_2, \ldots, j_k) \in F$, let

$$C_{j_1,j_2,\ldots,j_k} = B_{1,j_1} \cap B_{2,j_2} \cap \cdots \cap B_{k,j_k}.$$

Then each set $C_{j_1,j_2,\ldots,j_k}$, being a finite intersection of members of the semiring $\mathcal{S}$, is itself a member of $\mathcal{S}$.

Let $(j_1, j_2, \ldots, j_k)$ and $(j'_1, j'_2, \ldots, j'_k)$ be distinct k-tuples belonging to F. There is at least one integer i between 1 and k for which $j_i \neq j'_i$. Then

$$C_{j_1,j_2,\ldots,j_k} \subset B_{i,j_i} \quad \text{and} \quad C_{j'_1,j'_2,\ldots,j'_k} \subset B_{i,j'_i},$$

and $B_{i,j_i} \cap B_{i,j'_i} = \emptyset$, because the sets $B_{i,j}$ are pairwise disjoint for $1 \leq j \leq q(i)$, and therefore

$$C_{j_1,j_2,\ldots,j_k} \cap C_{j'_1,j'_2,\ldots,j'_k} = \emptyset.$$

It follows that the collection of subsets of X consisting of the sets $C_{j_1,j_2,\ldots,j_k}$ for $(j_1, j_2, \ldots, j_k) \in J$ is a finite collection of pairwise disjoint members of the semiring $\mathcal{S}$.

Let $x \in A_1 \cap A_2 \cap \cdots \cap A_k$. Then, for each integer i between 1 and k, there is exactly one integer j_i between 1 and $q(i)$ for which $x \in B_{i,j_i}$. Then $x \in C_{j_1,j_2,\ldots,j_k}$. We have therefore completed the verification that the intersection $A_1 \cap A_2 \cap \cdots \cap A_k$ of the sets $A_1, A_2, \ldots, A_k$ is a finite union of pairwise disjoint members of the semiring $\mathcal{S}$. The result follows. ∎

Proposition 6.8 *Let $\mathcal{S}$ be a semiring of subsets of a given set, and let $A_1, A_2, \ldots, A_k$ be a finite list of members of the semiring $\mathcal{S}$. Then there exists a finite list $B_1, B_2, \ldots, B_m$ of members of $\mathcal{S}$ satisfying the following properties:*

(i) the sets $B_1, B_2, \ldots, B_m$ are pairwise disjoint;

(ii) each member of the list $A_1, A_2, \ldots, A_k$ is expressible as a union of sets belonging to the list $B_1, B_2, \ldots, B_m$.

Proof We can prove this result by induction on the number k of members of A occurring in the list $A_1, A_2, \ldots, A_k$. The result is clearly true when $k = 1$. We may therefore suppose as our induction hypothesis that $k > 1$ and that there exists some finite list $C_1, C_2, \ldots, C_p$ of pairwise disjoint members of $\mathcal{S}$ such that each of the sets A_i with $i < k$ may be expressed as a union of members of $\mathcal{S}$ that occur in the list $C_1, C_2, \ldots, C_p$.

For each integer i between 1 and p for which $C_i \subset A_k$ let $q(i) = 1$. For all other integers i between 1 and p there exist pairwise disjoint sets $D_{i,2}, \ldots, D_{i,q(i)}$ belonging to $\mathcal{S}$, where $q(i) > 1$, such that

$$C_i \setminus A_k = \bigcup_{j=2}^{q(i)} D_{i,j}.$$

Let $D_{i,1} = C_i \cap A_k$ for all integers i between 1 and p. Then, for each integer i between 1 and p, the sets $D_{i,1}, \ldots, D_{i,q(i)}$ are pairwise disjoint members of the semiring $\mathcal{S}$ and

$$C_i = \bigcup_{j=1}^{q(i)} D_{i,j}.$$

Let J denote the set of ordered pairs of positive integers (i,j) for which $1 \leq i \leq p$ and $1 \leq j \leq q(i)$, and let $\mathcal{G}$ denote the finite collection of members of the semiring $\mathcal{S}$ consisting of the sets $D_{i,j}$ for which $(i,j) \in J$. Each set C_i is the union of those sets $D_{i,j}$ for which $(i,j) \in J$. If $(i,j) \in J$ and $(i',j') \in J$ and if $i \neq i'$ then $D_{i,j} \subset C_i$, $D_{i',j'} \subset C_{i'}$ and $C_i \cap C_{i'} = \emptyset$, and therefore $D_{i,j} \cap D_{i',j'} = \emptyset$. If $(i,j) \in J$ and $(i',j') \in J$, where $i = i'$ and $j \neq j'$ then $D_{i,j} \cap D_{i',j'} = \emptyset$, because the sets $D_{i,1}, \ldots, D_{i,q(i)}$ are pairwise disjoint. It follows that $\mathcal{G}$ is a finite collection of pairwise disjoint members of the semiring S, and moreover each of the sets C_i is expressible as a union of sets belonging to the collection $\mathcal{G}$. Now each set A_i with $i < k$ can be expressed as a union of sets belonging to the list $C_1, C_2, \ldots, C_p$, and moreover each set C_i in this list is in turn expressible as a union of sets belonging to the collection $\mathcal{G}$. It follows that each of the sets $A_1, \ldots, A_{k-1}$ is expressible as a union of sets belonging to $\mathcal{G}$. Moreover each set belonging to $\mathcal{G}$ is a subset of E, where $E = \bigcup\limits_{i=1}^{p} C_i$, and

$$\bigcup_{i=1}^{p} D_{i,1} = \bigcup_{i=1} (A_k \cap C_i) = A_k \cap \left(\bigcup_{i=1}^{p} C_i \right) = A_k \cap E.$$

It follows therefore that $A_k \cap E$ is expressible as a union of sets belonging to the collection $\mathcal{G}$.

We now show that $A_k \setminus E$ can be represented as the union of a finite collection of pairwise disjoint members of the semiring $\mathcal{S}$. Now

$$A_k \setminus E = A_k \setminus \bigcup_{i=1}^{p} C_i = \bigcap_{i=1}^{p} (A_k \setminus C_i).$$

Each of the sets $A_k \setminus C_i$ is expressible as a finite union of pairwise disjoint sets belonging to the semiring $\mathcal{S}$. The intersection of those sets is therefore also expressible as a finite union of pairwise disjoint sets belonging to the semiring $\mathcal{S}$. (see Lemma 6.7). Accordingly let $\mathcal{H}$ be a finite collection of pairwise disjoint members of the semiring $\mathcal{S}$ whose union is the set $A_k \setminus E$, and let $\mathcal{K} = \mathcal{G} \cup \mathcal{H}$, so that $\mathcal{K}$ is the consisting of those members of the semiring $\mathcal{S}$ that belong either to $\mathcal{G}$ or to $\mathcal{H}$. Now the sets belonging to $\mathcal{K}$ are pairwise disjoint, since those sets belonging to $\mathcal{G}$ are pairwise disjoint subsets of E and those sets belonging to $\mathcal{H}$ are pairwise disjoint subsets of the complement of E. Moreover

$$A_k = (A_k \cap E) \cup (A_k \setminus E),$$

and we have shown that the sets $A_k \cap E$ and $A_k \setminus E$ are expressible as unions of sets belonging to $\mathcal{G}$ and $\mathcal{H}$ respectively. It follows that A_k is expressible as a union of sets belonging to $\mathcal{K}$. We have shown that the sets $A_1, A_2, \ldots, A_{k-1}$ are also expressible as unions of sets belonging to $\mathcal{G}$. The result follows. ∎

Corollary 6.9 *Let $\mathcal{S}$ be a semiring of subsets of some given set X, and let $A_1, A_2, \ldots, A_k$ be subsets of X. Suppose that each of the sets $A_1, A_2, \ldots, A_k$ can be represented as a finite union of pairwise disjoint members of the semiring $\mathcal{S}$. Then the union $\bigcup_{i=1}^{k} A_i$ of those sets can also be so expressed.*

Proof There is a some finite list $C_1, C_2, \ldots, C_k$ of members of the semiring $\mathcal{S}$ with the property that, for each integer i between 1 and k, the set A_i is a union of some of the sets occurring in the list $C_1, C_2, \ldots, C_k$. There then exists a finite collection $\mathcal{G}$ consisting of pairwise disjoint members $D_1, D_2, \ldots, D_m$ of the semiring $\mathcal{S}$ with the property that, for each integer i between 1 and k, the set C_i is expressible as a union of members of $\mathcal{S}$ belonging to the collection $\mathcal{G}$ (see Proposition 6.8). Then each of the sets $A_1, A_2, \ldots, A_k$ is expressible as a union of members of $\mathcal{S}$ belonging to $\mathcal{G}$. It follows that $\bigcup_{i=1}^{k} A_k$ is also expressible as a union of members of $\mathcal{S}$ belonging to $\mathcal{G}$. The members of the collection $\mathcal{G}$ are pairwise disjoint. The result follows. ∎

Corollary 6.10 *Let X be a set, let $\mathcal{S}$ be a semiring of subsets of X, and let A and B be subsets of X each of which is expressible as a finite union of pairwise disjoint members of the semiring $\mathcal{S}$. Then $A \setminus B$ can also be so expressed.*

Proof There is some finite collection $\mathcal{G}$ consisting of of pairwise disjoint members $D_1, D_2, \ldots, D_m$ of the semiring $\mathcal{S}$ determined so as to ensure that each of the sets A and B is expressible as a union of members of $\mathcal{S}$ belonging to $\mathcal{G}$ (see Proposition 6.8).

If a set D_j in the collection $\mathcal{G}$ is not a subset of A then it must be disjoint from all those members of the collection $\mathcal{G}$ that are contained in A. But the union of those members of the collection $\mathcal{G}$ that are contained in A is the whole of the set A. It follows that, for each integer j between 1 and m, either $D_j \subset A$ or else $D_j \cap A = \emptyset$. Similarly, for each integer j between 1 and m, either $D_j \subset B$ or else $D_j \cap B = \emptyset$. But then $A \setminus B$ is the disjoint union of those sets D_j for which $D_j \subset A$ and $D_j \cap B = \emptyset$. Indeed let $x \in A \setminus B$. Then there is exactly one integer j between 1 and m for which $x \in D_j$, because A is a union of sets belonging to the list $D_1, D_2, \ldots, D_m$ of pairwise disjoint members of $\mathcal{S}$. Thus $D_j \cap A$ is non-empty, and therefore $D_j \subset A$. But also $x \notin B$. It follows that D_j cannot be a subset of B, and therefore $D_j \cap B = \emptyset$. The result follows. ∎

6.3 Rings of Sets

Definition Let X be a set. A collection $\mathcal{R}$ of subsets of X is said to be a *ring* of subsets of X if it is non-empty and $A \cap B$, $A \cup B$ and $A \setminus B$ are members of $\mathcal{R}$ for all members A and B of $\mathcal{R}$.

Example The *characteristic function* of a subset A of the set $\mathbb{R}$ of real numbers is the function $\chi_A: \mathbb{R} \to \{0, 1\}$ defined so that

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{if } x \notin A. \end{cases}$$

Let $\mathcal{C}$ be the collection of subsets of $\mathbb{R}$ consisting of those sets A whose characteristic functions χ_A satisfy the following conditions:

(i) $\chi_A(x) = 1$ for all $x \in A$, and $\chi_A(x) = 0$ for all $x \in \mathbb{R} \setminus A$;

(ii) $\{x \in \mathbb{R} : \chi_A(x) = 1\}$ is a bounded subset of $\mathbb{R}$;

(iii) The function χ_A has only finitely many points of discontinuity.

The collection $\mathcal{C}$ of subsets $\mathbb{R}$ thus consists of those bounded subsets of $\mathbb{R}$ whose characteristic functions have only finitely many points of discontinuity. Now, given members A and B of $\mathcal{C}$, the characteristic functions $\chi_{A\cap B}$, $\chi_{A\cup B}$ and $\chi_{A\setminus B}$ of $A \cap B$, $A \cup B$ and $A \setminus B$ respectively satisfy

$$\begin{aligned} \chi_{A\cap B}(x) &= \chi_A(x)\chi_B(x), \\ \chi_{A\cup B}(x) &= \chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x), \\ \chi_{A\setminus B}(x) &= \chi_A(x) - \chi_A(x)\chi_B(x) \end{aligned}$$

for all $x \in \mathbb{R}$. It follows easily from this that $\chi_{A\cap B}$, $\chi_{A\cup B}$ and $\chi_{A\setminus B}$ satisfy conditions (i), (ii) and (iii) stated above. It follows that $\mathcal{C}$ is a ring of subsets of $\mathbb{R}$.

Lemma 6.11 *Let $\mathcal{R}$ be a ring of subsets of a given set X. Then the empty set $\emptyset$ belongs to $\mathcal{R}$, and any finite union or intersection of members of the ring $\mathcal{R}$ is also a member of that ring.*

Proof The ring $\mathcal{R}$ is by definition non-empty. Let A be a member of $\mathcal{R}$. Then $A \setminus A = \emptyset$. It follows that the empty set $\emptyset$ is a member of the ring $\mathcal{R}$. Also it follows by induction on the number of sets involved that any finite union or intersection of members of the ring $\mathcal{R}$ must itself belong to that ring. ∎

Let X be a set. The *power set* $\mathcal{P}(X)$ of X is the set whose elements are the subsets of X. The power set $\mathcal{P}(X)$ of X is itself a ring of subsets of X, and any ring $\mathcal{Q}$ of subsets of X is itself a subset of the power set $\mathcal{P}(X)$ of X. The intersection of any collection of rings of subsets of X is the intersection of a collection of collections of subsets of X, and is thus itself a collection of subsets of X. Morever if sets A and B belong to all the rings in some collection of rings of subsets of X, then so do $A \cup B$, $A \cap B$ and $A \setminus B$. It follows directly that the intersection of any collection of rings of subsets of X is itself a ring of subsets of X.

In particular, let $\mathcal{C}$ be a collection of subsets of X and let $\mathcal{R}(C)$ be the intersection of all rings of subsets of X that contain the collection $\mathcal{C}$. (as a subcollection). Then $\mathcal{R}(\mathcal{C})$ is itself a ring of subsets of X. Moreover it is contained in any other ring of subsets of X that also contains the collection $\mathcal{C}$. Thus $\mathcal{R}(\mathcal{C})$ is the smallest ring of subsets of $\mathcal{X}$ that contains the collection $\mathcal{C}$ (as a subcollection). We refer to $\mathcal{R}(\mathcal{C})$ as the ring of subsets of X *generated* by the collection $\mathcal{C}$ of subsets of X.

Lemma 6.12 *Let $\mathcal{C}$ be the collection consisting of those bounded subsets of $\mathbb{R}$ whose characteristic functions have only finitely many points of discontinuity. Then $\mathcal{C}$ is the ring of subsets of $\mathbb{R}$ generated by the semiring whose members are the empty set, the singleton sets in $\mathbb{R}$ and the intervals in $\mathbb{R}$.*

Proof Let A be a non-empty member of the ring $\mathcal{C}$. Then the characteristic function χ_A of A has only finitely many points of discontinuity. Let those points of discontinuity be $u_0, u_1, u_2, \ldots, u_m$, where

$$u_0 < u_1 < \cdots < u_m.$$

Then A can be expressed as a finite union of pairwise disjoint sets, where each of the latter sets is either a singleton set taking the form $\{u_i\}$ for some integer i between 0 and m or else an open interval taking the form $\{x \in \mathbb{R} : u_{i-1} < x < u_i\}$ for some integer i between 1 and m. Thus each member of the ring $\mathcal{C}$ is expressible as a finite union of pairwise disjoint members of the semiring $\mathcal{J}$ whose non-empty members are the singleton sets and bounded intervals in $\mathbb{R}$. It follows from this any ring of sets that contains all subsets of $\mathbb{R}$ belonging to semiring $\mathcal{J}$ must also contain all subsets of $\mathbb{R}$ belonging to the ring $\mathcal{C}$. Therefore $\mathcal{C}$ is the ring of subsets of $\mathbb{R}$ generated by the semiring $\mathcal{J}$, as required. ∎

Proposition 6.13 *Let X be a set, let $\mathcal{S}$ be a semiring of subsets of X, and let $\mathcal{R}(\mathcal{S})$ be the ring of subsets of X generated by the semiring $\mathcal{S}$. Then $\mathcal{R}(\mathcal{S})$ consists of those subsets of X that are expressible as finite unions of pairwise disjoint subsets of X belonging to the semiring $\mathcal{S}$.*

Proof Let $\mathcal{T}$ be the collection of subsets of X consisting of all subsets of X that are expressible as finite unions of pairwise disjoint members of the semiring $\mathcal{S}$. Then $\mathcal{T} \subset \mathcal{R}(\mathcal{S})$. But it follows from Lemma 6.7 Corollary 6.9 and Corollary 6.10 that if A and B are subsets of X belonging to the collection $\mathcal{T}$ then so are $A \cap B$, $A \cup B$ and $A \setminus B$. It follows that $\mathcal{T}$ is itself a ring of subsets of X, and therefore $\mathcal{R}(\mathcal{S}) \subset \mathcal{T}$. Consequently $\mathcal{T} = \mathcal{R}(\mathcal{S})$. The result follows. ∎

6.4 Products of Semirings of Sets

Proposition 6.14 *Let $X_1, X_2, \ldots, X_n$ be sets, let $\mathcal{S}_i$ be a semiring of subsets of X_i for $i = 1, 2, \ldots, n$, and let $\mathcal{S}$ be the collection of subsets of the Cartesian product $X_1 \times X_2 \cdots \times X_n$ consisting of those subsets of this Cartesian product that can be expressed as product sets of the form*

$$A_1 \times A_2 \times \cdots \times A_n$$

in which A_i is a member of the semiring $\mathcal{S}_i$ for $i = 1, 2, \ldots, n$. Then $\mathcal{S}$ is a semiring of subsets of the Cartesian product $X_1 \times X_2 \cdots \times X_n$

Proof The empty set belongs to $\mathcal{S}$, because the empty set belongs to each semiring $\mathcal{S}_i$ and any Cartesian product of sets involving the empty set is itself equal to the empty set.

Let

$$X = X_1 \times X_2 \times \cdots \times X_n,$$

and let A and B be subsets of X belonging to $\mathcal{S}$. Then there exist subsets A_i and B_i of X_i belonging to $\mathcal{S}_i$ for $i = 1, 2, \ldots, n$ such that

$$A = A_1 \times A_2 \times \cdots \times A_n$$

and

$$B = B_1 \times B_2 \times \cdots \times B_n.$$

An element $(x_1, x_2, \ldots, x_n)$ of X belongs to the set A if and only if $x_i \in A_i$ for $i = 1, 2, \ldots, n$. That element belongs to the set B if and only if $x_i \in B_i$ for $i = 1, 2, \ldots, n$. It follows that $(x_1, x_2, \ldots, x_n)$ belongs to $A \cap B$ if and only if $x_i \in A_i \cap B_i$ for $i = 1, 2, \ldots, n$. We conclude from this that

$$A \cap B = (A_1 \cap B_1) \times (A_2 \cap B_2) \times \cdots \times (A_n \cap B_n).$$

Moreover $A_i \cap B_i \in \mathcal{S}_i$ for $i = 1, 2, \ldots, n$ because, for each integer i between 1 and n, $\mathcal{S}_i$ is a semiring of subsets of $\mathcal{S}$. It follows that $A \cap B$ belongs to the collection $\mathcal{S}$ of subsets of X.

Then, for each integer i between 1 and n, there exist pairwise disjoint subsets

$$D_{i,1}, D_{i,2}, \ldots, D_{i,q(i)}$$

of X_i belonging to the semiring $\mathcal{S}_i$ such that each of the sets A_i and B_i can be expressed as a finite union of some of these sets $D_{i,j}$ (see Proposition 6.8). Let J denote the set of all n-tuples $(j_1, j_2, \ldots, j_n)$ with $1 \leq j_i \leq q(i)$ for $i = 1, 2, \ldots, n$, let

$$F_{j_1, j_2, \ldots, j_n} = D_{1,j_1} \times D_{2,j_2} \times \cdots \times D_{n,j_n}$$

for all $(j_1, j_2, \ldots, j_n) \in J$, and let $\mathcal{G}$ denote the collection consisting of the sets $F_{j_1, j_2, \ldots, j_n}$ corresponding to the n-tuples $(j_1, j_2, \ldots, j_n)$ in the indexing set J.

Now, for each integer i between 1 and n, there are subsets K_i and L_i of $\{1, 2, \ldots, q(i)\}$ such that

$$A_i = \bigcup_{j \in K_i} D_{i,j} \quad \text{and} \quad B_i = \bigcup_{j \in L_i} D_{i,j},$$

because each of the sets A_i and B_i is expressible as a finite union of sets of the form $D_{i,j}$ with $1 \leq j \leq q(i)$. Let

$$\begin{aligned} K &= \{(j_1, j_2, \ldots, j_n) \in J : j_i \in K_i \text{ for } i = 1, 2, \ldots, n\}, \\ L &= \{(j_1, j_2, \ldots, j_n) \in J : j_i \in L_i \text{ for } i = 1, 2, \ldots, n\}. \end{aligned}$$

Now $F_{j_1,j_2,\ldots,j_n} \in A$ for all $(j_1, j_2, \ldots, j_n) \in K$. Also a element $(x_1, x_2, \ldots, x_n)$ of the set X belongs to the subset A if and only if $x_i \in A_i$ for $i = 1, 2, \ldots, n$. But then, for each integer i between 1 and n, there must exist $j_i \in K_i$ for which $x_i \in D_{i,j_i}$. Then $(j_1, j_2, \ldots, j_n) \in K$ and $(x_1, x_2, \ldots, x_n) \in F_{j_1,j_2,\ldots,j_n}$. These results ensure that

$$A = \bigcup_{(j_1,j_2,\ldots,j_n)\in K} F_{j_1,j_2,\ldots,j_n}.$$

Similarly

$$B = \bigcup_{(j_1,j_2,\ldots,j_n)\in L} F_{j_1,j_2,\ldots,j_n}.$$

It follows from that that

$$A \setminus B = \bigcup_{(j_1,j_2,\ldots,j_n)\in K\setminus L} F_{j_1,j_2,\ldots,j_n},$$

because the sets $F_{j_1,j_2,\ldots,j_n}$ for $(j_1, j_2, \ldots, j_n) \in J$ are pairwise disjoint. Each set $F_{j_1,j_2,\ldots,j_n}$ belongs to $\mathcal{S}$. Thus $A\setminus B$ expressible as a finite union of pairwise disjoint members of the collection $\mathcal{S}$.

We have now shown that the empty set belongs to the collection $\mathcal{S}$ of subsets of X. Also, given any members A and B of the collection $\mathcal{S}$, the intersection $A \cap B$ is a member of $\mathcal{S}$, and the difference $A \setminus B$ is expressible as a finite union of pairwise disjoint subsets of X belonging to $\mathcal{S}$. These results ensure that the collection $\mathcal{S}$ of subsets of X is a semiring, as required. ∎

Definition Let $X_1, X_2, \ldots, X_n$ be sets, and let $\mathcal{S}_i$ be a semiring of subsets of X_i for $i = 1, 2, \ldots, n$. The *product* of the semirings $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$ is defined to be the collection $\mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n$ of subsets of the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ consisting of those subsets of of the sets X_i that can be expressed as Cartesian products

$$A_1 \times A_2 \times \cdots \times A_n$$

of sets $A_1, A_2, \ldots, A_n$ in which A_i is a member of the semiring $\mathcal{S}_i$ for $i = 1, 2, \ldots, n$.

Proposition 6.14 ensures that any product of semirings of sets is itself a semiring of sets.

Corollary 6.15 *Let n be a positive integer, and let $\mathcal{J}_n$ be the ring of subsets of $\mathbb{R}^n$ that consists of the empty set together those subsets of $\mathbb{R}^n$ that are Cartesian products of subsets of $\mathbb{R}$ that are bounded intervals or singleton sets. Then $\mathcal{J}_n$ is a semiring of subsets of $\mathbb{R}^n$.*

Proof This result follows directly on applying Lemma 6.5 and Proposition 6.14. ∎

6.5 Content Functions on Semirings

Definition Let X be a set, let $\mathcal{S}$ be a semiring of subsets of X, and let $\lambda: \mathcal{S} \to [0, +\infty)$ be a function mapping each member A of the semiring $\mathcal{S}$ to a non-negative real number $\lambda(A)$. The function λ is said to be *finitely additive* if

$$\lambda(A_1 \cup A_2 \cup \cdots \cup A_s) = \sum_{r=1}^{s} \lambda(A_r)$$

whenever $A_1, A_2, \ldots, A_s$ are pairwise disjoint members of the semiring $\mathcal{S}$ whose union belongs to the semiring $\mathcal{S}$.

Definition A *content function* $\lambda: \mathcal{S} \to [0, +\infty)$ on a semiring $\mathcal{S}$ of subsets of a given set is a finitely additive function mapping each member A of the semiring $\mathcal{S}$ to a non-negative real number $\lambda(A)$.

Lemma 6.16 *Let $\mathcal{J}$ be the semiring of subsets of $\mathbb{R}$ consisting of the empty set, the singleton sets and the bounded intervals, and let the function $\lambda: \mathcal{J} \to [0, +\infty)$ be defined such that $\lambda(\emptyset) = 0$, $\lambda(\{c\}) = 0$ for all $c \in \mathbb{R}$, and*

$$\lambda([a, b]) = \lambda((a, b]) = \lambda([a, b)) = \lambda((a, b)) = b - a$$

for all real numbers a and b satisfying $a < b$. Then $\lambda: \mathcal{J} \to [0, +\infty)$ is a content function on the semiring $\mathcal{J}$.

Proof If a member A of the semiring $\mathcal{S}$ is a finite union of pairwise disjoint sets $A_1, A_2, \ldots, A_s$, and if $\lambda(A) = 0$, then $\lambda(A_r) = 0$ for $r = 1, 2, \ldots, s$, and therefore

$$\lambda(A) = \sum_{r=1}^{s} \lambda(A_r)$$

in all cases for which $\lambda(A) = 0$,

Now suppose that A is a bounded interval with endpoints a and b, where a and b satisfy $a < b$, and that

$$A = A_1 \cup A_2 \cup \cdots \cup A_s,$$

where the sets $A_1, A_2, \ldots, A_s$ are pairwise disjoint and each set A_k is either a singleton set or a bounded interval.

Let $u_0, u_1, \ldots, u_N$ be a list of real numbers satisfying

$$a = u_0 < u_1 < \cdots < u_N = b,$$

where the list $u_0, u_1, \ldots, u_N$ includes the endpoints of all the intervals occurring in the list $A_1, A_2, \ldots, A_s$ and also includes the elements of all the singleton sets occurring in this list. Then

$$\lambda(A) = b - a = \sum_{j=1}^{N} (u_j - u_{j-1}).$$

Now, for each integer j between 1 and N, and for each integer r between 1 and s, either $(u_{j-1}, u_j) \subset A_r$ or $(u_{j-1}, u_j) \cap A_r = \emptyset$, where

$$(u_{j-1}, u_j) = \{x \in \mathbb{R} : u_{j-1} < x < u_j\}.$$

The nature of intervals in the real line and the choice of $u_1, u_2, \ldots, u_N$ then ensures that $\lambda(A_r)$ is the sum of the quantities $u_j - u_{j-1}$ for which $(u_{j-1}, u_j) \subset A_r$. For each integer r between 1 and s, let K_r denote those integers j between 1 and N for which $(u_{j-1}, u_j) \subset A_r$. Then

$$\lambda(A_r) = \sum_{j \in K_r} (u_j - u_{j-1}).$$

Now the sets $A_1, A_2, \ldots, A_s$ are pairwise disjoint. It follows that, for each integer j between 1 and N, there is exactly one integer r between 1 and s for which $(u_{j-1}, u_j) \subset A_r$. It follows that

$$\lambda(A) = \sum_{j=1}^{N} (u_j - u_{j-1}) = \sum_{r=1}^{s} \sum_{j \in K_r} (u_j - u_{j-1}) = \sum_{r=1}^{s} \lambda(A_j).$$

The result follows. ∎

Lemma 6.17 *Let X be a set, let $\mathcal{S}$ be a semiring of subsets of X, and let $\lambda: \mathcal{S} \to [0, +\infty)$ be a content function on X. Then $\lambda(\emptyset) = 0$ and $\lambda(A) \leq \lambda(B)$ for all members A and B of the semiring $\mathcal{S}$ for which $A \subset B$.*

Proof Let A be a subset of X belonging to the semiring $\mathcal{S}$. Then the sets A and $\emptyset$ are disjoint. It follows from the finite additivity of the content function λ that

$$\lambda(A) = \lambda(A \cup \emptyset) = \lambda(A) + \lambda(\emptyset).$$

Subtracting $\lambda(A)$, we conclude that $\lambda(\emptyset) = 0$.

Now let A and B be members of the semiring $\mathcal{S}$, where $A \setminus B$. It follows from the definition of semirings that there exists a finite list of pairwise disjoint members $C_1, C_2, \ldots, C_k$ of the semiring $\mathcal{S}$ for which

$$B \setminus A = C_1 \cup C_2 \cup \cdots \cup C_k.$$

The finite additivity and non-negativity of the content function λ then ensures that

$$\lambda(B) = \lambda(A) + \lambda(B \setminus A) = \lambda(A) + \sum_{j=1}^{k} \lambda(C_j) \geq \lambda(A).$$

The result follows. ∎

Lemma 6.18 *Let X be a set, let $D_1, D_2, \ldots, D_q$ be pairwise disjoint subsets of X, and let A be a subset of X. Suppose that A is expressible as a finite union of sets included in the list $D_1, D_2, \ldots, D_q$. Then, for each integer j between 1 and q, either D_j is a non-empty subset of A or else $D_j \cap A = \emptyset$.*

Proof We may suppose, without loss of generality, that the set A is non-empty. Let K be the set consisting of those integers j between 1 and q for which D_j is non-empty and $D_j \subset A$. Then the set A is the union of those sets D_j for which $j \in K$. If $j \notin K$ then $D_j \cap D_p = \emptyset$ for all $p \in K$, and therefore $D_j \cap A = \emptyset$. The result follows. ∎

Proposition 6.19 *Let $\lambda: \mathcal{S} \to [0, +\infty)$ be a content function on a semiring $\mathcal{S}$ of subsets of some set X, and let A and $A_1, A_2, \ldots, A_s$ be members of the semiring $\mathcal{S}$. Suppose that $A \subset \bigcup\limits_{k=1}^{s} A_k$. Then $\lambda(A) \leq \sum\limits_{k=1}^{s} \lambda(A_k)$.*

Proof It follows from Proposition 6.8 that there is a finite list $D_1, D_2, \ldots, D_q$ of members of the semiring $\mathcal{S}$ such that $D_1, D_2, \ldots, D_q$ are pairwise disjoint and such that each of the set A, $A_1, A_2, \ldots, A_s$ is expressible as a union of sets in the list $D_1, D_2, \ldots, D_q$.

Let us define $\sigma_j(A)$ for $j = 1, 2, \ldots, q$ so that $\sigma_j(A) = 1$ whenever D_j is a non-empty subset of A and $\sigma_j(A) = 0$ in all other cases. Similarly, for each integer k between 1 and s, let us define $\sigma_j(A_k)$ so that $\sigma_j(A_k) = 1$ whenever D_j is a non-empty subset of A_k and $\sigma_j(A_k) = 0$ in all other cases. Then the set A is the union of those sets D_j for which $\sigma_j(A) = 1$. Suppose that $\sigma_j(A) = 1$ for some integer j between 1 and q. Then D_j is non-empty and $D_j \subset A$. Let $x \in D_j$. Then there is at least one integer k between 1 and s for which $x \in A_k$, because $x \in A$ and $A \subset \bigcup\limits_{k=1}^{s} A_k$. Thus if j is an integer between 1 and q, and if $\sigma_j(A) = 1$, then there is at least one integer k between 1 and s for which $\sigma_j(A_k) = 1$. It follows that

$$\sigma_j(A) \leq \sum_{k=1}^{s} \sigma_j(A_k)$$

for all integers j between 1 and q.

Now the content function λ is finitely additive. It follows that

$$\lambda(A) = \sum_{j=1}^{q} \sigma_j(A)\lambda(D_j).$$

Similarly

$$\lambda(A_k) = \sum_{j=1}^{q} \sigma_j(A_k)\lambda(D_j)$$

for $k = 1, 2, \ldots, s$. We have shown that

$$\sigma_j(A) \leq \sum_{k=1}^{s} \sigma_j(A_k)$$

for all integers j between 1 and q. It follows that

$$\lambda(A) = \sum_{j=1}^{q} \sigma_j(A)\lambda(D_j) \leq \sum_{k=1}^{s}\sum_{j=1}^{q} \sigma_j(A_k)\lambda(D_j) = \sum_{k=1}^{s} \lambda(A_k),$$

as required. ∎

Proposition 6.20 *Let $\lambda: \mathcal{S} \to [0, +\infty)$ be a content function on a semiring $\mathcal{S}$ of subsets of some set X, and let A and $A_1, A_2, \ldots, A_s$ be members of the semiring $\mathcal{S}$. Suppose that the sets $A_1, A_2, \ldots, A_s$ are disjoint and are contained in A. Then $\sum\limits_{k=1}^{s} \lambda(A_k) \leq \lambda(A)$.*

Proof It follows from Proposition 6.8 that there is a finite list $D_1, D_2, \ldots, D_q$ of members of the semiring $\mathcal{S}$ such that $D_1, D_2, \ldots, D_q$ are pairwise disjoint and such that each of the set A, $A_1, A_2, \ldots, A_s$ is expressible as a union of sets in the list $D_1, D_2, \ldots, D_q$.

Let us define $\sigma_j(A)$ for $j = 1, 2, \ldots, s$ so that $\sigma_j(A) = 1$ whenever D_j is a non-empty subset of A and $\sigma_j(A) = 0$ in all other cases. Similarly, for each integer k between 1 and s, let us define $\sigma_j(A_k)$ so that $\sigma_j(A_k) = 1$ whenever D_j is a non-empty subset of A_k and $\sigma_j(A_k) = 0$ in all other cases. Then the set A is the union of those D_j for which $\sigma_j(A) = 1$.

Now for each integer j between 1 and q, there is at most one set A_k in the list $A_1, A_2, \ldots, A_s$ for which $D_j \subset A_k$, because the sets $A_1, A_2, \ldots, A_s$ are pairwise disjoint. It follows that

$$\sum_{k=1}^{s} \sigma_j(A_k) \leq 1$$

for $j = 1, 2, \ldots, q$. Moreover, given any integer k between 1 and s, the identity $\sigma_j(A) = 1$ is satisfied by those integers j between 1 and q for which $\sigma_j(A_k) = 1$. It follows that

$$\sum_{k=1}^{s} \sigma_j(A_k) \leq \sigma_j(A)$$

for all integers j between 1 and q. Now the content function λ is finitely additive. It follows that

$$\lambda(A) = \sum_{j=1}^{q} \sigma_j(A)\lambda(D_j).$$

Similarly

$$\lambda(A_k) = \sum_{j=1}^{q} \sigma_j(A_k)\lambda(D_j)$$

for $k = 1, 2, \ldots, s$. We have shown that

$$\sum_{k=1}^{s} \sigma_j(A_k) \leq \sigma_j(A)$$

for all integers j between 1 and q. It follows that

$$\sum_{k=1}^{s} \lambda(A_k) = \sum_{k=1}^{s} \sum_{j=1}^{q} \sigma_j(A_k)\lambda(D_j) \leq \sum_{j=1}^{q} \sigma_j(A)\lambda(D_j) = \lambda(A),$$

as required. ∎

Proposition 6.21 *Let X be a set, let $\mathcal{S}$ be a semiring of subsets of X, and let $\lambda: \mathcal{S} \to [0, +\infty)$ be a content function on the semiring $\mathcal{S}$. Then the content function λ extends in a unique fashion to a content function $\overline{\lambda}: \mathcal{R}(\mathcal{S}) \to [0, +\infty)$ defined on the ring $\mathcal{R}(\mathcal{S})$ of subsets of X generated by the semiring $\mathcal{S}$.*

Proof It follows from Proposition 6.13 that any subset of X belonging to the ring $\mathcal{R}(\mathcal{S})$ of subsets of X generated by semiring $\mathcal{S}$ is expressible as a finite union of pairwise disjoint members of that semiring.

Let A be a member of the ring $\mathcal{R}(\mathcal{S})$ generated by the semiring $\mathcal{S}$, and let it be the case that

$$A = \bigcup_{j=1}^{p} B_j = \bigcup_{k=1}^{q} C_k$$

where $B_1, B_2, \ldots, B_p$ is a list of pairwise disjoint members of the semiring $\mathcal{S}$, and $C_1, C_2, \ldots, C_q$ is also a list of pairwise disjoint members of the semiring $\mathcal{S}$. Then, for each integer j between 1 and p,

$$B_j = B_j \cap A = \bigcup_{k=1}^{q} (B_j \cap C_k).$$

Now the sets $B_j \cap C_k$ for $k = 1, 2, \ldots, q$ are pairwise disjoint. It follows from the finite additivity of the content function λ on $\mathcal{S}$ that

$$\lambda(B_j) = \sum_{k=1}^{q} \lambda(B_j \cap C_k),$$

and therefore

$$\sum_{j=1}^{p} \lambda(B_j) = \sum_{j=1}^{p} \sum_{k=1}^{q} \lambda(B_j \cap C_k).$$

Similarly

$$\sum_{k=1}^{q} \lambda(C_k) = \sum_{j=1}^{p} \sum_{k=1}^{q} \lambda(B_j \cap C_k).$$

It follows that

$$\sum_{j=1}^{p} \lambda(B_j) = \sum_{k=1}^{q} \lambda(C_k).$$

We therefore define $\overline{\lambda}(A)$ to be the unique real number with the property that $\overline{\lambda}(A) = \sum_{j=1}^{p} \lambda(B_j)$ for all finite lists $B_1, B_2, \ldots, B_p$ of pairwise disjoint members of the semiring $\mathcal{S}$ for which $A = \bigcup_{j=1}^{p} B_j$. It then follows directly that

$$\overline{\lambda}\colon \mathcal{R}(\mathcal{S}) \to [0, +\infty)$$

is finitely additive, and is thus a content function on the ring $\mathcal{R}(\mathcal{S})$ generated by $\mathcal{S}$. Moreover this content function $\overline{\lambda}$ is clearly the only finitely additive function that extends λ. The result follows. ∎

Corollary 6.22 *Let $\mathcal{I}$ be the ring of subsets of the set $\mathbb{R}$ of real numbers consisting of the empty set together with those subsets of $\mathbb{R}$ that are representable as finite unions of singleton sets and bounded intervals. Then there is a well-defined content function*

$$\nu\colon \mathcal{I} \to [0, +\infty)$$

that satisfies $\nu(\{c\}) = 0$ for all $c \in \mathbb{R}$ and

$$\nu([a, b]) = \nu((a, b]) = \nu([a, b)) = \nu((a, b)) = b - a$$

for all real numbers a and b satisfying $a < b$.

Proof This result follows immediately on applying Proposition 6.21 to extend the content function on the semiring of subsets of $\mathbb{R}$, characterized by the conditions stated in the corollary, whose existence was established by Lemma 6.16. $\blacksquare$

6.6 Content Functions on Products of Semirings of Sets

Proposition 6.23 *Let $X_1, X_2, \ldots, X_n$ be sets, and, for each integer i between 1 and n, let $\mathcal{S}_i$ be a semiring of subsets of X_i, and let $\lambda_i : \mathcal{S}_i \to [0, +\infty)$ be a content function on the semiring $\mathcal{S}_i$. Also let*

$$\lambda(A_1 \times A_2 \times \cdots \times A_n) = \lambda_1(A_1)\lambda_2(A_2) \cdots \lambda_n(A_n)$$

for all product sets $A_1 \times A_2 \times \cdots \times A_n$ for which A_i is a member of the semiring $\mathcal{S}_i$ for $i = 1, 2, \ldots, n$. Then

$$\lambda : \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \mathcal{S}_n \to [0, +\infty)$$

is finitely additive, and is therefore a content function on the product semiring $\mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \mathcal{S}_n$.

Proof Let $X = X_1 \times X_2 \times \cdots \times X_n$ and $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n$. Let A be a subset of X belonging to the product semiring $\mathcal{S}$, and suppose that $A = \bigcup_{r=1}^{s} A^{(r)}$, where $A^{(1)}, A^{(2)}, \ldots, A^{(s)}$ are pairwise disjoint members of the semiring $\mathcal{S}$. Then, for each integer i between 1 and n, there exist subsets A_i, $A_i^{(1)}, A_i^{(2)}, \ldots, A_i^{(s)}$ of X_i, all belonging to the semiring $\mathcal{S}_i$, such that

$$A = A_1 \times A_2 \times \cdots A_n$$

and

$$A^{(r)} = A_1^{(r)} \times A_2^{(r)} \times \cdots \times A_n^{(r)}$$

for $r = 1, 2, \ldots, s$. Then, for each integer i between 1 and n there exists a positive integer $q(i)$ and pairwise disjoint members $D_{i,1}, D_{i,2}, \ldots, D_{i,q(i)}$ of the semiring $\mathcal{S}_i$ such that each of the sets A_i, $A_i^{(1)}, \ldots, A_i^{(s)}$ can be expressed as a union of sets included in the list $D_{i,1}, D_{i,2}, \ldots, D_{i,q(i)}$ (Proposition 6.8). Then, for each integer i between 1 and n, subsets K_i, $K_i^{(1)}, \ldots, K_i^{(s)}$ of $\{j \in \mathbb{Z} : 1 \leq j \leq q(i)\}$ can be found so that

$$A_i = \bigcup_{j \in K_i} D_{i,j}$$

and

$$A_i^{(r)} = \bigcup_{j \in K_i^{(r)}} D_{i,j}$$

for $r = 1, 2, \ldots, s$.

Let

$$K = \{(j_1, j_2, \ldots, j_n) : j_i \in K_i \text{ for } i = 1, 2, \ldots, n\}$$

and

$$K^{(r)} = \{(j_1, j_2, \ldots, j_n) : j_i \in K_i^{(r)} \text{ for } i = 1, 2, \ldots, n\}.$$

and let

$$F_{j_1,j_2,\ldots,j_n} = D_{1,j_1} \times D_{2,j_2} \times \cdots \times D_{n,j_n}$$

for each n-tuple $(j_1, j_2, \ldots, j_n)$ of integers that satisfies $1 \leq j_i \leq q(i)$ for $i = 1, 2, \ldots, n$, and let $\mathcal{G}$ denote the collection consisting of these sets $F_{j_1,j_2,\ldots,j_n}$. Then the subsets of X belonging to the collection $\mathcal{G}$ are all members of the semiring $\mathcal{S}$. Also the sets $F_{j_1,j_2,\ldots,j_n}$ are pairwise disjoint,

$$A = \bigcup_{(j_1,j_2,\ldots,j_n) \in K} F_{j_1,j_2,\ldots,j_n},$$

and

$$A^{(r)} = \bigcup_{(j_1,j_2,\ldots,j_n) \in K^{(r)}} F_{j_1,j_2,\ldots,j_n}$$

for $r = 1, 2, \ldots, s$. (These results follow from a direct application of Corollary 6.4.)

We now investigate the behaviour of the content functions on the relevant semirings. Now $A = A_1 \times A_2 \times \cdots \times A_n$, where each set A_i is the disjoint union of the sets D_{i,j_i} for which $j_i \in K_i$. It follows that

$$\lambda_i(A_i) = \sum_{j_i \in K_i} \lambda_i(D_{i,j_i}),$$

Now it follows from the definition of the function λ that

$$\lambda(A) = \lambda_1(A_1)\lambda_2(A_2)\cdots\lambda_n(A_n)$$

and

$$\lambda(F_{j_1,j_2,\ldots,j_n}) = \lambda_1(D_{1,j_1})\lambda_2(D_{2,j_2})\cdots\lambda_n(D_{n,j_n})$$

for each $(j_1, j_2, \ldots, j_n) \in K$. It follows (applying the Distributive Law) that

$$\begin{aligned}\lambda(A) &= \sum_{j_1\in K_1}\sum_{j_2\in K_2}\cdots\sum_{j_n\in K_n}\lambda_1(D_{i,j_1})\lambda_2(D_{2,j_2})\cdots\lambda_n(D_{n,j_n})\\ &= \sum_{(j_1,j_2,\ldots,j_n)\in K}\lambda(F_{j_1,j_2,\ldots,j_n}).\end{aligned}$$

Similarly

$$\begin{aligned}\lambda(A^{(r)}) &= \sum_{j_1\in K_1^{(r)}}\sum_{j_2\in K_2^{(r)}}\cdots\sum_{j_n\in K_n^{(r)}}\lambda_1(D_{i,j_1})\lambda_2(D_{2,j_2})\cdots\lambda_n(D_{n,j_n})\\ &= \sum_{(j_1,j_2,\ldots,j_n)\in K^{(r)}}\lambda(F_{j_1,j_2,\ldots,j_n}).\end{aligned}$$

Now the set A is by assumption the union of the pairwise disjoint sets $A^{(1)}, A^{(2)}, \ldots, A^{(r)}$. It is also the union of the pairwise disjoint sets $F_{j_1,j_2,\ldots,j_n}$ for which $(j_1, j_2, \ldots, j_n) \in K$, and each $A^{(r)}$ is the union of the pairwise disjoint sets $F_{j_1,j_2,\ldots,j_n}$ for which $(j_1, j_2, \ldots, j_n) \in K^{(r)}$. Thus the indexing set K is the disjoint union of the sets $K^{(1)}, K^{(2)}, \ldots, K^{(s)}$, and therefore

$$\begin{aligned}\lambda(A) &= \sum_{(j_1,j_2,\ldots,j_n)\in K}\lambda(F_{j_1,j_2,\ldots,j_n})\\ &= \sum_{r=1}^{s}\sum_{(j_1,j_2,\ldots,j_n)\in K^{(r)}}\lambda(F_{j_1,j_2,\ldots,j_n})\\ &= \sum_{r=1}^{s}\lambda(A^{(r)}).\end{aligned}$$

Thus the function $\lambda\colon \mathcal{S} \to [0, +\infty)$ is finitely-additive, and is thus a content function on the semiring $\mathcal{S}$, as required. ∎

Corollary 6.24 *Let n be a positive integer, and let $\mathcal{B}_n$ be the ring of subsets of $\mathbb{R}^n$ that consists of the empty set together with all subsets of $\mathbb{R}^n$ representable as finite unions of Cartesian products of subsets of $\mathbb{R}$ that are bounded intervals or singleton sets. Then there is a well-defined (finitely additive) content function $\lambda\colon \mathcal{B}_n \to [0, +\infty)$ characterized by the property that*

$$\lambda(I_1 \times I_2 \times \cdots \times I_n) = (b_1 - a_1)(b_2 - a_2)\cdots(b_n - a_n)$$

for all subsets $I_1, I_2, \ldots, I_n$ of $\mathbb{R}$ that are bounded intervals or singleton sets, where $a_i = \inf I_i$ and $b_i = \sup I_i$ for $i = 1, 2, \ldots, n$.

Proof It follows from Lemma 6.16 and Proposition 6.23 that there is a content function on the product semiring of subsets of $\mathbb{R}^n$ consisting of the empty set together with those subsets of $\mathbb{R}^n$ that are expressible as Cartesian products of subsets of $\mathbb{R}$ that are bounded intervals and singleton sets. It then follows from Proposition 6.21 that the resultant content function on the semiring extends to a content function on the ring of subsets of $\mathbb{R}^n$ generated by the product semiring. Morever Proposition 6.13 establishes that the subsets of $\mathbb{R}^n$ that belong to the ring of subsets generated by the product semiring are those subsets of $\mathbb{R}^n$ that are finite unions of Cartesian products of bounded intervals and singleton sets. The result follows. ∎