Course MAU22200: Michaelmas Term 2020.

Assignment 1.

To be handed in by Thursday 19th March, 2020.

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Module MAU22200—Advanced Analysis,
Hilary Term 2020.
Assignment I.

Name (please print): . . . .

Student number: . . . .

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Course MAU22200: Hilary Term 2020. Assignment 1.

Stieltjes Measures

1. Throughout this question, let $F: \mathbb{R} \to \mathbb{R}$ be a non-decreasing function of a real variable. The function F thus has the property that $F(u) \leq F(v)$ for all real numbers s and t satisfying $u \leq v$. For each real number s, let $F(s^+)$ and $F(s^-)$ be defined so that

$$\begin{aligned} F(s^+) &= \lim_{x \to s^+} F(x) = \inf\{F(x) : x > s\}, \\ F(s^-) &= \lim_{x \to s^-} F(x) = \sup\{F(x) : x < s\}. \end{aligned}$$

Note that $F(s)$ is a lower bound for the set $\{F(x) : x > s\}$ and an upper bound for the set $\{F(x) : x < s\}$ for each real number s, because the function F is non-decreasing. It follows that $F(s^-) \leq F(s) \leq F(s^+)$ for all real numbers s. Also $F(u^+) \leq F(v) \leq F(w^-)$ for all real numbers u, v and w satisfying $u < v < w$. The definition of $F(s^+)$ and $F(s^-)$ also ensures that, given any positive real number ε, there exist real numbers q and r satisfying $q < s < r$ for which $F(q) > F(s^-) - \varepsilon$ and $F(r) < F(s^+) + \varepsilon$.

We define the *Stieltjes content* $m_F(I)$ of each bounded interval or singleton set I contained in $\mathbb{R}$ so that

$$\begin{aligned} m_F(\{v\}) &= F(v^+) - F(v^-), \\ m_F([u, v]) &= F(v^+) - F(u^-), \\ m_F([u, v)) &= F(v^-) - F(u^-), \\ m_F((u, v]) &= F(v^+) - F(u^+), \\ m_F((u, v)) &= F(v^-) - F(u^+) \end{aligned}$$

for all real numbers u and v satisfying $u < v$.

(a) Let a and b be real numbers satisfying $a < b$, and let $u_0, u_1, \ldots, u_N$ be a list of real numbers with the property that

$$a = u_0 < u_1 < u_2 < \cdots < u_N = b.$$

For each integer j between 0 and N, let $D_j = \{u_j\}$, and, for each integer j between 1 and N, let

$$E_j = (u_{j-1}, u_j) = \{x \in \mathbb{R} : u_{j-1} < x < u_j\}.$$

Prove that

$$m_F((a,b)) = \sum_{j=1}^{N-1} m_F(D_j) + \sum_{j=1}^{N} m_F(E_j),$$

where $m_F(D_j)$ and $m_F(E_j)$ denote the Stieltjes content of the sets D_j and E_j respectively. Also determine and write down corresponding expressions for $m_F([a,b))$, $m_F((a,b])$, $m_F([a,b])$, expressing each of these as a finite sum whose summands are the Stieltjes measures of sets, each of which included amongst the singleton sets D_j and the open intervals E_j.

(b) Let a, b, $u_0, u_1, \ldots, u_N$, $D_0, D_1, \ldots, D_N$ and $E_1, E_2, \ldots, E_N$ be defined as set out in (a). Let J be an interval or singleton set whose endpoints are included in the list $u_0, u_1, \ldots, u_N$, and let

$$\begin{aligned} S(J) &= \{j \in \mathbb{Z} : 0 \leq j \leq N \text{ and } D_j \subset J\}, \\ T(J) &= \{j \in \mathbb{Z} : 1 \leq j \leq N \text{ and } E_j \subset J\} \end{aligned}$$

(Note that an integer j between 0 and N belongs to $S(J)$ if and only if $u_j \in J$, and and integer j between 1 and N belongs to $T(J)$ if and only if $(u_{j-1}, u_j) \subset J$.) Prove that

$$m_F(J) = \sum_{j \in S(J)} m_F(D_j) + \sum_{j \in T(J)} m_F(E_j).$$

(c) Let a, b, $u_0, u_1, \ldots, u_N$, $D_0, D_1, \ldots, D_N$ and $E_1, E_2, \ldots, E_N$ be defined as set out in (a). Let J, $J^{(1)}, J^{(2)}, \ldots, J^{(s)}$ be intervals or singleton sets whose endpoints are included in the list $u_0, u_1, \ldots, u_N$. Suppose that $J^{(1)}, J^{(2)}, \ldots, J^{(s)}$ are pairwise disjoint and that $J = \bigcup_{r=1}^{s} J^{(r)}$. Using the result of (b), or otherwise, prove that

$$m_F(J) = \sum_{r=1}^{s} m_F(J^{(r)}).$$

(d) Let $\{v\}$ be a singleton set in the real line. Prove that, given any positive real number ε, there exists an open interval V such that $v \in V$ and $m_F(V) < m_F(\{v\}) + \varepsilon$.

(e) Let J be a bounded interval of positive length in the real line, and let $a = \inf J$ and $b = \sup J$. (It then follows that $a < b$, and J

coincides with exactly one of the intervals (a,b), $[a,b)$, $(a,b]$ and $[a,b]$.) Considering individually all relevant cases that result, prove that given any positive real number ε, there exists an open interval V such that $J \subset V$ and $m_F(V) < m_F(J) + \varepsilon$.

(f) Let J be a bounded interval in $\mathbb{R}$, and let $J^{(1)}, J^{(2)}, J^{(3)}, \ldots$ be an infinite sequence of bounded intervals in $\mathbb{R}$. Suppose that $J \subset \bigcup_{r=1}^{+\infty} J^{(r)}$. By applying the one-dimensional Heine-Borel theorem and using the results obtained in (d) and (e) above, or otherwise, prove that $m_F(J) \leq \sum_{r=1}^{+\infty} m_F(J^{(r)})$. [Hint: it is suggested that the proof be modelled on that of Proposition 7.8 of the module notes.]

We follow the above question with a review of some results that follow from what has been established in the previous parts of this question. Let $\mathcal{J}$ be the semiring of subsets of the real line consisting of the empty set together with all singleton sets and bounded intervals contained in the set $\mathbb{R}$ of real numbers. Also let the empty set be assigned Stieltjes content equal to zero, so that $m_F(\emptyset) = 0$. Note that the result stated in (c) then ensures that Stieltjes measure determines a finitely additive content function $m_F \colon \mathcal{J} \to [0, +\infty)$ on the semiring $\mathcal{J}$. The result of (f) ensures that this content function is countably subadditive. One could then adapt the approach of Subsection 7.2 of the module notes in order to define *Lebesgue-Stieltjes outer measure* μ_F^* on the real line $\mathbb{R}$ and show that it is indeed an outer measure. The general theory of outer measures and measurable sets then ensures the existence of *Lebesgue-Stieltjes measure* μ_F, defined on the class of *Lebesgue-Stieltjes-measurable* subsets of the real line $\mathbb{R}$. The general theory of Lebesgue integration developed in the module notes provides the appropriate definition of the *Lebesgue-Stieltjes integral* of a Lebesgue-Stieltjes-measurable function g over a Lebesgue-Stieltjes-measurable subset E of the real line. This Lebesgue-Stieltjes integral may be denoted by $\int_E g \, d\mu_F$, or simply by $\int_E g \, dF$.

A particular case of Stieltjes integration is that in which the non-decreasing function determining the Stieltjes measure is the *Heaviside function* H defined in accordance with the following requirements: $H(x) = 1$ if $x > 0$; $H(x) = 0$ if $x < 0$; $H(0)$ is some determined value chosen to satisfy $0 \leq H(0) \leq 1$. If E is a subset of the real line $\mathbb{R}$, and if $0 \in E$, then $\int_E g(x) \, dH = g(0)$. (Applied mathematicians typically consider the "derivative" of the Heaviside function H at zero to be the "Dirac delta function".)