A Note on the Barycentric Subdivision of a Simplex

Let σ be a simplex. Then the collection K_σ of all faces of σ is a simplicial complex (see MA3486, notes for Hilary Term 2018, example in subsection 4.1, page 50). This simplicial complex has a first barycentric subdivision K'_σ. It is convenient to refer to this simplicial complex K'_σ as the *barycentric subdivision* or *first barycentric subdivision* of the simplex σ.

The definition of the barycentric subdivision of a simplicial complex (see MA3486, notes for Hilary Term 2018, definition in subsection 4.2, page 52) determines what are the simplices of the first barycentric subdivision K'_σ of σ. The vertices of an r-simplex of the first barycentric subdivision can be listed in the form $\tau_0, \tau_1, \ldots, \tau_r$, where each τ_j is a face of σ and τ_{j-1} is a face of τ_j for $2 \leq j \leq r$.

To make the discussion less abstract, we consider a particular example, making arbitrary choices of the dimensions of the simplices involved (subject to the requirements of the definition). Accordingly let us suppose that σ is a 12-dimensional simplex and that its vertices are

$$\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{12}.$$

Let τ be a 3-dimensional simplex of the first barycentric subdivision of σ. Then the vertices of τ are the barycentres of a 'chain' of faces of σ, where each face in the 'chain' other than the last is a face of the next face in the chain. Making fairly arbitrary choices of the vertices spanning the faces, we may consider, for example, the instance where τ has vertices $\hat{\tau}_0$, $\hat{\tau}_1$, $\hat{\tau}_2$ and $\hat{\tau}_3$, where

τ_0 has vertices $\mathbf{v}_3$ and $\mathbf{v}_9$,
τ_1 has vertices $\mathbf{v}_3$, $\mathbf{v}_7$ and $\mathbf{v}_9$,
τ_2 has vertices $\mathbf{v}_0$, $\mathbf{v}_3$, $\mathbf{v}_5$, $\mathbf{v}_7$ and $\mathbf{v}_9$,
τ_3 has vertices $\mathbf{v}_0$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_5$, $\mathbf{v}_7$, $\mathbf{v}_9$, $\mathbf{v}_{11}$ and $\mathbf{v}_{12}$.

Let the vertices of τ be denoted by $\mathbf{w}_0$, $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$, where $\mathbf{w}_j$ is the barycentre $\hat{\tau}_j$ of the simplex τ_j for $j = 0, 1, 2, 3$. It follows from the definition of barycentres of simplices (see MA3486, notes for Hilary Term 2018, subsection 4.2, page 52) that

$$\begin{aligned}
\mathbf{w}_0 &= \tfrac{1}{2}(\mathbf{v}_3 + \mathbf{v}_9),\\
\mathbf{w}_1 &= \tfrac{1}{3}(\mathbf{v}_3 + \mathbf{v}_7 + \mathbf{v}_9),\\
\mathbf{w}_2 &= \tfrac{1}{5}(\mathbf{v}_0 + \mathbf{v}_3 + \mathbf{v}_5 + \mathbf{v}_7 + \mathbf{v}_9),\\
\mathbf{w}_3 &= \tfrac{1}{8}(\mathbf{v}_0 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_5 + \mathbf{v}_7 + \mathbf{v}_9 + \mathbf{v}_{11} + \mathbf{v}_{12}).
\end{aligned}$$

The points of the simplex τ of the barycentric subdivision can each be expressed uniquely in the form

$$u_0\mathbf{w}_0 + u_1\mathbf{w}_1 + u_2\mathbf{w}_2 + u_3\mathbf{w}_3,$$

where u_0, u_1, u_2 and u_3 are real numbers between 0 and 1 for which $u_0 + u_1 + u_2 + u_3 = 1$. Moreover, according to the definition of the *interior* of a simplex, (see MA3486, notes for Hilary Term 2018, definition in subsection 3.5, page 41) a point of τ belongs to the interior of τ if and only those real numbers u_0, u_1, u_2 and u_3 are strictly positive. The requirement that these barycentric coordinates sum to one then ensure that, for a point in the interior of τ, the barycentric coordinates u_0, u_1, u_2, u_3 of that point with respect to the vertices of τ satisfy $0 < u_j < 1$ for $j = 0, 1, 2, 3$. Conversely when these strict inequalities are satisfied by these barycentric coordinates, the point determined by them belongs to the interior of the simplex τ.

So let $\mathbf{x}$ be a point of τ, and let

$$\mathbf{x} = u_0\mathbf{w}_0 + u_1\mathbf{w}_1 + u_2\mathbf{w}_2 + u_3\mathbf{w}_3,$$

where $0 \leq u_j \leq 1$ for $j = 0, 1, 2, 3$ and $u_0 + u_1 + u_3 + u_4$, and let $\mathbf{x} = \sum_{k=0}^{12} t_k\mathbf{v}_k$, where $\sum_{k=0}^{12} t_k = 1$. Then $t_0, t_1, \ldots, t_{12}$ are the barycentric coordinates of the point $\mathbf{x}$ with respect to the vertices of σ.

First note that

$$t_1 = t_4 = t_6 = t_8 = t_{10} = 0.$$

Indeed the vertices of σ associated with these barycentric coordinates are not vertices of any of the faces τ_0, τ_1, τ_2, and τ_3.

Next note that, because the coefficients u_0, u_1, u_2 and u_3 are non-negative, the barycentric coordinates associated with the vertices $\mathbf{v}_0$, $\mathbf{v}_3$, $\mathbf{v}_5$, $\mathbf{v}_7$ and $\mathbf{v}_9$ will be at least as large in numerical value as those associated with the other vertices $\mathbf{v}_2$, $\mathbf{v}_{11}$ and $\mathbf{v}_{12}$. Indeed

$$t_2 = t_{11} = t_{12} = \tfrac{1}{8}u_3,$$

whilst t_0, t_3, t_5, t_7 and t_9 must equal or exceed $\frac{1}{8}u_3$. Similarly

$$t_0 = t_5 = \tfrac{1}{8}u_3 + \tfrac{1}{5}u_2,$$

whilst t_3, t_7 and t_9 must equal or exceed the common value of t_0 and t_1.

Finally we note that

$$t_7 = \tfrac{1}{8}u_3 + \tfrac{1}{5}u_2 + \tfrac{1}{3}u_1$$

and

$$t_3 = t_9 = \tfrac{1}{8}u_3 + \tfrac{1}{5}u_2 + \tfrac{1}{3}u_1 + \tfrac{1}{2}u_0.$$

Summarizing the equalities and inequalities, we see that

$$0 = t_1 = t_4 = t_6 = t_8 = t_{10} \le t_2 = t_{11} = t_{12} \le t_0 = t_5 \le t_7 \le t_3 = t_9.$$

Moreover the point $\mathbf{x}$ belongs to the interior of the simplex τ if and only if

$$0 = t_1 = t_4 = t_6 = t_8 = t_{10} < t_2 = t_{11} = t_{12} < t_0 = t_5 < t_7 < t_3 = t_9,$$

because these strict inequalities are the necessary and sufficient conditions to ensure that u_0, u_1, u_2 and u_3 are strictly positive.

We have above determined the ordering (in terms of numerical size) that characterizes the barycentric coordinates, with respect to the vertices of σ, of points in the interior of the particular simplex τ of the barycentric subdivision of σ. It should be clear how, once the barycentric coordinates of a given point of σ are ranked in increasing numerical order, one can write down the vertices of the unique simplex of the barycentric subdivision of σ that contains the given point in its interior.

We now return to our particular example. We show that if $\sum_{j=0}^{3} u_j = 1$ then $\sum_{k=0}^{12} t_k = 1$. Now the equations expressing $t_0, t_1, \ldots, t_{12}$ in terms of u_0, u_1, u_2, u_3 are determined by a matrix $A_{k,j}$ so that $t_k = \sum_{j=0}^{3} A_{k,j} u_j$, where the matrix A with coefficient $A_{k,j}$ in the kth row and jth column is the following 13×4 matrix:

$$A = \begin{pmatrix} \frac{1}{8} & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{8} & 0 & 0 & 0 \\ \frac{1}{8} & \frac{1}{5} & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ \frac{1}{8} & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{8} & \frac{1}{5} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{8} & \frac{1}{5} & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ \frac{1}{8} & 0 & 0 & 0 \\ \frac{1}{8} & 0 & 0 & 0 \end{pmatrix}.$$

Note that the columns of this matrix sum up to one. Thus $\sum_{k=0}^{12} A_{k,j} = 1$. It follows that

$$\sum_{k=0}^{12} t_k = \sum_{k=0}^{12} \sum_{j=0}^{3} A_{k,j} u_j = \sum_{j=0}^{3} \left(\sum_{k=0}^{12} A_{k,j} \right) u_j = \sum_{j=0}^{3} u_j = 1.$$

Now let, for example,

$$\mathbf{x} = \frac{17}{240}\mathbf{v}_0 + \frac{1}{48}\mathbf{v}_2 + \frac{77}{240}\mathbf{v}_3 + \frac{17}{240}\mathbf{v}_5 + \frac{37}{240}\mathbf{v}_7 + \frac{77}{240}\mathbf{v}_9 + \frac{1}{48}\mathbf{v}_{11} + \frac{1}{48}\mathbf{v}_{12}.$$

In ranking the barycentric coordinates in numerical order, we note that $\frac{1}{48} = \frac{5}{240}$. So, denoting the coefficient of $\mathbf{v}_k$ by t_k, as above, we find that

$$t_1 = t_4 = t_6 = t_8 = t_{10} < t_2 = t_{11} = t_{12} < t_0 = t_5 < t_7 < t_3 = t_9.$$

Thus the point $\mathbf{x}$ belongs to the interior of the simplex τ. Now

$$\begin{aligned} \frac{17}{240} - \frac{1}{48} &= \frac{17-5}{240} = \frac{12}{240} = \frac{1}{20}, \\ \frac{37}{240} - \frac{17}{240} &= \frac{20}{240} = \frac{1}{12}, \\ \frac{77}{240} - \frac{37}{240} &= \frac{40}{240} = \frac{1}{6}. \end{aligned}$$

It follows that

$$\begin{aligned}\mathbf{x} &= \frac{1}{48}(\mathbf{v}_0 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_5 + \mathbf{v}_7 + \mathbf{v}_9 + \mathbf{v}_{11} + \mathbf{v}_{12}) \\ &\quad + \frac{1}{20}(\mathbf{v}_0 + \mathbf{v}_3 + \mathbf{v}_5 + \mathbf{v}_7 + \mathbf{v}_9) \\ &\quad + \frac{1}{12}(\mathbf{v}_3 + \mathbf{v}_7 + \mathbf{v}_9) \\ &\quad + \frac{1}{6}(\mathbf{v}_3 + \mathbf{v}_9) \\ &= \frac{1}{3}\mathbf{w}_0 + \frac{1}{4}\mathbf{w}_1 + \frac{1}{4}\mathbf{w}_2 + \frac{1}{6}\mathbf{w}_3\end{aligned}$$