Course MA3486: Hilary Term 2018.

Solutions to Revision Problems.

1. *Let $F\colon \mathbb{R} \rightrightarrows \mathbb{R}$ and Let $G\colon \mathbb{R} \rightrightarrows \mathbb{R}$ be the correspondences defined such that*

$$F(x) = \begin{cases} \{y \in \mathbb{R} : y \geq e^x + 1\} & \text{if } x < 0, \\ \{y \in \mathbb{R} : y \geq e^{-x} - x^2\} & \text{if } x \geq 0, \end{cases}$$

$$G(x) = \begin{cases} \{y \in \mathbb{R} : y \geq e^{-x} + x^2\} & \text{if } x < 0; \\ \{y \in \mathbb{R} : y \geq e^x + 1\} & \text{if } x \geq 0. \end{cases}$$

Make the following determinations, justifying your answer in each case.

(a) *Determine whether or not the correspondence F is upper hemicontinuous at $x = 0$.*

This correspondence is upper hemicontinuous at $x = 0$. Note that $F(0) = [1, +\infty)$. Let V be an open set in $\mathbb{R}$ for which $F(0) \subset V$. Then $1 \in V$, and V is open in $\mathbb{R}$, and therefore there exists some real number s satisfying $s < 1$ for which $(s, 1] \subset V$. Then $(s, +\infty) \subset V$. Now $e^x > 0$ and therefore $F(x) \subset (1, \infty)$ whenever $x < 0$. It follows that $F(x) \subset V$ whenever $x < 0$. In order to complete the verification of upper hemicontinuity, we need to show that $F(x) \subset V$ for all positive values of x that lie sufficiently close to zero. Now the function $x \mapsto e^{-x} - x^2$ is decreasing for non-negative values of x. Nevertheless the continuity of this function ensures the existence of a positive real number δ with the property that $e^{-x} - x^2 > s$ whenever $0 \leq x < \delta$. Then $F(x) \subset (s, +\infty) \subset V$ for all real numbers x satisfying $|x| < \delta$. We conclude that the correspondence F is upper hemicontinuous at zero.

[N.B., it would, in principle, be possible to quantify matters, determining, given any value of s satisfying $s < 1$, a value of the positive real number δ that ensures that $F(x) \subset V$ whenever $0 \leq x < \delta$. But this would just make unnecessary work. An appeal to the (ε–δ) continuity of the relevant (single-valued) functions is sufficient.]

(b) *Determine whether or not the correspondence F is lower hemicontinuous at $x = 0$.*

The correspondence F is *not* lower hemicontinuous $x = 0$. Note that $F(0) = [1, +\infty)$ whereas $F(x) \subset (\frac{3}{2}, \infty)$ whenever $x < 0$. Thus the set $F(x)$ in some sense "abruptly collapses" as x moves away from zero in the negative direction. To get a formal counter-example we note that there exists a negative real number u such that $e^x + 1 > \frac{3}{2}$ whenever $u < x < 0$. Thus if, for example, $V = \{y \in \mathbb{R} : \frac{1}{2} < y < \frac{3}{2}$, then V is open in $\mathbb{R}$ and $F(0) \cap V \neq \emptyset$, but $F(x) \cap V = \emptyset$ for all real numbers x satisfying $u < x < 0$. Thus there cannot possibly exist any positive real number δ with the property that $F(x) \cap V \neq \emptyset$ whenever $|x| < \delta$. This concludes the verification that the correspondence F is not lower hemicontinuous at $x = 0$.

(c) *Determine whether or not the correspondence G is upper hemicontinuous at $x = 0$.*

The correspondence G is not upper hemicontinuous at $x = 0$. In outline $G(0) = [2, +\infty)$, and $G(x) \subset [2, +\infty)$ whenever $x \geq 0$. But as x moves from "leftwards" from zero to negative values, the set $G(x)$ "abruptly inflates" to intervals whose lower endpoint is close to 1.

To get an explicit counter-example, take $V = \{y \in \mathbb{R} : y > \frac{3}{2}$. Then V is open in $\mathbb{R}$, and $G(0) \subset V$. Now the function $x \mapsto e^{-x} + x^2$ is continuous at $x = 0$, where it takes the value 1. It follows from continuity that there exists some negative real number u such that $e^{-x} + x^2 < \frac{3}{2}$ whenever $u < x < 0$. It follows that if $u < x < 0$ then $G(x) \cap [0, \frac{3}{2}] \neq \emptyset$. Therefore $G(x)$ is not a subset of V for any real number x satisfying $u < x < 0$. Therefore the correspondence G cannot be upper hemicontinuous at $x = 0$.

(d) *Determine whether or not the correspondence G is lower hemicontinuous at $x = 0$.*

The correspondence G is lower hemicontinuous at $x = 0$. In summary, any open set V that has non-empty intersection with $G(0)$ must contain some real number greater than 2, and this real number will be in $G(x)$ for all values of x sufficiently close to 0.

Indeed let V be an open set in $\mathbb{R}$ with the property that $V \cap G(0) \neq \emptyset$. Now $G(0) = [2, +\infty)$, and if $2 \in V$ then there exists $\varepsilon_0 > 0$ such that $y \in V$ for all real numbers y satisfying $2 - \varepsilon_0 < y < 2 + \varepsilon_0$. Thus $V \cap [2, +\infty) \neq \emptyset$ if and only if $V \cap (2, +\infty) \neq \emptyset$. It follows that if V is an open set in $\mathbb{R}$, and if $V \cap G(0) \neq \emptyset$, then there exists $v \in V$ satisfying $v > 2$. It follows from the continuity of the functions $x \mapsto e^x + x^2$ and $x \mapsto e^{-x} + 1$ that there exists some positive real number δ such that $e^x + x^2 < v$ for all real numbers x satisfying $0 < x < \delta$ and $e^{-x} + 1 < v$ for all real numbers x satisfying $0 \leq x < \delta$. Then $v \in G(x)$ for all real numbers x satisfying $|x| < \delta$, and thus $G(x) \cap V \neq \emptyset$ for all real numbers x satisfying $|x| < \delta$.

2. *Let σ be the simplex with vertices $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, where*

$$\mathbf{v}_0 = (20, 30, 40), \quad \mathbf{v}_1 = (40, 60, 70),$$

$$\mathbf{v}_2 = (70, 70, 80), \quad \mathbf{v}_3 = (20, 50, 40),$$

and let $\mathbf{x} = (34, 50, 54)$. Determine the barycentric coordinates of $\mathbf{x}$ with respect to the vertices of σ, and hence determine the vertices of the unique simplex in the barycentric subdivision of σ that contains the point $\mathbf{x}$ in its interior.

Let t_0, t_1, t_2 and t_3 denote the barycentric coordinates of $\mathbf{x}$ with respect to the vertices of σ, so that

$$\mathbf{x} = t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3.$$

and $t_0 + t_1 + t_2 + t_3 = 1$. On substituting $t_0 = 1 - t_1 - t_2 - t_3$, we find that

$$\mathbf{x} - \mathbf{v}_0 = t_1(\mathbf{v}_1 - \mathbf{v}_0) + t_2(\mathbf{v}_2 - \mathbf{v}_0) + t_3(\mathbf{v}_3 - \mathbf{v}_0).$$

Thus

$$\begin{pmatrix} 14 \\ 20 \\ 17 \end{pmatrix} = t_1 \begin{pmatrix} 20 \\ 30 \\ 30 \end{pmatrix} + t_2 \begin{pmatrix} 50 \\ 40 \\ 40 \end{pmatrix} + t_3 \begin{pmatrix} 0 \\ 20 \\ 0 \end{pmatrix}.$$

This can be written as a set of simultaneous linear equations in the form

$$\left\{ \begin{array}{rcl} 20t_1 + 50t_2 & = & 14 \\ 30t_1 + 40t_2 + 20t_3 & = & 20 \\ 30t_1 + 40t_2 & = & 14 \end{array} \right.$$

Subtracting the third equation from the second, we find that $20t_3 = 6$. Thus $t_3 = 0.3$. Multiplying the first and third equations by 3 and 2 respectively, and then subtracting, we find that

$$60t_1 + 150t_2 = 42 \quad \text{and} \quad 60t_1 + 80t_2 = 28,$$

and therefore $70t_2 = 14$. Thus $t_2 = 0.2$. But then $20t_1 = 14 - 50t_2 = 14 - 10 = 4$, and therefore $t_1 = 0.2$. Finally $t_0 = 1 - t_1 - t_2 - t_3 = 0.3$. Thus

$$t_0 = 0.3, \quad t_1 = 0.2, \quad t_2 = 0.2, \quad t_3 = 0.3.$$

The unique simplex in the barycentric subdivision of σ containing the point $\mathbf{x}$ in its interiors as that whose vertices are the barycentres $\mathbf{w}_0$ and $\mathbf{w}_1$ of τ_0 and τ_1 respectively, where τ_0 has vertices $\mathbf{v}_0$ and $\mathbf{v}_3$ and τ_1 has vertices $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. Then

$$\mathbf{w}_0 = \tfrac{1}{2}(\mathbf{v}_0 + \mathbf{v}_3) = (20, 40, 40)$$

and

$$\mathbf{w}_1 = \tfrac{1}{4}(\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = (37.5, 52.5, 57.5).$$

With these values we find that $\mathbf{x} = 0.2\mathbf{w}_0 + 0.8\mathbf{w}_1$.

3. *Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_6$ be the vertices of a 7-simplex σ, let τ_0 be the face of σ spanned by $\mathbf{v}_3$ and $\mathbf{v}_5$, let τ_1 be the face of σ, spanned by $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_5$, and let τ_3 be the face of σ spanned by $\mathbf{v}_0$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_5$ and $\mathbf{v}_6$, and let $\mathbf{w}_0$, $\mathbf{w}_1$ and $\mathbf{w}_2$ denote the barycentres $\hat{\tau}_0$, $\hat{\tau}_1$ and $\hat{\tau}_2$ of the simplices τ_0, τ_1 and τ_2 respectively. Let $t_0, t_1, \ldots, t_6$ be the barycentric coordinates of some point $\mathbf{x}$ of σ with respect to $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_6$, so that*

$$\mathbf{x} = t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \cdots + t_6\mathbf{v}_6$$

where $0 \le t_i \le 1$ for $i = 0, 1, \ldots, 6$ and $\sum_{i=0}^{6} t_i = 1$. Determine necessary and sufficient conditions which, if satisfied by $t_0, t_1, \ldots, t_6$ ensure that the point $\mathbf{x}$ belongs to the 2-simplex of the first barycentric subdivision of σ spanned by vertices $\mathbf{w}_0$, $\mathbf{w}_1$ and $\mathbf{w}_2$. [Appropriately justify your answer.]

The necessary and sufficient conditions are that

$$0 = t_1 = t_4 \le t_0 = t_6 \le t_2 \le t_3 = t_5 \le 1$$

and $\sum_{i=0}^{6} t_i = 1$. [The latter condition on the sum of the t_i can be presumed to be satisfied without needing to be explicitly stated, given the inclusion of this condition in the statement of the problem.] Indeed the point $\mathbf{x}$ belongs to the simplex spanned by $\mathbf{w}_0$, $\mathbf{w}_1$ and $\mathbf{w}_2$ if and only if there exist non-negative real numbers u_0, u_1 and u_2 such that

$$\mathbf{x} = u_0\mathbf{w}_0 + u_1\mathbf{w}_1 + u_2\mathbf{w}_2,$$

where

$$\mathbf{w}_0 = \tfrac{1}{2}(\mathbf{w}_3 + \mathbf{w}_5), \quad \mathbf{w}_1 = \tfrac{1}{3}(\mathbf{w}_2 + \mathbf{w}_3 + \mathbf{w}_5),$$

$$\mathbf{w}_2 = \tfrac{1}{5}(\mathbf{w}_0 + \mathbf{w}_2 + \mathbf{w}_3 + \mathbf{w}_5 + \mathbf{w}_6)$$

and $u_0 + u_1 + u_2 = 1$. Let $\mathbf{x}$ be expressible in this fashion, and let $\mathbf{x} = \sum_{t=0}^{6} t_i \mathbf{v}_i$. It then follows from the affine independence of $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_6$ that

$$t_1 = t_4 = 0, \quad t_0 = t_6 = \tfrac{1}{5}u_2, \quad t_2 = \tfrac{1}{5}u_2 + \tfrac{1}{3}u_1,$$
$$t_3 = t_5 = \tfrac{1}{5}u_2 + \tfrac{1}{3}u_1 + \tfrac{1}{2}u_0.$$
$$0 = t_1 = t_4 \leq t_0 = t_6 \leq t_2 \leq t_3 = t_5 \leq 1,$$

because $u_0 \geq 0$, $u_1 \geq 0$ and $u_2 \geq 0$, and $\sum_{i=0}^{6} t_i = u_1 + u_2 + u_3 = 1$. Conversely, given $t_0, t_1, \ldots, t_6$ satisfying these conditions, let

$$u_2 = 5t_0, \quad u_1 = 3(t_2 - t_0), \quad u_0 = 2(t_3 - t_2).$$

Then $u_0 \geq 0$, $u_1 \geq 0$, $u_2 \geq 0$,

$$u_0 + u_1 + u_2 = 2t_0 + t_2 + 2t_3 = \sum_{i=0}^{6} t_i = 1.$$

Moreover

$$t_1 = t_4 = 0, \quad t_0 = t_6 = \tfrac{1}{5}u_2, \quad t_2 = \tfrac{1}{3}u_1 + t_0 = \tfrac{1}{3}u_1 + \tfrac{1}{5}u_2,$$
$$t_3 = t_5 = \tfrac{1}{2}u_0 + t_2 = \tfrac{1}{2}u_0 + \tfrac{1}{3}u_1 + \tfrac{1}{5}u_2,$$

and thus if $\mathbf{x} = \sum_{i=0}^{6} t_i \mathbf{v}_i$ then

$$\begin{aligned} \mathbf{x} &= \tfrac{1}{2}(\mathbf{v}_3 + \mathbf{v}_5) + \tfrac{1}{3}(\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_5) + \tfrac{1}{5}(\mathbf{v}_0 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_5 + \mathbf{v}_6) \\ &= u_0\mathbf{w}_0 + u_1\mathbf{w}_1 + u_2\mathbf{w}_2. \end{aligned}$$

Thus the conditions of $t_0, t_1, \ldots, t_6$ are indeed necessary and sufficient to ensure that $\mathbf{x}$ belongs to the simplex spanned by $\mathbf{w}_0$, $\mathbf{w}_1$ and $\mathbf{w}_2$.

4. *This question has the objective of demonstrating, by fairly direct calculation, that the main conclusions of Perron's Theorem (Theorem 6.12 in 2017/18) hold for positive* 2×2 *matrices. Let*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be such a matrix, where a, b, c *and* d *are strictly positive real numbers.*

(a) *Determine the characteristic polynomial of this matrix, and determine the roots of this characteristic polynomial in terms of a, b, c and d. Hence, by means of these calculations, show that the roots of the characteristic polynomial are both real, and are simple roots, and that the maximum of the two roots has absolute value greater than the minimum of these roots.*

The characteristic polynomial is $\chi(\lambda)$, where

$$\chi(\lambda) = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = \lambda^2 - (a+d)\lambda + ad - bc.$$

The roots of the characteristic polynomial are therefore $\frac{1}{2}(a + d + \sqrt{D})$ and $\frac{1}{2}(a + d - \sqrt{D})$, where by the formula

$$\tfrac{1}{2}\left((a+d) \pm \sqrt{(a+d)^2 - 4ad + 4bc}\right).$$

Now

$$\begin{aligned} D &= (a+d)^2 - 4ad + 4bc = a^2 + d^2 + 2ad - 4ad + 4bc \\ &= a^2 + d^2 - 2ad + 4bc = (d-a)^2 + 4bc, \end{aligned}$$

Now a, b, c and d are all strictly positive real numbers. It follows that $D > 0$. Thus the roots of the characteristic polynomial are real and distinct. Moreover the average of those two roots is the strictly positive real number $\frac{1}{2}(a + d)$. It follows that the two roots are simple roots, and the maximum of the two roots is strictly positive and has absolute value greater than that of the other root.

(b) *The Perron root (or Perron-Frobenius eigenvalue) μ is the maximum of the two eigenvalues of the given 2×2 matrix. Determine the coefficients of a corresponding eigenvector having at least one strictly positive coefficient, and verify that both coefficients of this eigenvector are then strictly positive.*

The Perron-Frobenius eigenvalue μ is given by the equation

$$\mu = \tfrac{1}{2}(a + d + \sqrt{D}),$$

where $D = (d-a)^2 + 4bc$ (see (a)). To find an eigenvector corresponding to the Perron-Frobenius eigenvalue μ, we must solve the vector equation

$$\begin{pmatrix} \mu - a & -b \\ -c & \mu - d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

to find (up to a scalar multiple) the values of the real numbers u and v in terms of a, b, c and d. We therefore require that

$$(\mu - a)u = bv$$

It follows that

$$\left(\frac{b}{\mu - a}, 1\right)$$

is an eigenvector corresponding to the Perron-Frobenius eigenvalue. This eigenvector will have positive coefficients if and only if $\mu - a > 0$. Now $\sqrt{D} = \sqrt{(d-a)^2 + 4bc} > |d - a|$. It follows that

$$\mu - a = \tfrac{1}{2}(d - a + \sqrt{D}) > \tfrac{1}{2}(d - a + |d - a|) \geq 0.$$

It follows that the eigenvector corresponding to the Perron-Frobenius eigenvalue specified above does have strictly positive coefficients, which accords with the general result guaranteed by Perron's Theorem in this situation.

5. *Let n be a positive integer, and let*

$$\Delta = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0} \textit{ and } \sum_{i=1}^{n} (\mathbf{x})_i = 1\}.$$

Let T be an $n \times n$ matrix with strictly positive coefficients. (It then follows that $T\mathbf{v} >> 0$ for all $\mathbf{v} \in \Delta$.) Let $f\colon \Delta \to \Delta$ be defined so that

$$f(\mathbf{v}) = \frac{1}{\sum\limits_{1=1}^{n} (T\mathbf{v})_i} T\mathbf{v}$$

for all $\mathbf{v} \in \mathbb{R}$. Show that an element $\mathbf{v}$ is an eigenvector of T if and only if it is a fixed point of the continuous map $f\colon \Delta \to \Delta$ (i.e., show that $\mathbf{v}$ is an eigenvector of T if and only if $f(\mathbf{v}) = \mathbf{v}$). Show also that if $\mathbf{v}$ is an eigenvector of T, and if $\mathbf{v} \in \Delta$, then $\mathbf{v} >> \mathbf{0}$. (Note that this result, combined with the Brouwer Fixed Point Theorem, guarantees that every positive matrix has at least one eigenvector with strictly positive coefficients.)

Let $\mathbf{v} \in \Delta$. Then $T\mathbf{v} >> 0$, because the non-negative vector $\mathbf{v}$ has at least one strictly positive coefficient and the $n \times n$ matrix T has strictly positive coefficients. Suppose that $\mathbf{v}$ is an eigenvector of T. Then there exists some real number λ for which $T\mathbf{v} = \lambda\mathbf{v}$. It then follows from the definition of the simplex Δ that

$$\sum_{i=1}^{n} (T\mathbf{v})_i = \lambda \sum_{i=1}^{n} (\mathbf{v})_i = \lambda.$$

Thus if $\mathbf{v} \in \Delta$ is an eigenvector of T then $\mathbf{v} = f(\mathbf{v})$. Conversely it follows immediately from the definition of the function f that if $f(\mathbf{v}) = \mathbf{v}$ then $T\mathbf{v} = \lambda\mathbf{v}$, where $\lambda = \sum_{i=1}^{n} (T\mathbf{v})_i$. Thus an element $\mathbf{v}$ of the simplex Δ is an eigenvector of T if and only if it is a fixed point of T.

Now let $\mathbf{v}$ be an eigenvector of T with eigenvalue λ. Then $\lambda > 0$ and $\mathbf{v} = \lambda^{-1}T\mathbf{v}$. But $T\mathbf{v} >> \mathbf{0}$. It follows that $\mathbf{v} >> \mathbf{0}$. Thus all coefficients of the eigenvector $\mathbf{v}$ have strictly positive coefficients.

In view of the above, the Brouwer Fixed Point Theorem guarantees that every square matrix with positive coefficients has at least one eigenvector with strictly positive coefficients.