Course MA3486: Hilary Term 2018.

Revision Problems.

1. Let $F\colon \mathbb{R} \rightrightarrows \mathbb{R}$ and Let $G\colon \mathbb{R} \rightrightarrows \mathbb{R}$ be the correspondences defined such that

$$F(x) = \begin{cases} \{y \in \mathbb{R} : y \geq e^x + 1\} & \text{if } x < 0, \\ \{y \in \mathbb{R} : y \geq e^{-x} - x^2\} & \text{if } x \geq 0, \end{cases}$$

$$G(x) = \begin{cases} \{y \in \mathbb{R} : y \geq e^{-x} + x^2\} & \text{if } x < 0; \\ \{y \in \mathbb{R} : y \geq e^x + 1\} & \text{if } x \geq 0. \end{cases}$$

Make the following determinations, justifying your answer in each case.

(a) Determine whether or not the correspondence F is upper hemicontinuous at $x = 0$.

(b) Determine whether or not the correspondence F is lower hemicontinuous at $x = 0$.

(c) Determine whether or not the correspondence G is upper hemicontinuous at $x = 0$.

(d) Determine whether or not the correspondence G is lower hemicontinuous at $x = 0$.

2. Let σ be the simplex with vertices $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, where

$$\mathbf{v}_0 = (20, 30, 40), \quad \mathbf{v}_1 = (40, 60, 70),$$

$$\mathbf{v}_2 = (70, 70, 80), \quad \mathbf{v}_3 = (20, 50, 40),$$

and let $\mathbf{x} = (34, 50, 54)$. Determine the barycentric coordinates of $\mathbf{x}$ with respect to the vertices of σ, and hence determine the vertices of the unique simplex in the barycentric subdivision of σ that contains the point $\mathbf{x}$ in its interior.

3. Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_6$ be the vertices of a 7-simplex σ, let τ_0 be the face of σ spanned by $\mathbf{v}_3$ and $\mathbf{v}_5$, let τ_1 be the face of σ, spanned by $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_5$, and let τ_3 be the face of σ spanned by $\mathbf{v}_0$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_5$ and $\mathbf{v}_6$, and let $\mathbf{w}_0$, $\mathbf{w}_1$ and $\mathbf{w}_2$ denote the barycentres $\hat{\tau}_0$, $\hat{\tau}_1$ and $\hat{\tau}_2$ of the simplices τ_0, τ_1 and τ_2 respectively. Let $t_0, t_1, \ldots, t_6$ be the barycentric coordinates of some point $\mathbf{x}$ of σ with respect to $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_6$, so that

$$\mathbf{x} = t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \cdots + t_6\mathbf{v}_6$$

where $0 \leq t_i \leq 1$ for $i = 0, 1, \ldots, 6$ and $\sum\limits_{i=0}^{6} t_i = 1$. Determine necessary and sufficient conditions which, if satisfied by $t_0, t_1, \ldots, t_6$ ensure that the point $\mathbf{x}$ belongs to the 2-simplex of the first barycentric subdivision of σ spanned by vertices $\mathbf{w}_0$, $\mathbf{w}_1$ and $\mathbf{w}_2$. [Appropriately justify your answer.]

4. This question has the objective of demonstrating, by fairly direct calculation, that the main conclusions of Perron's Theorem (Theorem 6.12 in 2017/18) hold for positive 2×2 matrices. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be such a matrix, where a, b, c and d are strictly positive real numbers.

(a) Determine the characteristic polynomial of this matrix, and determine the roots of this characteristic polynomial in terms of a, b, c and d. Hence, by means of these calculations, show that the roots of the characteristic polynomial are both real, and are simple roots, and that the maximum of the two roots has absolute value greater than the minimum of these roots.

(b) The Perron root (or Perron-Frobenius eigenvalue) μ is the maximum of the two eigenvalues of the given 2×2 matrix. Determine the coefficients of a corresponding eigenvector having at least one strictly positive coefficient, and verify that both coefficients of this eigenvector are then strictly positive.

5. Let n be a positive integer, and let

$$\Delta = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0} \text{ and } \sum_{i=1}^{n} (\mathbf{x})_i = 1\}.$$

Let T be an $n \times n$ matrix with strictly positive coefficients. (It then follows that $T\mathbf{v} >> 0$ for all $\mathbf{v} \in \Delta$.) Let $f \colon \Delta \to \Delta$ be defined so that

$$f(\mathbf{v}) = \frac{1}{\sum\limits_{1=1}^{n} (T\mathbf{v})_i} T\mathbf{v}$$

for all $\mathbf{v} \in \mathbb{R}$. Show that an element $\mathbf{v}$ is an eigenvector of T if and only if it is a fixed point of the continuous map $f \colon \Delta \to \Delta$ (i.e., show that $\mathbf{v}$ is an eigenvector of T if and only if $f(\mathbf{v}) = \mathbf{v}$). Show also that if $\mathbf{v}$ is an eigenvector of T, and if $\mathbf{v} \in \Delta$, then $\mathbf{v} >> \mathbf{0}$. (Note that this result, combined with the Brouwer Fixed Point Theorem, guarantees that every positive matrix has at least one eigenvector with strictly positive coefficients.)