Worked Solutions to Non-Bookwork Questions on the MA3486 Annual Paper 2017

1. (a) *Let $\Phi: \mathbb{R} \rightrightarrows \mathbb{R}$ be the correspondence defined such that*

$$\Phi(x) = \begin{cases} \{y \in \mathbb{R} : x \leq y \leq x + 2\} & \text{if } x \leq 0; \\ \{y \in \mathbb{R} : 0 \leq xy^2 \leq 1\} & \text{if } 0 < x \leq 1; \\ \{y \in \mathbb{R} : x - 2 \leq y \leq x\} & \text{if } x > 1. \end{cases}$$

Answer the following, fully justifying your answers:—

(i) *Is $\Phi: \mathbb{R} \to \mathbb{R}$ upper hemicontinuous at $x = 0$?*

The correspondence Φ is not upper hemicontinuous at $x = 0$. Note that $\Phi(0) = [0, 2]$. Let $V = (-1, 3) = \{y \in \mathbb{R} : -1 \leq y < 3\}$. Then V is open in $\mathbb{R}$ and $\Phi(0) \subset V$. But $4 \not\in V$, though $4 \in \Phi(x)$ for all real numbers x satisfying $0 < x < \frac{1}{16}$. It follows that there cannot exist any positive real number δ with the property that $\Phi(x) \subset V$ for all real numbers x satisfying $|x| < \delta$.

(ii) *Is $\Phi: \mathbb{R} \to \mathbb{R}$ lower hemicontinuous at $x = 0$?*

The correspondence Φ is lower hemicontinuous at 0. Let V be an open set in $\mathbb{R}$ for which $\Phi(0) \cap V \neq \emptyset$. Now $\Phi(0) = [0, 2]$. Therefore the open set V contains at least one real number in the interval $[0, 2]$. Moreover, because V is an open set, if $2 \in V$ then V also contains positive real numbers less than 2. Let $s \in V$, where $0 < v < 2$. If $v - 2 \leq x \leq 0$ then $v \in \Phi(x)$. Also if $0 < x < \frac{1}{4}$ then $0 \leq xv^2 < 1$ and therefore $v \in \Phi(x)$. Thus if $\delta \leq 2 - v$ and $\delta \leq \frac{1}{4}$ then $v \in \Phi(x) \cap V$ for all real numbers x satisfying $|x| < \delta$. It follows that $\Phi: \mathbb{R} \rightrightarrows \mathbb{R}$ is lower hemicontinuous at 0.

(iii) *Is $\Phi: \mathbb{R} \to \mathbb{R}$ upper hemicontinuous at $x = 1$?*

The correspondence Φ is upper hemicontinuous at 0. First note that $\Phi(1) = [-1, 1]$. Let V be an open set in $\mathbb{R}$ for which $[-1, 1] \subset V$. Then there exists $s > 1$ for which $[-s, s] \subset V$. If $1 < x < s$ then $[x - 2, x] \subset [-1, s) \subset V$. Also if $x > 1/s^2$ then $\Phi(x) \subset [-s, s] \subset V$. Thus if δ is the minimum of $s - 1$ and $1 - 1/s^2$ then $\Phi(x) \subset V$ for all real numbers x satisfying $|x - 1| < \delta$. It follows that the correspondence Φ is upper hemicontinuous at 1.

(iv) *Is $\Phi: \mathbb{R} \to \mathbb{R}$ lower hemicontinuous at $x = 1$?*

The correspondence Φ is lower hemicontinuous at 0. Let V be an open set in $\mathbb{R}$ that intersects $\Phi(1)$. Now $\Phi(1) = [-1, 1]$. If $1 \in V$, or if $-1 \in V$ then $v \in V$ for some real number v satisfying $-1 < v < 1$. It follows that an open set V which has non-empty intersection with $\Phi(1)$ must always contain some real number v satisfying $-1 < v < 1$. Then $v \in \Phi(x)$ for all real numbers x satisfying $1 < x \leq v + 2$ (where $v + 2 > 1$), and also $v \in \Phi(x)$ for all real numbers x satisfying $0 < x \leq 1$. Thus if $0 < \delta < 1$ and $\delta \leq v + 1$, and if $|x - 1| < \delta$ then $0 < x < v + 2$, and therefore $v \in \Phi(x)$. Thus $\Phi(x) \cap V \neq \emptyset$ for all real numbers x satisfying $|x - 1| < \delta$.

2. (c) *The point* $(6, 7)$ *of* $\mathbb{R}^2$ *belongs to the triangle in* $\mathbb{R}^2$ *with vertices* $(0, 0)$, $(6, 6)$ *and* $(8, 10)$. *Determine the barycentric coordinates of* $(6, 7)$ *with regard to the vertices of this triangle.*

We need to determine real numbers t_0, t_1 and t_2 between 0 and 1 such that

$$t_0(0, 0) + t_1(6, 6) + t_2(8, 10) = (6, 7)$$

and $t_0 + t_1 + t_2 = 1$. Then $6t_1 + 8t_2 = 6$ and $6t_1 + 10t_2 = 7$. Subtracting the second equation from the first, we find that $2t_2 = 1$ and therefore $t_2 = \frac{1}{2}$. Then $6t_1 + 4 = 6$ and therefore $t_1 = \frac{1}{3}$. Using the identity $t_0 + t_1 + t_2 = 1$, we deduce that $t_0 = \frac{1}{6}$. Thus the barycentric coordinates of the point $(6, 7)$ with respect to the given vertices are (in the given order) $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}$.

(d) *Let* σ *be a 5-dimensional simplex with vertices* $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, *let* K *be the simplicial complex consisting of the simplex* σ *together with all its faces, let* K' *be the first barycentric subdivision of* σ, *let*

$$\mathbf{x} = \tfrac{1}{5}\mathbf{v}_0 + \tfrac{7}{15}\mathbf{v}_1 + \tfrac{2}{15}\mathbf{v}_3 + \tfrac{1}{5}\mathbf{v}_4,$$

and let τ *be the unique simplex of* K' *that contains the point* $\mathbf{x}$ *in its interior (so that* $\mathbf{x}$ *belongs to* τ *but does not belong to any proper face of* τ*). Determine the barycentric coordinates of the vertices of* τ *with respect to the vertices of* σ, *and determine the barycentric coordinates of the point* $\mathbf{x}$ *with respect to the vertices of* τ.

The ordering of the values of the barycentric coordinates of $\mathbf{x}$ with respect to the vertices of σ is as follows:

$$\frac{2}{15}, \quad \frac{1}{5}, \quad \frac{7}{15}.$$

(Note that $\frac{1}{5} = \frac{3}{15}$.) It follows that

$$\begin{aligned} \mathbf{x} &= \tfrac{2}{15}(\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_3 + \mathbf{v}_4) + \tfrac{1}{15}(\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_4) + \tfrac{4}{15}\mathbf{v}_1 \\ &= \tfrac{4}{15}\mathbf{w}_0 + \tfrac{1}{5}\mathbf{w}_1 + \tfrac{8}{15}\mathbf{w}_2, \end{aligned}$$

where

$$\mathbf{w}_0 = \mathbf{v}_1, \quad \mathbf{w}_1 = \tfrac{1}{3}(\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_4), \quad \mathbf{w}_2 = \tfrac{1}{4}(\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_3 + \mathbf{v}_4).$$

The simplex τ has vertices $\mathbf{w}_0$, $\mathbf{w}_1$ and $\mathbf{w}_2$, as specified above. The barycentric coordinates of the vertices of τ with respect to the vertices of σ, in the given order, are as follows: for $\mathbf{w}_0$, $0, 0, 0, 0, 1$; for $\mathbf{w}_1$, $\frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3}$; for $\mathbf{w}_2$, $\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}, \frac{1}{4}$. Moreover the barycentric coordinates of $\mathbf{x}$ with respect to the vertices of τ are, in the given order, $\frac{4}{15}, \frac{1}{5}, \frac{8}{15}$. (Note that these barycentric coordinates do indeed add up to 1, as they are required to do.)