Module MA3486: Annual Examination 2018
Worked solutions

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Module Website

The module website, with online lecture notes, problem sets. etc. are located at

`http://www.maths.tcd.ie/~dwilkins/Courses/MA3486/`

Notes

The examiner generally assesses the length of solutions, and thus judges the time taken to complete them on the basis of the length of the solution, which would normally approximate to one-and-a-half pages when typeset with LaTeX.

There are exceptions. If the worked solution comes out longer (as tends to happening with linear programming problems in another module taught in other years), then the examiner often tests the problem, writing out by hand. This consideration does not seem applicable to the current paper.

If the problem requires the candidate to come up with proof-style arguments that are unseen, then this would suggest a shorter worked solution. The solution to question 1 on this paper is shorter than the others, but the examiner suggests that, because the problem requires unseen proofs, as opposed to routine calculations, it may not be advisable to consider extending the length of this question by adding any further parts. The examiner further notes that the "Sample Paper" and the actual annual paper available to candidates from 1986 contains specific examples that are essentially being generalized in the context of the current question 1, and the question from the 2016 "Sample Paper" has already been discussed in a "tutorial" session, and worked solutions to the 2016 Annual Examination paper have already been made available to the class on the module website. Thus well-prepared candidates may well have particular instances of the more general results near the surface of their minds to reflect on when attempting the question.

The answer to question 2(b) may be longer, but it seems more diffuse and calculational. The general result has been included in one of the appendices to the official lecture notes, and was covered in a sparsely-attended class at 9am on Friday of the sixth week of term. (The length of the printed proof seems surprisingly long in relation to the depth and difficulty of the general result.) Currently it is intended that a broadly similar problem will be included in a "tutorial" in a lecture slot within the next four weeks, probably set up in the context of the first barycentric subdivision of a tetrahedron (or maybe a 5-simplex). Assuming this happens, the corresponding example problem discussed in class may well be embedded in a sequence of problems in some future problem set so as not to seem obviously destined for the examination paper.

Question 3 is bookwork. Sperner's Lemma has come up fairly regularly, both in the small number of old MA3486 papers, and also in old Algebraic Topology MA421 papers. It seems that the application of Sperner's Lemma to prove the non-existence of a continuous retraction from an n-simplex to its boundary is here appearing *on an MA3486 paper* for the first time.

Question 4 is also bookwork. Practically all lecture material from 2016 has already been covered, and therefore almost everything intended for the final weeks after Study Week is, at the time of writing, vapourware.

A set of draft notes on Perron-Frobenius Theory, reaching Perron's original theorem for positive square matrices, has already been written, based on and amplifying the rather sketchy account in Appendix C of J.W.S. Cassels, *Economics for Mathematicians*, (L.M.S. Lecture Note Series 62). The plan is to discuss applications of these results to Leontieff Models (discussed in Cassels's Chapter 5, entitled *Linear Economic Models*). The draft notes written to date should support two to three potential bookwork-style examination questions, of which that included on the current MA3486 paper is one.

1. [Seen analogous problems, but with $f(x)$ and $g(x)$ replaced by explicit given functions. The more general problem here should be unseen.]

   (a) The correspondence is upper hemicontinuous at 0 in this case. Let V be an open set for which $\Phi(0) \subset V$. Then $[0, g(x)] \subset V$. It follows from the openness of V that there exists a real number v satisfying $v > g(x)$ for which $[0, v) \subset V$. Then $f(0) \leq g(0) < v$. It follows from the continuity of the function f that there exist positive real numbers δ_1 and δ_2 such that $f(x) < v$ whenever $-\delta_1 < x \leq 0$ and $g(x) < v$ whenever $0 \leq x < \delta_2$. Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and if $-\delta < x < \delta$ then $\Phi(x) \subset [0, v]$ and therefore $\Phi(x) \subset V$. Thus Φ is upper hemicontinuous at 0 in this case.

   (b) The correspondence is not lower hemicontinuous at 0 in this case. There exist real numbers u and v for which $f(0) < u < v < g(0)$. It follows from the continuity of f that there exists a positive real number δ such that $f(x) < u$ whenever $-\delta < x \leq 0$. Let $V = (u, v)$. Then V is open in $\mathbb{R}$, $V \cap \Phi(0) \neq \emptyset$, but $V \cap \Phi(x) = \emptyset$ for all real numbers x satisfying $-\delta < x < 0$. Thus Φ is not lower hemicontinuous at 0 in this case.

   (c) The correspondence is not upper hemicontinuous at 0 in this case. There exists a real number v satisfying $g(x) < v < f(x)$. The continuity of f then ensures the existence of a positive real number δ such that $f(x) > v$ whenever $0 \leq x < \delta$. Let $V = (-1, v)$. Then V is an open set in $\mathbb{R}$. Now $\Phi(0) = [0, g(x)]$. It follows that $\Phi(0) \subset V$. But $\Phi(x) = [0, f(x)]$ when $0 < x < \delta$, and therefore $\Phi(x) \not\subset V$ when $0 < x < \delta$. Thus Φ is not upper hemicontinuous at 0 in this case.

   (d) The correspondence is lower hemicontinuous at 0 in this case. Let V be an open set for which $V \cap \Phi(0) \neq \emptyset$. If $g(0) \in V$ then there exists some real number u satisfying $0 < u < g(0)$ for which $u \in V$, because $g(x) > 0$ and the set V is open. If $g(0) \notin V$ then the condition $V \cap \Phi(0) \neq \emptyset$ ensures the existence of a real number u satisfying $0 \leq u < g(0)$ for which $u \in V$. It follows from the continuity of f and g that there exist positive real numbers δ_1 and δ_2 such that $f(x) > u$ whenever $-\delta_1 < x \leq -0$ and $g(x) > u$ whenever $0 \leq x \leq \delta_2$. Let δ be the minimum of δ_1 and δ_2. if $-\delta < x < \delta$ then $u \in \Phi(x)$, and thus $V \cap \Phi(x) \neq \emptyset$. Thus Φ is lower hemicontinuous at 0 in this case.

2. (a) [Definitions.] A *simplex* in $\mathbb{R}^k$ of dimension q with vertices

$$\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$$

is defined to be a set of the form

$$\left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j : 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^{q} t_j = 1 \right\},$$

where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent points of $\mathbb{R}^k$. Let $\mathbf{x}$ be a point of this simplex. Then $\mathbf{x} = \sum_{j=0}^{q} t_j \mathbf{v}_j$ where $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$. The coefficients t_j of the vertices in this expression are the barycentric coordinate of the point $\mathbf{x}$. The barycentre of the simplex is the point whose barycentric coordinates are all equal to $1/(q+1)$, where q is the dimension of the simplex.

(b) [Seen similar.] The definition of simplices ensures that

$$\sigma = \{t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 : t_0, t_1, t_2 \in [0,1],\ \ t_0 + t_1 + t_2 = 1\}$$

and

$$\tau = \{u_0\hat{\sigma}_0 + u_1\hat{\sigma}_1 + u_2\hat{\sigma}_2 : u_0, u_1, u_2 \in [0,1],\ \ u_0 + u_1 + u_2 = 1\}.$$

Now

$$\hat{\sigma}_0 = \mathbf{v}_2, \quad \hat{\sigma}_1 = \tfrac{1}{2}(\mathbf{v}_0 + \mathbf{v}_2), \quad \hat{\sigma}_2 = \tfrac{1}{3}(\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2).$$

It follows that

$$\begin{aligned} \tau \ = \ & \{(\tfrac{1}{3}u_2 + \tfrac{1}{2}u_1)\mathbf{v}_0 + \tfrac{1}{3}u_2\mathbf{v}_1 + (\tfrac{1}{3}u_2 + \tfrac{1}{2}u_1 + u_0)\mathbf{v}_2 : \\ & \quad u_0, u_1, u_2 \in [0,1] \text{ and } u_0 + u_1 + u_2 = 1\}. \end{aligned}$$

So let $u_0, u_1, u_2 \in [0,1]$ satisfy $u_0 + u_1 + u_2 = 1$, and let t_0, t_1 and t_2 be the real numbers determined by the equations

$$t_0 = \tfrac{1}{3}u_2 + \tfrac{1}{2}u_1, \quad t_1 = \tfrac{1}{3}u_2 \quad \text{and} \quad t_2 = \tfrac{1}{3}u_2 + \tfrac{1}{2}u_1 + u_0.$$

Then $t_0 \geq 0$, $t_1 \geq 0$, $t_2 \geq 0$ and $t_0 + t_1 + t_2 = 1$. It then follows that $t_0, t_1, t_2 \in [0,1]$. Moreover $t_1 \leq t_0 \leq t_2$. We conclude that

$$\begin{aligned} \tau \ \subset \ & \{t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 : \\ & \quad t_0, t_1, t_2 \in [0,1],\ \ t_0 + t_1 + t_2 = 1 \text{ and } t_1 \leq t_0 \leq t_2\}, \end{aligned}$$

Now let t_0, t_1 and t_2 be real numbers in the interval $[0, 1]$ that satisfy the conditions $t_0 + t_1 + t_2 = 1$ and $t_1 \leq t_0 \leq t_2$. We seek to determine real numbers u_0, u_1 and u_2 for which

$$t_0 = \tfrac{1}{3}u_2 + \tfrac{1}{2}u_1, \quad t_1 = \tfrac{1}{3}u_2 \quad \text{and} \quad t_2 = \tfrac{1}{3}u_2 + \tfrac{1}{2}u_1 + u_0.$$

Clearly $u_0 = t_2 - t_0$ $u_1 = 2(t_0 - t_1)$ and $u_3 = 3t_1$. The conditions $t_1 \geq 0$, $t_1 \geq 0$ and $t_1 \leq t_0 \leq t_2$ ensure that $u_0 \geq 0$, $u_1 \geq 0$ and $u_2 \geq 0$. Moreover

$$u_0 + u_1 + u_2 = t_2 - t_0 + 2(t_0 - t_1) + 3t_1 = t_0 + t_1 + t_2 = 1.$$

Also

$$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 = u_0\hat{\sigma}_0 + u_1\hat{\sigma}_1 + u_2\hat{\sigma}_2.$$

It follows that

$$\begin{aligned} \tau \supset \; & \{t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 : \\ & \quad t_0, t_1, t_2 \in [0, 1], \;\; t_0 + t_1 + t_2 = 1 \;\text{ and }\; t_1 \leq t_0 \leq t_2\}, \end{aligned}$$

Therefore

$$\begin{aligned} \tau = \; & \{t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 : \\ & \quad t_0, t_1, t_2 \in [0, 1], \;\; t_0 + t_1 + t_2 = 1 \;\text{ and }\; t_1 \leq t_0 \leq t_2\}, \end{aligned}$$

as required.

3. (a) [Bookwork.] Given integers $i_0, i_1, \ldots, i_q$ between 0 and n, let $N(i_0, i_1, \ldots, i_q)$ denote the number of q-simplices of K whose vertices are labelled by $i_0, i_1, \ldots, i_q$ (where an integer occurring k times in the list labels exactly k vertices of the simplex). We must show that $N(0, 1, \ldots, n)$ is odd.

We prove the result by induction on the dimension n of the simplex Δ; it is clearly true when $n = 0$. Suppose that the result holds in dimensions less than n. For each simplex σ of K of dimension n, let $p(\sigma)$ denote the number of $(n-1)$-faces of σ labelled by $0, 1, \ldots, n-1$. If σ is labelled by $0, 1, \ldots, n$ then $p(\sigma) = 1$; if σ is labelled by $0, 1, \ldots, n-1, j$, where $j < n$, then $p(\sigma) = 2$; in all other cases $p(\sigma) = 0$. Therefore

$$\sum_{\substack{\sigma\in K\\ \dim\sigma=n}} p(\sigma) = N(0, 1, \ldots, n) + 2\sum_{j=0}^{n-1} N(0, 1, \ldots, n-1, j).$$

Now the definition of Sperner labellings ensures that the only $(n-1)$-face of Δ containing simplices of K labelled by $0, 1, \ldots, n-1$ is that with vertices labelled by $0, 1, \ldots, n-1$. Thus if M is the number of $(n-1)$-simplices of K labelled by $0, 1, \ldots, n-1$ that are contained in this face, then $N(0, 1, \ldots, n-1) - M$ is the number of $(n-1)$-simplices labelled by $0, 1, \ldots, n-1$ that intersect the interior of Δ. It follows that

$$\sum_{\substack{\sigma\in K\\ \dim\sigma=n}} p(\sigma) = M + 2\big(N(0, 1, \ldots, n-1) - M\big),$$

since any $(n-1)$-simplex of K that is contained in a proper face of Δ must be a face of exactly one n-simplex of K, and any $(n-1)$-simplex that intersects the interior of Δ must be a face of exactly two n-simplices of K. On combining these equalities, we see that $N(0, 1, \ldots, n) - M$ is an even integer. But the induction hypothesis ensures that Sperner's Lemma holds in dimension $n-1$, and thus M is odd. It follows that $N(0, 1, \ldots, n)$ is odd, as required.

(b) [Bookwork.] Suppose that such a map $r\colon \Delta \to \partial\Delta$ were to exist. It would then follow from the Simplicial Approximation Theorem that there would exist a simplicial approximation $s\colon K \to L$ to the map r, where L is the simplicial complex consisting of all of the proper faces of Δ, and K is the jth barycentric subdivision, for some sufficiently large j, of the simplicial complex consisting of the simplex Δ together with all of its faces.

If $\mathbf{v}$ is a vertex of K belonging to some proper face Σ of Δ then $r(\mathbf{v}) = \mathbf{v}$, and hence $s(\mathbf{v})$ must be a vertex of Σ, since $s\colon K \to L$ is a simplicial approximation to $r\colon \Delta \to \partial\Delta$. In particular $s(\mathbf{v}) = \mathbf{v}$ for all vertices $\mathbf{v}$ of Δ. Thus if $\mathbf{v} \mapsto m(\mathbf{v})$ is a labelling of the vertices of Δ by the integers $0, 1, \ldots, n$, then $\mathbf{v} \mapsto m(s(\mathbf{v}))$ is a Sperner labelling of the vertices of K. Thus Sperner's Lemma guarantees the existence of at least one n-simplex σ of K labelled by $0, 1, \ldots, n$. But then $s(\sigma) = \Delta$, which is impossible, since Δ is not a simplex of L. We conclude therefore that there cannot exist any continuous map $r\colon \Delta \to \partial\Delta$ satisfying $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$.

4. (a) [Bookwork.] Suppose that the matrix T is positive. Then $T_{j,k} > 0$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Let $\mathbf{v} \in \mathbb{R}^n$ satisfy both $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{v} \geq \mathbf{0}$. Then

$$(T\mathbf{v})_j = \sum_{k=1}^{n} (T)_{j,k}(\mathbf{v})_k > 0$$

for each integer j between 1 and m, because $(T)_{j,k}(\mathbf{v})_k \geq 0$ for $k = 1, 2, \ldots, n$ and $(T)_{j,k}(\mathbf{v})_k > 0$ for at least one integer k between 1 and n. Therefore $T\mathbf{v} >> \mathbf{0}$.

Conversely suppose that T is an $m \times n$ matrix with with real coefficients which has the property that if and only if $T\mathbf{v} >> 0$ for all non-zero n-dimensional vectors $\mathbf{v}$ with non-negative real coefficients. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$ with

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, \ldots, 0), \ldots, \quad \mathbf{0}_n = (0, 0, \ldots, 1).$$

Then $T\mathbf{e}_k >> \mathbf{0}$ for $k = 1, 2, \ldots, n$, and therefore $(T)_{j,k} = (T\mathbf{e}_k)_j > 0$ for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$. The result follows.

(b) [Bookwork.] The definition of the Perron root μ of T ensures that there exists a non-zero vector $\mathbf{b}$ with the properties that $\mathbf{b} \geq \mathbf{0}$ and $T\mathbf{b} \geq \mu\mathbf{b}$. Suppose it were the case that $T\mathbf{b} \neq \mu\mathbf{b}$. Let $\mathbf{v} = T\mathbf{b}$. Then

$$T\mathbf{v} - \mu\mathbf{v} = T(T\mathbf{b} - \mu\mathbf{b}) >> \mathbf{0},$$

because $T\mathbf{b} - \mu\mathbf{b} \geq \mathbf{0}$, $T\mathbf{b} - \mu\mathbf{b} \neq \mathbf{0}$ and $T >> 0$ (by (a)). But then there would exist real numbers ρ satisfying $\rho > \mu$ that were sufficiently close to μ to ensure that $T\mathbf{v} - \rho\mathbf{v} >> 0$ and thus $T\mathbf{v} \geq \rho\mathbf{v}$. This would contradict the condition on the statement of the proposition that characterizes the value of μ. We conclude therefore that $T\mathbf{b} = \mu\mathbf{b}$. Now $T\mathbf{b} >> 0$, because $T >> 0$ and $\mathbf{b} \geq 0$. It follows that $\mu > 0$ and $\mathbf{b} >> \mathbf{0}$.

(c) [Bookwork.] Let $\mathbf{u}$ be an n-dimensional vector with real coefficients for which $T\mathbf{u} \geq \mu\mathbf{u}$. If s is positive and sufficiently large then then $s\mathbf{b} - \mathbf{u} >> 0$. On the other hand if s is negative and $|s|$ is sufficiently large then then $s\mathbf{b} - \mathbf{u} << \mathbf{0}$. It follows from this that there exists a well-defined real number t defined so that

$$t = \inf\{s \in \mathbb{R} : s\mathbf{b} - \mathbf{u} \geq \mathbf{0}\}.$$

Then $t\mathbf{b}-\mathbf{u} \geq 0$, and moreover there exists some integer j between 1 and n for which $t(\mathbf{b})_j - (\mathbf{u})_j = 0$. Now

$$T(t\mathbf{b} - \mathbf{u}) = \mu t\mathbf{b} - T\mathbf{u} \leq \mu(t\mathbf{b} - \mathbf{u}),$$

and therefore $(T(t\mathbf{b} - \mathbf{u}))_j \leq 0$. If it were the case that $t\mathbf{b} - \mathbf{u} \neq 0$ then the inequalities $t\mathbf{b} - \mathbf{u} \geq 0$ and $T >> 0$ would ensure that $T(t\mathbf{b} - \mathbf{u}) >> 0$ (by (a)), from which it would follow that $(T(t\mathbf{b} - \mathbf{u}))_j > 0$. Because this latter inequality does not hold, it must be the case that $t\mathbf{b} - \mathbf{u} = 0$, and thus $\mathbf{u} = t\mathbf{b}$. The result follows.