Module MA3486: Fixed Point Theorems and Economic Equilibria
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Part III (Sections 6 to 8)

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Contents

6 Perron-Frobenius Theory

6.1 Eigenvectors of Non-Negative Matrices

We establish some notation that will be used throughout this section.

Let m and n be positive integers. Given any $m \times n$ matrix T, we denote by $(T)_{i,j}$ the coefficient in the ith row and jth column of the matrix T for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Also given any n-dimensional vector $\mathbf{v}$, we denote by $(\mathbf{v})_j$ the jth coefficient of the vector j for $j = 1, 2, \ldots, n$.

Definition A matrix T is said to be *non-negative* if all its coefficients are non-negative real numbers.

Definition A matrix T is said to be *positive* if all its coefficients are strictly positive real numbers.

Let S and T be $m \times n$ matrices. If $(S)_{i,j} \leq (T)_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, then we denote this fact by writing $S \leq T$, or by writing $T \geq S$. If $(S)_{i,j} < (T)_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, then we denote this fact by writing $S << T$, or by writing $T >> S$.

Let $\mathbf{u}$ and $\mathbf{u}$ be n-dimensional vectors. If $(\mathbf{u})_j \leq (\mathbf{v})_j$ for $j = 1, 2, \ldots, n$, then we denote this fact by writing $\mathbf{u} \leq \mathbf{v}$, or by writing $\mathbf{v} \geq \mathbf{u}$. If $(\mathbf{u})_j < (\mathbf{v})_j$ for $j = 1, 2, \ldots, n$, then we denote this fact by writing $\mathbf{u} << \mathbf{v}$, or by writing $\mathbf{v} >> \mathbf{u}$.

A matrix T with real coefficients is thus *non-negative* if and only if $T \geq 0$. A matrix T with real coefficients is *positive* if and only if $T >> 0$.

Lemma 6.1 *Let T be an $m \times n$ matrix with real coefficients. Then T is a non-negative matrix if and only if $T\mathbf{v} \geq 0$ for all $\mathbf{v} \in \mathbb{R}^n$ satisfying $\mathbf{v} \geq \mathbf{0}$.*

Proof Suppose that the matrix T is non-negative. Let $\mathbf{v} \in \mathbb{R}^n$ satisfy $\mathbf{v} \geq \mathbf{0}$. Then

$$(T\mathbf{v})_j = \sum_{k=1}^{n} (T)_{j,k} (\mathbf{v})_k \geq 0$$

for each integer j between 1 and m, because $(T)_{j,k}(\mathbf{v})_k \geq 0$ for $k = 1, 2, \ldots, n$. Therefore $T\mathbf{v} \geq \mathbf{0}$.

Conversely suppose that T is an $m \times n$ matrix with with real coefficients which has the property that if and only if $T\mathbf{v} \geq 0$ for all non-zero n-dimensional vectors $\mathbf{v}$ with non-negative real coefficients. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$ with

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, \ldots, 0), \ldots, \quad \mathbf{0}_n = (0, 0, \ldots, 1).$$

Then $T\mathbf{e}_k \geq \mathbf{0}$ for $k = 1, 2, \ldots, n$, and therefore $(T)_{j,k} = (T\mathbf{e}_k)_j \geq 0$ for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$. The result follows. ∎

Lemma 6.2 *Let T be an $m \times n$ matrix with real coefficients. Then T is a positive matrix if and only if $T\mathbf{v} >> \mathbf{0}$ for all $\mathbf{v} \in \mathbb{R}^n$ satisfying both $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{v} \geq \mathbf{0}$.*

Proof Suppose that the matrix T is positive. Then $T_{j,k} > 0$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Let $\mathbf{v} \in \mathbb{R}^n$ satisfy both $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{v} \geq \mathbf{0}$. Then

$$(T\mathbf{v})_j = \sum_{k=1}^{n} (T)_{j,k}(\mathbf{v})_k > 0$$

for each integer j between 1 and m, because $(T)_{j,k}(\mathbf{v})_k \geq 0$ for $k = 1, 2, \ldots, n$ and $(T)_{j,k}(\mathbf{v})_k > 0$ for at least one integer k between 1 and n. Therefore $T\mathbf{v} >> \mathbf{0}$.

Conversely suppose that T is an $m \times n$ matrix with with real coefficients which has the property that if and only if $T\mathbf{v} >> \mathbf{0}$ for all non-zero n-dimensional vectors $\mathbf{v}$ with non-negative real coefficients. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$ with

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, \ldots, 0), \ldots, \quad \mathbf{0}_n = (0, 0, \ldots, 1).$$

Then $T\mathbf{e}_k >> \mathbf{0}$ for $k = 1, 2, \ldots, n$, and therefore $(T)_{j,k} = (T\mathbf{e}_k)_j > 0$ for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$. The result follows. ∎

Proposition 6.3 *Let T be a non-negative $n \times n$ (square) matrix. Then there exists a well-defined non-negative real number μ (referred to as the* Perron root *of T) that may be characterized as the greatest real number ρ for which there exists a non-zero vector $\mathbf{v}$ with real coefficients satisfying the conditions $\mathbf{v} \geq \mathbf{0}$ and $T\mathbf{v} \geq \rho\mathbf{v}$.*

Proof Let

$$\Delta = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \geq \mathbf{0}, \ \sum_{j=1}^{n} (\mathbf{v})_j = 1\},$$

and, for each non-negative real number ρ, let E_ρ be the subset of Δ defined so that

$$E_\rho = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \geq \mathbf{0}, \ \sum_{j=1}^{n} (\mathbf{v})_j = 1 \ \text{ and } \ T\mathbf{v} \geq \rho\mathbf{v}\}.$$

Clearly $E_0 = \Delta$. Also if ρ exceeds the largest coefficient of the matrix T then clearly E_ρ is the empty set. Let

$$I = \{\rho \in \mathbb{R} : \rho \geq 0 \quad \text{and} \quad E_\rho \neq \emptyset\}.$$

Then I is a non-empty set of real numbers which is bounded above. It follows from the Least Upper Bound Principle that the set I has a least upper bound $\sup I$. Let $\mu = \sup I$.

Let ρ be a real number satisfying $0 \leq \rho < \mu$. Then there exists $\rho' \in I$ satisfying $\rho < \rho' \leq \mu$. The set $E_{\rho'}$ must then be non-empty, and moreover $E_{\rho'} \subset E_\rho$. It follows that $E_\rho \neq \emptyset$, and thus $\rho \in I$. It follows that

$$\{\rho \in \mathbb{R} : 0 \leq \rho < \mu\} \subset I,$$

and thus the subset I of $\mathbb{R}$ is an interval. We next prove that $\mu \in I$.

Now the characterization of the non-negative real number μ as the least upper bound of the interval I ensures the existence of an infinite sequence $\rho_1, \rho_2, \rho_3, \ldots$ of real numbers belonging to I for which $\lim_{s \to +\infty} \rho_s = \mu$. Then $E_{\rho_s} \neq \emptyset$ for all positive integers s, and therefore there exists an infinite sequence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ of vectors belonging to the simplex Δ such that $\mathbf{v}_s \in E_{\rho_s}$ for all positive integers s. Then $T\mathbf{v}_s \geq \rho_s \mathbf{v}_s$ for all positive integers s. Now the sequence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ of vectors belonging to the simplex Δ is a bounded sequence of vectors, because Δ is a bounded set. The multidimensional Bolzano-Weierstrass Theorem (Theorem 1.4) now ensures the existence of a subsequence $\mathbf{v}_{s_1}, \mathbf{v}_{s_2}, \mathbf{v}_{s_3}, \ldots$ of the sequence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ which converges to some vector $\mathbf{u}$. Moreover $\mathbf{u} \in \Delta$, because Δ is a closed set.

Now

$$T\mathbf{u} = \lim_{r \to +\infty} T\mathbf{v}_{s_r}.$$

Also

$$T\mathbf{v}_{s_r} - \rho_{s_r} \mathbf{v}_{s_r} \geq \mathbf{0}$$

for all positive integers r. Taking limits as $r \to +\infty$, we find that

$$T\mathbf{u} - \mu\mathbf{u} \geq 0,$$

and thus $T\mathbf{u} \geq \mu\mathbf{u}$.

The vector $\mathbf{u}$ is then a non-zero vector with non-negative coefficients, and $T\mathbf{u} \geq \rho\mathbf{u}$ for all real numbers ρ satisfying $0 \leq \rho \leq \mu$.

Now every non-zero n-dimensional vector with non-negative real coefficients is a scalar multiple of some vector belonging to the simplex Δ. We conclude therefore that if ρ is a non-negative real number, if $\mathbf{v}$ is a non-zero vector with non-negative coefficients, and if $T\mathbf{v} \geq \rho\mathbf{v}$ then $\rho \leq \mu$. The result follows. ∎

Definition Let T be a non-negative square matrix. The *Perron root* (or *Perron-Frobenius eigenvalue*) of T is the unique non-negative real number μ

of T that can be characterized as the greatest real number for which there exists a non-zero vector $\mathbf{v}$ with real coefficients satisfying the conditions $\mathbf{v} \geq \mathbf{0}$ and $T\mathbf{v} \geq \mu\mathbf{v}$.

Remark Proposition 6.3 ensures that every non-negative square matrix has a well-defined Perron root. The alternative name *Perron-Frobenius eigenvalue* for the Perron root seems to imply that the Perron root of a non-negative square matrix is an eigenvalue of that matrix. This result is indeed true. It will be proved for positive square matrices in Proposition 6.4, and the result will be extended to non-negative square matrices in Proposition 6.5. The eigenvalues of a square matrix over the field of complex numbers are the roots of the characteristic polynomial of that matrix.

Proposition 6.4 *Let T be a positive square matrix, and let μ be the Perron root of T. Then $\mu > 0$, and there exists $\mathbf{b} \in \mathbb{R}^n$ satisfying the conditions $\mathbf{b} >> \mathbf{0}$ and $T\mathbf{b} = \mu\mathbf{b}$. Moreover, given any $\mathbf{u} \in \mathbb{R}^n$ satisfying $T\mathbf{u} \geq \mu\mathbf{u}$, there exists a real number t for which $\mathbf{u} = t\mathbf{b}$, and thus $T\mathbf{u} = \mu\mathbf{u}$.*

Proof The definition of the Perron root μ of T ensures that there exists a non-zero vector $\mathbf{b}$ with the properties that $\mathbf{b} \geq \mathbf{0}$ and $T\mathbf{b} \geq \mu\mathbf{b}$. Suppose it were the case that $T\mathbf{b} \neq \mu\mathbf{b}$. Let $\mathbf{v} = T\mathbf{b}$. Then

$$T\mathbf{v} - \mu\mathbf{v} = T(T\mathbf{b} - \mu\mathbf{b}) >> \mathbf{0},$$

because $T\mathbf{b} - \mu\mathbf{b} \geq \mathbf{0}$, $T\mathbf{b} - \mu\mathbf{b} \neq \mathbf{0}$ and $T >> 0$ (see Lemma 6.2). But then there would exist real numbers ρ satisfying $\rho > \mu$ that were sufficiently close to μ to ensure that $T\mathbf{v} - \rho\mathbf{v} >> \mathbf{0}$ and thus $T\mathbf{v} \geq \rho\mathbf{v}$. This would contradict the condition on the statement of the proposition that characterizes the value of μ. We conclude therefore that $T\mathbf{b} = \mu\mathbf{b}$.

Moreover $T\mathbf{b} >> \mathbf{0}$, because $T >> 0$ and $\mathbf{b} \geq 0$. It follows that $\mu > 0$ and $\mathbf{b} >> \mathbf{0}$.

Next let $\mathbf{u}$ be an n-dimensional vector with real coefficients for which $T\mathbf{u} \geq \mu\mathbf{u}$. If s is positive and sufficiently large then then $s\mathbf{b} - \mathbf{u} >> 0$. On the other hand if s is negative and $|s|$ is sufficiently large then then $s\mathbf{b} - \mathbf{u} << \mathbf{0}$. It follows from this that there exists a well-defined real number t defined so that

$$t = \inf\{s \in \mathbb{R} : s\mathbf{b} - \mathbf{u} \geq \mathbf{0}\}.$$

Then $t\mathbf{b} - \mathbf{u} \geq 0$, and moreover there exists some integer j between 1 and n for which $t(\mathbf{b})_j - (\mathbf{u})_j = 0$. Now

$$T(t\mathbf{b} - \mathbf{u}) = \mu t\mathbf{b} - T\mathbf{u} \leq \mu(t\mathbf{b} - \mathbf{u}),$$

and therefore $(T(t\mathbf{b} - \mathbf{u}))_j \leq 0$. If it were the case that $t\mathbf{b} - \mathbf{u} \neq 0$ then the inequalities $t\mathbf{b} - \mathbf{u} \geq 0$ and $T >> 0$ would ensure that $T(t\mathbf{b} - \mathbf{b}) >> \mathbf{0}$ (Lemma 6.2), from which it would follow that $(T(t\mathbf{b} - \mathbf{b}))_j > 0$. Because this latter inequality does not hold, it must be the case that $t\mathbf{b} - \mathbf{u} = 0$, and thus $\mathbf{u} = t\mathbf{b}$. The result follows. ∎

Proposition 6.5 *Let T be a non-negative square matrix, and let μ be the Perron root of T. Then μ is an eigenvalue of T, and there exists a non-negative eigenvector $\mathbf{b}$ associated with the eigenvalue μ.*

Proof Let T be an $n \times n$ matrix. Then there exists an infinite sequence $T_1, T_2, T_3, \ldots$ of positive $n \times n$ matrices such that $T_r >> T$ for all positive integers r and $T_r \to T$ as $r \to +\infty$. Let μ_r be the Perron root of T_r and let $\mathbf{b}_r$ be the associated positive eigenvector, normalized to satisfy the condition $\sum_{j=1}^{n} (\mathbf{b}_r)_j = 1$.

The multidimensional Bolzano-Weierstrass Theorem (Theorem 1.4) ensures the existence of an infinite subsequence $T_{r_1}, T_{r_2}, T_{r_3}, \ldots$, a real number μ' and a vector $\mathbf{b} \in \mathbb{R}^n$ such that $\mu_{r_s} \to \mu$ and $\mathbf{b}_{r_s} \to \mathbf{b}$. Replacing the original sequence T_1, T_2, T_3 by a subsequence, if necessary, we may assume, without loss of generality, that $\mu_r \to \mu'$ and $\mathbf{b}_r \to \mathbf{b}$ as $r \to +\infty$. Then $\mu' \geq 0$, $(\mathbf{b})_j \geq 0$ for $j = 1, 2, \ldots, n$ and $\sum_{j=1}^{n} (\mathbf{b})_j = 1$. Then

$$T\mathbf{b} - \mu'\mathbf{b} = \lim_{r \to +\infty} (T_r \mathbf{b}_r - \mu_r \mathbf{b}_r) = \mathbf{0}.$$

Thus μ' is an eigenvalue of T, and $\mathbf{b}$ is a non-zero non-negative eigenvector of T associated to the eigenvalue μ'.

It remains to show that $\mu' = \mu$. Let ρ be a non-negative real number. Suppose that there exists a non-zero vector $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{v} \geq \mathbf{0}$ and $T\mathbf{v} \geq \rho\mathbf{v}$. Then, for each integer r, $T_r\mathbf{v} \geq \rho\mathbf{v}$, because $T_r >> T$, and therefore $\rho \leq \mu_r$. It follows that $\rho \leq \mu'$, because $\mu' = \lim_{r \to +\infty} \mu_r$. Also $T\mathbf{b} = \mu'\mathbf{b}$. It follows that μ' is the largest real number for which there exists a non-zero vector $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{v} \geq \mathbf{0}$ and $T\mathbf{v} \geq \rho\mathbf{v}$. Thus $\mu' = \mu$. The result follows. ∎

Lemma 6.6 *Let T be a non-negative $n \times n$ (square) matrix, let λ be a complex number, let $\mathbf{u}$ be an non-zero n-dimensional vector with complex coefficients, and let $\mathbf{v}$ be the n-dimensional vector with non-negative real coefficients defined such that $(\mathbf{v})_j = |(\mathbf{u})_j|$ for $j = 1, 2, \ldots, n$. Suppose that $\mathbf{u}$ is an eigenvector of T with eigenvalue λ, so that $T\mathbf{u} = \lambda\mathbf{u}$. Then $T\mathbf{v} \geq |\lambda|\mathbf{v}$. Moreover if $T >> 0$, and if $T\mathbf{v} = |\lambda|\mathbf{v}$, then λ is a positive real number, and there exists some complex number ω satisfying $|\omega| = 1$ for which $\mathbf{u} = \omega\mathbf{v}$.*

Proof There exist real numbers $\theta_1, \theta_2, \ldots, \theta_n$ and φ such that $u_j = e^{i\theta_j} v_j$ for $j = 1, 2, \ldots, n$ and $\lambda = e^{i\varphi}|\lambda|$, where $i = \sqrt{-1}$. (Here $e^{i\alpha} = \cos\alpha + i\,\sin\alpha$ for all real numbers α.) The identity $T\mathbf{u} = \lambda\mathbf{u}$ ensures that

$$|\lambda|v_j = e^{-i\varphi - i\theta_j}\lambda u_j = e^{-i\varphi - i\theta_j}\sum_{k=1}^{n} T_{j,k}u_k = \sum_{k=1}^{n} e^{-i\varphi + i\theta_k - i\theta_j} T_{j,k} v_k.$$

Taking real parts, we see that

$$|\lambda|v_j = \sum_{k=1}^{n} \cos(-\varphi + \theta_k - \theta_j) T_{j,k} v_k \leq \sum_{k=1}^{n} T_{j,k} v_k.$$

It follows that $T\mathbf{v} \geq |\lambda|\mathbf{v}$. Moreover if $T\mathbf{v} = |\lambda|\mathbf{v}$ then $\cos(-\varphi + \theta_k - \theta_j) = 1$ for all integers j and k between 1 and n for which $v_k > 0$ and $T_{j,k} > 0$.

Now suppose that $T >> 0$ and $T\mathbf{v} = |\lambda|\mathbf{v}$. Then $\mathbf{v} \neq 0$, because $\mathbf{u} \neq 0$. Also $\mathbf{v} \geq 0$. Therefore $T\mathbf{v} >> \mathbf{0}$ (Lemma 6.2). It follows that $\lambda \neq 0$ and $\mathbf{v} >> \mathbf{0}$. Then $T_{j,k} > 0$ and $v_k > 0$ for all integers j and k between 1 and n, and therefore $\cos(-\varphi + \theta_k - \theta_j) = 1$ for all integers j and k. Applying this result with $j = k$, we find that $\cos(-\varphi) = 1$, and therefore φ is an integer multiple of 2π. It then follows that $\theta_j - \theta_k$ is an integer multiple of 2π for all j and k. But these real numbers φ, θ_j and θ_k are only determined up to addition of an integer multiple of 2π. Let $\omega = e^{i\theta_1}$. Then $e^{i\varphi} = 1$ and and $e^{i\theta_j} = \omega$ for $j = 1, 2, \ldots, n$. It follows that λ is real and positive, and $\mathbf{u} = \omega\mathbf{v}$, where ω is a complex number satisfying $|\omega| = 1$, as required. ∎

Proposition 6.7 *Let T be a non-negative square matrix, and let μ be the Perron root of T. Then every eigenvalue λ of T satisfies the inequality $|\lambda| \leq \mu$.*

Proof Let λ be an eigenvalue of T, and let $\mathbf{u}$ be an eigenvector of T with eigenvalue λ. The number λ and the coefficients of the vector $\mathbf{u}$ may be real or complex. Let $\mathbf{v} \in \mathbb{R}^n$ be defined such that $(\mathbf{v})_j = |(\mathbf{u})_j)|$ for $j = 1, 2, \ldots, n$. Now $T\mathbf{u} = \lambda\mathbf{u}$. It follows from Lemma 6.6 that $T\mathbf{v} \geq |\lambda|\mathbf{v}$. The definition of the Perron root μ then ensures that $|\lambda| \leq \mu$, as required. ∎

Proposition 6.8 *Let T be a non-negative $n \times n$ (square) matrix, and let μ denote the Perron root of T. Let I denote the identity $n \times n$ matrix. Then, given any σ is a non-negative real number satisfying $\mu\sigma < 1$, the matrix $I - \sigma T$ is invertible and $(1 - \sigma T)^{-1}$ is a non-negative matrix.*

Proof We use some basic results of linear algebra and complex analysis. Let z be a complex number. Then the eigenvectors of the matrix $I - zT$ are the same as those of the matrix T, and therefore the eigenvalues of $I - zT$ are of the form $1 - z\lambda$ as λ ranges of the eigenvales of T.

Now the modulus of any eigenvalue of the non-negative matrix T is bounded above by the Perron root of T (Proposition 6.7). Therefore the eigenvalues of $I - zT$ have real part not less than $1 - |z|\mu$. A square matrix is invertible if zero is not an eigenvalue of that matrix. It follows that the matrix $I - zT$ is invertible for all complex numbers z satisfying $\mu|z| < 1$.

The determinant of the matrix $I - zT$ is a polynomial function of z. It follows that if $\mu > 0$ then all coefficients of the matrix $(I - zT)^{-1}$ are holomorphic functions of the complex variable z throughout the disk $\{z \in \mathbb{C} : |z| < \mu^{-1}\}$, and if $\mu = 0$ then all coefficients of the matrix $(I - zT)^{-1}$ are holomorphic functions of the complex variable z throughout entire complex plane. A basic theorem of complex analysis therefore ensures that each coefficient of the matrix $(I - zT)^{-1}$ may be represented as a power series in the complex plane z that converges for all complex numbers z satisfying $\mu|z| < 1$.

Now

$$(1 - zT)(1 + zT + z^2T^2 + z^3T^3 + \cdots + z^kT^k) = 1 - z^{k+1}T^{k+1},$$

and thus

$$(1 - zT)^{-1} \quad = \quad 1 + zT + z^2T^2 + z^3T^3 + \cdots + z^kT^k + z^{k+1}T^{k+1}(I - zT)^{-1}$$

when $\mu|z| < 1$.

Now it has already been shown that $(1 - zT)^{-1}$ can be represented by a power series in z that converges whenever $\mu|z| < 1$. we can therefore conclude that

$$(1 - zT)^{-1} = 1 + zT + z^2T^2 + z^3T^3 + \cdots$$

for all complex numbers z satisfying $\mu|z| < 1$. In particular

$$(1 - \sigma T)^{-1} = 1 + \sigma T + \sigma^2T^2 + \sigma^3T^3 + \cdots$$

for all non-negative real numbers σ satisfying $\mu\sigma < 1$. But each summand on the right side of this power series representation of $(1 - \sigma T)^{-1}$ is a non-negative matrix. It follows that $I - \sigma T$ is invertible and $(1 - \sigma T)^{-1}$ is a non-negative matrix for all non-negative real numbers σ satisfying $\sigma\rho < 1$, as required. ∎

Proposition 6.9 *Let T be a non-negative $n \times n$ (square) matrix, let μ denote the Perron root of T. Then the Perron root of the transpose T^T is equal to the Perron root μ of T, and there exists a non-zero vector $\mathbf{p} \in \mathbb{R}^n$ satisfying $\mathbf{p} \geq \mathbf{0}$ and $\mathbf{p}^T T = \mu \mathbf{p}^T$, where $\mathbf{p}^T$, the transpose of $\mathbf{p}$ is the row vector components are the components of the column vector $\mathbf{p}$.*

Proof The transpose T^T of the non-negative square matrix T is itself a non-negative square matrix with the same characteristic polynomial as T, and thus with the same eigenvalues as T. The Perron root of the transpose T^T of T is a non-negative real eigenvalue of T^T (Proposition 6.5), and moreover it is an upper bound on the modulus of every eigenvalue of T^T (Proposition 6.7. It follows that the non-negative square matrix T and its transpose T^T have the same Perron root. Moreover the Perron root is an eigenvalue of T^T, and therefore there exists a non-zero vector $\mathbf{p} \in \mathbb{R}^n$ for which $\mathbf{p} \geq 0$ and $T^T\mathbf{p} = \mu\mathbf{p}$. Taking the transpose of this equation, we find that $\mathbf{p}^T T = \mu\mathbf{p}^T$, as required. ∎

Proposition 6.10 *Let T be a non-negative $n \times n$ (square) matrix, let μ denote the Perron root of T, and let σ is a non-negative real number. Then there exists a non-zero vector $\mathbf{w} \in \mathbb{R}^n$ satisfying $\mathbf{w} \geq \mathbf{0}$ and $\mathbf{w} >> \sigma T\mathbf{w}$ if and only if $\mu\sigma < 1$.*

Proof Let $\mathbf{v} \in \mathbb{R}^n$ satisfy $\mathbf{v} >> \mathbf{0}$, and let $\mathbf{w} = (I - \sigma T)^{-1}\mathbf{v}$, where I denotes the identity $n \times n$ matrix. It follows from Proposition 6.8 that if $\mu\sigma < 1$ then $(I - \sigma T)^{-1}$ a non-negative matrix, and therefore $\mathbf{w} \geq 0$. Also

$$\mathbf{w} - \sigma T\mathbf{w} = (I - \sigma T)\mathbf{w} = \mathbf{v} >> \mathbf{0},$$

and therefore $\mathbf{w} >> \sigma T\mathbf{w}$. We have thus shown that if $\mu\sigma < 1$ then there exists a vector $\mathbf{w}$ with the required properties.

Conversely suppose that σ is a non-negative real number and that $\mathbf{w} \in \mathbb{R}^n$ is a non-zero vector for which $\mathbf{w} \geq 0$ and $\mathbf{w} >> \sigma T\mathbf{w}$. It follows from Proposition 6.9 that there exists a non-zero vector $\mathbf{p} \in \mathbb{R}^n$ satisfying $\mathbf{p} \geq \mathbf{0}$ and $\mathbf{p}^T T = \mu\mathbf{p}^T$, where $\mathbf{p}^T$ denotes the transpose of $\mathbf{p}$. Then

$$(1 - \sigma\mu)\mathbf{p}^T\mathbf{w} = \mathbf{p}^T\mathbf{w} - \sigma\mu\mathbf{p}^T\mathbf{w} = \mathbf{p}^T(\mathbf{w} - \sigma T\mathbf{w}) > 0.$$

It follows that $\mathbf{p}^T\mathbf{w} > 0$ and $\sigma\mu < 1$, as required. This completes the proof. ∎

6.2 Perron's Theorem for Positive Matrices

In 1907 Oskar Perron (1880–1975) proved a fundamental theorem concerning the eigenvalues and eigenvectors of a positive square matrix, in particular showing that the positive real number now referred to as the *Perron root* (or *Perron-Frobenius eigenvalue*) of the matrix is a simple eigenvector, with a one-dimensional eigenspace spanned by a positive eigenvector, and that any other eigenvalues of the matrix has a modulus strictly less than the Perron root. In 1912, Georg Frobenius (1849-1917) generalized Perron's Theorem to a particular class of non-negative square matrices that are said to be *unzerlegbar* (i.e., "indecomposible" or "irreducible"). These discoveries initiated the development of a body of results concerning non-negative square matrices that is today referred to as *Perron-Frobenius Theory*

Before stating and proving Perron's Theorem, we review (without proof) some standard results from linear algebra, related to the Jordan normal form of a square matrix, that are relevant to the proof of Perron's Theorem.

Let T be a linear operator defined on a finite-dimensional complex vector space V. Then the vector space V can be decomposed as a direct sum of subspaces that are invariant under the action of T and cannot be further decomposed as direct sums of invariant subspaces. Then

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$$

where, for each integer r between 1 and m, the linear operator T maps the subspace V_r of V into itself. Moreover the subspace V_r has no proper non-zero vector subspace that is invariant under the action of T. Associated with each subspace V_r is a complex number λ_r that is the unique eigenvalue of the restriction of the linear operator T to V_r.

The *characteristic polynomial* χ of T on V is defined such that $\chi(z) = \det(zI_V - T)$, where I_V denotes the identity operator on V. It can be shown that

$$\chi(z) = \prod_{r=1}^{m} (z - \lambda_r)^{d_r},$$

where $d_r = \dim_{\mathbb{C}} V_r$ for $r = 1, 2, \ldots, m$. It follows that a complex number λ is a simple root of the characteristic polynomial χ of T if and only if the following two conditions are satisfied: there exists exactly one integer r between 1 and m for which $\lambda = \lambda_r$; for this value of r, $d_r = 1$.

The theory of the Jordan Normal Form ensures that each subspace V_r has a basis of the form

$$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{d_r},$$

with the property that $T\mathbf{e}_1 = \lambda_r \mathbf{e}_s$ and $T\mathbf{e}_s = \lambda_r \mathbf{e}_s + \mathbf{e}_{s-1}$ for $1 < s \leq d_r$. All eigenvectors of T contained in V_r are scalar multiples of $\mathbf{e}_1$. Moreover if $d_r > 1$ then $(T - \lambda_r I_{V_r})^2 \mathbf{e}_2 = \mathbf{0}$ but $T\mathbf{e}_2 \neq \lambda_r \mathbf{e}_2$.

These results of linear algebra, summarized without detailed proof, yield the result stated in the following lemma.

Lemma 6.11 *Let T be a linear operator acting on a finite-dimensional complex vector space V, and let λ be an eigenvalue of T. Then λ is a simple root of the characteristic polynomial of T if and only if the following two conditions are satisfied:*

- *the eigenspace associated with the eigenvalue* λ *is one-dimensional;*
- *if* $\mathbf{v} \in V$ *satisfies the identity* $(T - \lambda I_V)^2 \mathbf{v} = \mathbf{0}$ *then* $T\mathbf{v} = \lambda \mathbf{v}$*.*

Theorem 6.12 (Perron) *Let* T *be a positive square matrix, and let* μ *be the Perron root of* T*. Then the following properties are satisfied:—*

(i) there exists an eigenvector of T *with associated eigenvalue* μ *whose coefficients are all strictly positive;*

(ii) the eigenvalue μ *is a simple root of the characteristic polynomial of* T*, and the corresponding eigenspace is therefore one-dimensional;*

(iii) all eigenvalues λ *(real or complex) of* T *that are distinct from* μ *satisfy the inequality* $|\lambda| < \mu$*.*

Proof Let the positive square matrix T be an $n \times n$ matrix, and let μ denote the Perron root of T. Proposition 6.4 establishes that the Perron root μ of T is well-defined and is an eigenvalue of T with which is associated an eigenvector $\mathbf{b}$ with positive coefficients. Moreover Proposition 6.4 ensures that the following properties are then satisfied:—

(iv) $\mathbf{b} >> \mathbf{0}$ and $T\mathbf{b} = \mu\mathbf{b}$;

(v) if ρ is a non-negative real number, if $\mathbf{v}$ is a non-zero n-dimensional vector with non-negative coefficients, and if $T\mathbf{v} \geq \rho\mathbf{v}$, then $\rho \leq \mu$.

(vi) given any n-dimensional vector $\mathbf{u}$ with real coefficients for which $T\mathbf{u} \geq \mu\mathbf{u}$, there exists a real number t for which $\mathbf{u} = t\mathbf{b}$, and thus $T\mathbf{u} = \mu\mathbf{u}$.

Now because the coefficients of the matrix T are all real, and μ is also a real number, the real and imaginary parts of any eigenvector of T with associated eigenvalue μ must themselves be eigenvectors with eigenvalue μ. The result just obtained therefore ensures that any convex eigenvector of T with eigenvalue μ must be a complex scalar multiple of the eigenvector $\mathbf{b}$. Thus the eigenspace of T associated with the eigenvalue μ is one-dimensional, when considered over the field of complex numbers.

Let I denote the identity $n \times n$ matrix, and let $\mathbf{v}$ be real n-dimensional vector for which $(T - \mu I)^2 \mathbf{v} = \mathbf{0}$. Then $T\mathbf{v} - \mu\mathbf{v}$ is an eigenvector of T with associated eigenvalue μ. It follows from property (vi) above that there must exist some real number α for which $T\mathbf{v} - \mu\mathbf{v} = \alpha\mathbf{b}$. Now $\mathbf{b} >> \mathbf{0}$. It follows that if $\alpha \geq 0$ then $T\mathbf{v} \geq \mu\mathbf{v}$. But property (vi) stated at the commencement of the proof then ensures that $\mathbf{v} = t\mathbf{b}$ for some real number t. But then $T\mathbf{v} = \mu\mathbf{v}$ and $\alpha = 0$. Similarly if $\alpha \leq 0$ then $T(-\mathbf{v}) \geq \mu(-\mathbf{v})$, and this also ensures that $\alpha = 0$. It follows that if $\mathbf{v}$ is a real n-dimensional vector satisfying $(T - \mu I)^2\mathbf{v} = \mathbf{0}$ then $T\mathbf{v} = \mu\mathbf{v}$. The criterion stated in Lemma 6.11 therefore establishes that μ is a simple root of the characteristic polynomial of T.

We have now verified (i) and (ii). It remains to verify that all eigenvalues λ of T distinct from μ satisfy the inequality $|\lambda| < \mu$.

Now it follows from Proposition 6.7 that all eigenvalues λ of T satisfy the inequality $|\lambda| \leq \mu$.

Now suppose that $|\lambda| = \mu$. It then follows from property (vi), stated at the commencement of the proof, that $T\mathbf{v} = \mu\mathbf{v} = |\lambda|\mathbf{v}$. It then follows from Lemma 6.6 that λ is a positive real number, and therefore $\lambda = \mu$. This completes the proof of (iii), and therefore completes the proof of the theorem. ∎

7 Game Theory and Nash Equilibria

7.1 Zero-Sum Two-Person Games

Example Consider the following hand game. This is a zero-sum two-person game. At each go, the two players present simultaneously either and open hand or a fist. If both players present fists, or if both players present open hands, then no money changes hands. If one player presents a fist and the other player presents an open hand then the player presenting the fist receives ten cents from the player presenting the open hand.

The payoff for the first player can be represented by the following payoff matrix:

$$\begin{pmatrix} 0 & -10 \\ 10 & 0 \end{pmatrix}.$$

In this matrix the entry in the first row represent the payoffs when the first player presents an open hand; those in the second row represent the payoffs when the first player presents a fist. The entries in the first column represent the payoff when the second player presents an open hand; those in the second column represent the payoffs when the second player presents a fist. In this game the second player, choosing the best strategy, is always going to play a fist, because that reduces the payoff for the first player, whatever the first player chooses to play. Similarly the first player, choosing the best strategy, is going to play a fist, because that maximizes the payoff for the first player whatever the second player does. Thus in this game, both players choosing the best strategies, play fists.

It should be noticed that, in this situation, if the second player always plays a fist, the first player would not be tempted to move from a strategy of always playing a fist in order get a better payoff. Similarly if the first player always plays a fist, then the second player would not be tempted to move from a strategy of always playing a fist in order to reduce the payoff to the first player. This is a very simple example of a *Nash Equilibrium.* This equilibrium arises because the element in the second row and second column of the payoff matrix is simultaneously the largest element in its column and the smallest element in its row. Matrix elements with this property as said to be *saddle points* of the matrix.

Example Now consider the game of *Rock, Paper, Scissors.* This game has a long history, and versions of this game were well-established in China and Japan in particular for many centuries.

Two players simultaneously present hand symbols representing *Rock* (a closed fist), *Paper* (a flat hand), or *Scissors* (first two fingers outstretched in a 'V'). Paper beats Rock, Scissors beats Paper, Rock beats Scissors. If both players present the same hand symbol then that round is a draw.

Ordering the strategies for the playes in the order *Rock* (1st), *Paper* (2nd) and *Scissors* (3rd), the payoff matrix for the first player is the following:—

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

The entry in the ith row and jth column of this payoff matrix represents the return to the first player on a round of the game if the first player plays strategy i and the second player plays strategy j.

A *pure strategy* would be one in which a player presents the same hand symbol in every round. But it is not profitable for any player in this game to adopt a pure strategy. If the first player adopts a strategy of playing *Paper*, then the second player, on observing this, would adopt a strategy of always playing *Scissors*, and would beat the first player on every round. A preferable strategy, for each player, is the *mixed strategy* of playing *Rock*, *Paper* and *Scissors* with equal probability, and seeking to ensure that the sequence of plays is as random as possible.

Let us denote by M the payoff matrix above. A *mixed strategy* for the first player is one in which, on any given round *Rock* is played with probability p_1, *Paper* is played with probability p_2 and *Scissors* is played with probability p_3. The mixed strategies for the first player can therefore be represented by points of a triangle Δ_P, where

$$\Delta_P = \left\{ (p_1, p_2, p_3) \in \mathbb{R}^3 : p_1 \geq 0,\ \ p_2 \geq 0,\ \ p_3 \geq 0,\ \ p_1 + p_2 + p_3 = 1 \right\}.$$

A mixed strategy for the second player is one in which *Rock* is played with probability q_1, *Paper* with probability q_2 and *Scissors* with probability q_3. The mixed strategies for the second player can therefore be represented by points of a triangle Δ_Q, where

$$\Delta_Q = \left\{ (q_1, q_2, q_3) \in \mathbb{R}^3 : q_1 \geq 0,\ \ q_2 \geq 0,\ \ q_3 \geq 0,\ \ q_1 + q_2 + q_3 = 1 \right\}.$$

Let $\mathbf{p} \in \Delta_P$ represent the mixed strategy chosen by the first player, and $\mathbf{q} \in \Delta_Q$ the mixed strategy chosen by the second player, where

$$\mathbf{p} = (p_1, p_2, p_3), \quad \mathbf{q} = (q_1, q_2, q_3).$$

Let M_{ij} the payoff for the first player when the first player plays strategy i and the second player plays strategy j. Then M_{ij} is the entry in the ith row and jth column of the payoff matrix M. In matrix equations we consider $\mathbf{p}$ and $\mathbf{q}$ to be column vectors, denoting their transposes by the row matrices $\mathbf{p}^T$ and $\mathbf{q}^T$. The *expected payoff* for the first player is then $f(\mathbf{p}, \mathbf{q})$, where

$$f(\mathbf{p}, \mathbf{q}) = \mathbf{p}^T M \mathbf{q} = \sum_{i=1}^{3} \sum_{j=1}^{3} p_i M_{ij} q_j.$$

Let $\mathbf{p}^* = (p_1^*, p_2^*, p_3^*)$, where

$$p_1^* = p_2^* = p_3^* = \tfrac{1}{3}.$$

Then $\mathbf{p}^{*T} M = (0, 0, 0)$, and therefore

$$f(\mathbf{p}^*, \mathbf{q}) = 0$$

for all $\mathbf{q} \in \Delta_Q$. Similarly let $\mathbf{q}^* = (q_1^*, q_2^*, q_3^*)$, where

$$q_1^* = q_2^* = q_3^* = \tfrac{1}{3}.$$

Then

$$f(\mathbf{p}, \mathbf{q}^*) = 0$$

for all $\mathbf{p} \in \Delta_Q$. Thus the inequalities

$$f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q})$$

are satisfied for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_q$, because each of the quantities occurring is equal to zero.

Were the first player to adopt a mixed strategy $\mathbf{p}$, where $\mathbf{p} = (p_1, p_2, p_3)$, $p_i \geq 0$ for $i = 1, 2, 3$ and $p_1 + p_2 + p_3 = 1$, the second player could adopt mixed strategy $\mathbf{q}$, where $\mathbf{q} = (q_1, q_2, q_3) = (p_3, p_1, p_2)$. The payoff $f(\mathbf{p}, \mathbf{q})$ is then

$$\begin{aligned} f(\mathbf{p}, \mathbf{q}) &= -p_1 q_2 + p_1 q_3 - p_2 q_3 + p_2 q_1 - p_3 q_1 + p_3 q_2 \\ &= -p_1^2 + p_1 p_2 - p_2^2 + p_2 p_3 - p_3^2 + p_3 p_1 \\ &= -\tfrac{1}{6}\Big((2p_1 - p_2 - p_3)^2 + (2p_2 - p_3 - p_1)^2 \\ &\qquad + (2p_3 - p_1 - p_2)^2\Big) \\ &\leq 0. \end{aligned}$$

Moreover if $f(\mathbf{p}, \mathbf{q}) = 0$, where $q_1 = p_3$, $q_2 = p_1$ and $q_3 = p_2$, then

$$(2p_1 - p_2 - p_3)^2 + (2p_2 - p_3 - p_1)^2 + (2p_3 - p_1 - p_2)^2 = 0$$

and therefore $2p_1 = p_2 + p_3$, $2p_2 = p_3 + p_1$ and $2p_3 = p_1 + p_2$. But then

$$3p_1 = 3p_2 = 3p_3 = p_1 + p_2 + p_3 = 1,$$

and thus $\mathbf{p} = \mathbf{p}^*$. It follows that if $\mathbf{p} \in \Delta_Q$ and $\mathbf{p} \neq \mathbf{p}^*$ then there exists $\mathbf{q} \in \Delta_Q$ for which $f(\mathbf{p}, \mathbf{q}) < 0$. Thus if the first player adopts a mixed strategy other than the strategy $\mathbf{p}^*$ in which *Rock*, *Paper*, *Scissors* are played with equal probability on each round, there is a mixed strategy for the second player that ensures that the average payoff for the first player is negative, and thus the first player will lose in the long run over many rounds. Thus strategy $\mathbf{p}^*$ is the only sensible mixed strategy that the first player can adopt. The corresponding strategy $\mathbf{q}^*$ is the only sensible mixed strategy that the second player can adopt. The average payoff for each player is then equal to zero.

7.2 Von Neumann's Minimax Theorem

In 1920, John Von Neumann published a paper entitled "Zur Theorie der Gesellschaftsspielle" (*Mathematische Annalen*, Vol. 100 (1928), pp. 295–320). The title translates as "On the Theory of Social Games". This paper included a proof of the following "Minimax Theorem", which made use of the Brouwer Fixed Point Theorem. An alternative proof using results concerning convexity was presented in the book *On the Theory of Games and Economic Behaviour* by John Von Neumann and Oskar Morgenstern (Princeton University Press, 1944). George Dantzig, in a paper published in 1951, showed how the theorem could be solved using linear programming methods (see Joel N. Franklin, *Methods of Mathematical Economics*, (Springer Verlag, 1980, republished by SIAM in 1982).

Theorem 7.1 (Von Neumann's Minimax Theorem) *Let M be an $m \times n$ matrix, let*

$$\begin{aligned} \Delta_P &= \Big\{(p_1, p_2, \ldots, p_m) \in \mathbb{R}^m : p_i \geq 0 \textit{ for } i = 1, 2, \ldots, m, \textit{ and} \\ &\qquad \sum_{i=1}^m p_i = 1\Big\}, \\ \Delta_Q &= \Big\{(q_1, q_2, \ldots, q_n) \in \mathbb{R}^n : q_i \geq 0 \textit{ for } i = 1, 2, \ldots, n, \textit{ and} \\ &\qquad \sum_{j=1}^n q_j = 1\Big\}, \end{aligned}$$

and let

$$f(\mathbf{p}, \mathbf{q}) = \mathbf{p}^T M \mathbf{q} = \sum_{i=1}^{m} \sum_{j=1}^{n} M_{i,j} p_i q_j$$

for all $\mathbf{p} \in \Delta_P$ *and* $\mathbf{q} \in \Delta_Q$. *Then there exist* $\mathbf{p}^* \in \Delta_P$ *and* $\mathbf{q}^* \in \Delta_Q$ *such that*

$$f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q})$$

for all $\mathbf{p} \in \Delta_P$ *and* $\mathbf{q} \in \Delta_Q$.

Proof Let $f(\mathbf{p}, \mathbf{q}) = \mathbf{p}^T M \mathbf{q}$ for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$. Given $\mathbf{q} \in \Delta_Q$, let

$$\mu_P(\mathbf{q}) = \sup\{f(\mathbf{p}, \mathbf{q}) : \mathbf{p} \in \Delta_P\}$$

and let

$$P(\mathbf{q}) = \{\mathbf{p} \in \Delta_P : f(\mathbf{p}, \mathbf{q}) = \mu_P(\mathbf{q})\}.$$

Similarly given $\mathbf{p} \in \Delta_P$, let

$$\mu_Q(\mathbf{p}) = \inf\{f(\mathbf{p}, \mathbf{q}) : \mathbf{q} \in \Delta_Q\}$$

and let

$$Q(\mathbf{p}) = \{\mathbf{q} \in \Delta_Q : f(\mathbf{p}, \mathbf{q}) = \mu_Q(\mathbf{q})\}.$$

An application of Berge's Maximum Theorem (Theorem 2.23) ensures that the functions $\mu_P \colon \Delta_P \to \mathbb{R}$ and $\mu_Q \colon \Delta_Q \to \mathbb{R}$ are continuous, and that the correspondences $P \colon \Delta_Q \rightrightarrows \Delta_P$ and $Q \colon \Delta_P \rightrightarrows \Delta_Q$ are non-empty, compact-valued and upper hemicontinuous. These correspondences therefore have closed graphs (see Proposition 2.11). Morever $P(\mathbf{q})$ is convex for all $\mathbf{q} \in \Delta_Q$ and $Q(\mathbf{p})$ is convex for all $\mathbf{p} \in \Delta_P$. Let $X = \Delta_P \times \Delta_Q$, and let $\Phi \colon X \rightrightarrows X$ be defined such that

$$\Phi(\mathbf{p}, \mathbf{q}) = P(\mathbf{q}) \times Q(\mathbf{p})$$

for all $(\mathbf{p}, \mathbf{q}) \in X$. Kakutani's Fixed Point Theorem (Theorem 5.4) then ensures that there exists $(\mathbf{p}^*, \mathbf{q}^*) \in X$ such that $(\mathbf{p}^*, \mathbf{q}^*) \in \Phi(\mathbf{p}^*, \mathbf{q}^*)$. Then $\mathbf{p}^* \in P(\mathbf{q}^*)$ and $\mathbf{q}^* \in Q(\mathbf{p}^*)$ and therefore

$$f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q})$$

for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$, as required. ∎

7.3 Quasiconvex Functions

Definition Let K be a convex set in some real vector space. A real-valued function $f\colon K \to \mathbb{R}$ is said to be *quasiconvex* if

$$f((1-t)\mathbf{u} + t\mathbf{v}) \leq \max\Big(f(\mathbf{u}), f(\mathbf{v})\Big)$$

for all $\mathbf{u}, \mathbf{v} \in K$ and for all real numbers t satisfying $0 \leq t \leq 1$.

Definition Let K be a convex set in some real vector space. A real-valued function $f\colon K \to \mathbb{R}$ is said to be *quasiconcave* if

$$f((1-t)\mathbf{u} + t\mathbf{v}) \geq \min\Big(f(\mathbf{u}), f(\mathbf{v})\Big)$$

for all $\mathbf{u}, \mathbf{v} \in K$ and for all real numbers t satisfying $0 \leq t \leq 1$.

Linear functionals are quasiconvex and quasiconcave.

A function $f\colon K \to \mathbb{R}$ defined over a compact subset K of a real vector space is quasiconcave if and only if the function $-f$ is quasiconvex.

Lemma 7.2 *Let K be a convex set in a real vector space, and let $f\colon K \to \mathbb{R}$ be a quasiconcave function. Then, for each real number s, the preimage $f^{-1}([s, +\infty))$ of the interval $[s, +\infty)$ is a convex subset of K, where*

$$f^{-1}([s, +\infty)) = \{\mathbf{x} \in K : f(\mathbf{x}) \geq s\}.$$

Proof Let $\mathbf{u}, \mathbf{v} \in f^{-1}([s, +\infty))$, and let t be a real number satisfying $0 \leq t \leq 1$. Then $f(\mathbf{u}) \geq s$ and $f(\mathbf{v}) \geq s$. It follows from the definition of quasiconcavity that

$$f((1-t)\mathbf{u} + t\mathbf{v}) \geq \min\Big(f(\mathbf{u}), f(\mathbf{v})\Big) \geq s,$$

and therefore $(1-t)\mathbf{u} + t\mathbf{v} \in f^{-1}([s, +\infty))$, as required. ∎

7.4 Nash Equilibria

We consider a *game* with n players. Each player choses a strategy from an appropriate *strategy sets*. The strategies chosen by the players in the game constitute a *strategy profile*. The *utility*, or *payoff*, of the game, for each player is determined by the strategy profile chosen by the players in the game. The technical details involved are explored and specified in more detail in the following discussion.

We suppose that, in an n-player game, the ith player choses strategies from a *strategy set* S_i, where S_i is represented as a non-empty compact convex set in $\mathbb{R}^{m_i}$ for some positive integer m_i. (The convexity requirement would typically be satisfied in games where players can adopt mixed strategies.) We let $S = S_1 \times S_1 \times \cdot \times S_n$. The elements of the set S are referred to as *strategy profiles*. The *strategy profile set* S is a compact convex subset of $\mathbb{R}^m$, where

$$m = m_1 + m_2 + \cdots + m_n.$$

For each integer i between 1 and n let us define

$$\begin{array}{rcl} S_{-1} & = & S_2 \times S_3 \times S_4 \times \cdots \times S_n, \\ S_{-2} & = & S_1 \times S_3 \times S_4 \times \cdots \times S_n, \\ S_{-3} & = & S_1 \times S_2 \times S_4 \times \cdots \times S_n, \\ & \vdots & \\ S_{-n} & = & S_1 \times S_2 \times S_3 \times \cdots \times S_{n-1}, \end{array}$$

so that

$$S_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$$

for all integers i between 1 and n (making the appropriate interpretation of the right hand side of this expression, as specified above, in the cases $i = 1$ and $i = n$). The set S_{-i} is then a compact convex subset of $\mathbb{R}^{m-m_i}$ for $i = 1, 2, \ldots, n$.

We define projections $\pi_i \colon S \to S_i$ and $\pi_{-i} \colon S \to S_{-i}$ for $i = 1, 2, \ldots, n$ in the obvious fashion so that

$$\pi_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \mathbf{x}_i$$

and

$$\begin{array}{rcl} \pi_{-1}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) & = & (\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \ldots, \mathbf{x}_n), \\ \pi_{-2}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) & = & (\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4, \ldots, \mathbf{x}_n), \\ \pi_{-3}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) & = & (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4, \ldots, \mathbf{x}_n), \\ & \vdots & \\ \pi_{-n}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) & = & (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_{n-1}). \end{array}$$

We now consider the utility, or payoff, of the game for the players. We suppose that, for each integer i between 1 and n, the *utility* of the game, from the perspective of the ith player, is determined by a utility function $u_i \colon S_i \times S_{-i} \to \mathbb{R}$ defined so that, for each element $\mathbf{x}_{-i}$ of S_{-i} representing a choice of strategies by players of the game other than the ith player, the real number $u_i(\mathbf{x}_i, \mathbf{x}_{-i})$ represents the utility, or payoff, for the ith player on adopting the strategy **i**. We impose the following two requirements on these utility functions:

- the utility function $u_i\colon S_i \times S_{-i} \to \mathbb{R}$ is continuous on $S_i \times S_{-i}$;
- for fixed $\mathbf{x}_{-i}$, the function sending $\mathbf{x}_i$ to $u_i(\mathbf{x}_i, \mathbf{x}_{-i})$ is quasiconcave on S_i, and thus

$$u_i((1-t)\mathbf{x}'_i + t\mathbf{x}''_i, \mathbf{x}_{-i}) \geq \min\Big(u_i(\mathbf{x}'_i, \mathbf{x}_{-i}), u_i(\mathbf{x}''_i, \mathbf{x}_{-i})\Big)$$

  for all $\mathbf{x}'_i, \mathbf{x}''_i \in S_i$, $\mathbf{x}_{-i} \in S_{-i}$ and real numbers t satisfying $0 \leq t \leq 1$.

Let $\mathbf{x}'_i$ and $\mathbf{x}''_i$ elements of the strategy set S_i, representing strategies for the ith player, and let $\mathbf{x}_{-i}$ be an element of S_{-i}, representing a profile of strategies adopted by the other players. Then the ith player actively prefers the outcome of strategy profile $\mathbf{x}''_i$ to that of strategy profile $\mathbf{x}'_i$ if and only if

$$u_i(\mathbf{x}'_i, \mathbf{x}_{-i}) < u_i(\mathbf{x}''_i, \mathbf{x}_{-i}).$$

The ith player is indifferent between the outcomes of the strategy profiles $\mathbf{x}'_i$ and $\mathbf{x}''_i$ if and only if

$$u_i(\mathbf{x}'_i, \mathbf{x}_{-i}) = u_i(\mathbf{x}''_i, \mathbf{x}_{-i}).$$

Definition In an n-player game, let $S_1, S_2, \ldots, S_n$ denote the strategy sets for the players in the game, and let $u_i\colon S_i \times S_{-i} \to \mathbb{R}$ denote the utility function for the ith player in the game (where the set S_{-i} is defined for $i = 1, 2, \ldots, n$ as described above). A strategy profile

$$(\mathbf{x}^*_1, \mathbf{x}^*_2, \ldots, \mathbf{x}^*_n)$$

is said to be a *Nash equilibrium* for the game if

$$u_i(\mathbf{x}_i, \mathbf{x}^*_{-i}) \leq u_i(\mathbf{x}^*_i, \mathbf{x}^*_{-i}).$$

for all integers i between 1 and n and for all $\mathbf{x}_i \in S_i$.

Given any element $\mathbf{x}_{-i}$ of S_{-i} (representing a choice of strategies that might be adopted by the other players of the game), there will be a subset $B_i(\mathbf{x}_{-i})$ of S_i that represents the best strategies that the ith player can adopt when the other players are adopting the strategies represented by the element $\mathbf{x}_{-i}$ of S_{-i}. These best strategies are those strategies that maximize the utility function for the ith player, and we denote the value of the utility function u_i for those best strategies by $b_i(\mathbf{x}_{-i})$. Accordingly

$$\begin{aligned} b_i(\mathbf{x}_{-i}) &= \sup\{u_i(\mathbf{x}_i, \mathbf{x}_{-i}) : \mathbf{x}_i \in S_i\}, \\ B_i(\mathbf{x}_{-i}) &= \{\mathbf{x}_i \in S_i : u_i(\mathbf{x}_i, \mathbf{x}_{-i}) = b(\mathbf{x}_{-i})\}. \end{aligned}$$

We obtain in this fashion a single-valued function $b_i \colon S_{-i} \to S_i$ and a correspondence $B_i \colon S_{-i} \rightrightarrows S_i$.

Now, for each integer i between 1 and n, the constant correspondence that sends each element of S_{-i} to the strategy set S_i is clearly both upper hemicontinuous and lower hemicontinuous. The function $u_i \colon S_i \times S_{-i} \to \mathbb{R}$ is required to be continuous. Moreover, for each $\mathbf{x}_{i-1} \in S_{-i}$, the Extreme Value Theorem ensures that the set $B_i(\mathbf{x}_{-i})$ is non-empty, and the continuity of the utility function u_i ensures that $B_i(\mathbf{x}_{-i})$ is a closed subset of the compact set S_i. It follows that the the correspondence $B \colon S_{-i} \rightrightarrows S_i$ is both non-empty and compact. It therefore follows from Berge's Maximum Theorem (Theorem 2.23) that the function $b \colon S_{-i} \to \mathbb{R}$ is continuous on S_{-i}, $B_i(\mathbf{x}_{-i})$ is a compact subset of S_i for all $\mathbf{x}_{-i} \in S_{-i}$, and the correspondence $B \colon S_{-i} \rightrightarrows S_i$ is upper hemicontinuous in S_{-i}. Now every upper hemicontinuous closed-valued correspondence has a closed graph (Proposition 2.11). We conclude therefore that the correspondence $B \colon S_{-i} \rightrightarrows S_i$ has a closed graph.

Now, for each i, and for each $\mathbf{x}_{-i} \in S_{-i}$, the quasiconcavity requirement imposed on the utility function i ensures that the non-empty compact set $B_i(\mathbf{x}_{-i})$ is convex. Indeed the definition of $b_i(\mathbf{x}_{-i})$ and $B_i(\mathbf{x}_{-i})$ ensures that $u_i(\mathbf{z}, \mathbf{x}_{-i}) \leq b_i(\mathbf{x}_{-i})$ for all $\mathbf{z} \in S_i$, and $u_i(\mathbf{z}, \mathbf{x}_{-i}) = b_i(\mathbf{x}_{-i})$ for all $\mathbf{z} \in B_i(\mathbf{x}_{-i})$. It follows that

$$B_i(\mathbf{x}_{-i}) = \{\mathbf{z} \in S_i : u_i(\mathbf{z}, \mathbf{x}_{-i}) \geq b(\mathbf{x}_{-i})\}.$$

The quasiconcavity condition on the function u_i ensures that, for all $\mathbf{z}, \mathbf{z}' \in B_i(\mathbf{x}_{-i})$ and for all real numbers t satisfing $0 \leq t \leq 1$,

$$u_i((1-t)\mathbf{z}' + t\mathbf{z}'', \mathbf{x}_{-i}) \geq \min\Big(u_i(\mathbf{z}', \mathbf{x}_{-i}), u_i(\mathbf{z}'', \mathbf{x}_{-i})\Big) \geq b(\mathbf{x}_{-i}),$$

and therefore $(1-t)\mathbf{z}' + t\mathbf{z}'' \in B_i(\mathbf{x}_{-i})$. (This justification of the convexity of $B_i(\mathbf{x}_{-i})$ essentially repeats the argument presented in the proof of Lemma 7.2.)

We have now shown that, for each integer i between 1 and n, the correspondence $B_i \colon S_{-i} \to S_i$ that assigns to each element $\mathbf{x}_{-i}$ of S_{-i} the set of best strategies that the ith player can adopt in the event that the other players adopt the strategies represented by $\mathbf{x}_{-i}$ has closed graph, and maps each element $\mathbf{x}_{-i}$ of S_{-i} to a subset $B_i(\mathbf{x}_{-i})$ that is non-empty, compact and convex.

Now the Kakutani Fixed Point Theorem (Theorem 5.4) applies to correspondences with closed graph that map elements of a non-empty, compact and convex subset to non-empty convex subsets of that set. Thus in order to obtain a proof of the existence of Nash equilibria that utilizes the Kakutani Fixed Point Theorem, we must construct such a correspondence from a non-empty compact convex set to itself.

We recall that the *strategy profile set* S is defined to be the Cartesian product $S_1 \times S_2 \times \cdots \times S_n$ of the strategy sets for the players of the game. Let $\Phi\colon S \rightrightarrows S$ be the correspondence from the strategy profile set S to itself defined so that

$$\Phi(\mathbf{x}) = \Big(B_1(\pi_{-1}(\mathbf{x})), B_2(\pi_{-2}(\mathbf{x})), \cdots B_n(\pi_{-n}(\mathbf{x}))\Big)$$

for $i = 1, 2, \ldots, n$. Then

$$\{(\mathbf{x}, \mathbf{y}) \in S \times S : \mathbf{y} \in \Phi(\mathbf{x})\} = \bigcap_{i=1}^{n} G_i,$$

where

$$G_i = \{(\mathbf{x}, \mathbf{y}) \in S \times S : \pi_i(\mathbf{y}) \in B_i(\pi_{-i}(\mathbf{x}))\}$$

for $i = 1, 2, \ldots, n$. Now, for each i, the set

$$\{(\mathbf{x}_{-i}, \mathbf{y}_i) \in S_{-i} \times S_i : \mathbf{y}_i \in B_i(\mathbf{x}_{-i})\}$$

is closed in $S_{-i} \times S_i$, because the correspondence $B_i\colon S_{-i} \rightrightarrows S_i$ has closed graph. It follows that each set G_i is closed in $S \times S$, because the set G_i is the preimage of a closed set under the continuous mapping from $S \times S$ to $S_{-i} \times S_i$ that sends each ordered pair $(\mathbf{x}, \mathbf{y})$ in $S \times S$ to $(\pi_{-i}(\mathbf{x}), \pi_i(\mathbf{y}))$. The graph of the correspondence Φ is the intersection of the closed sets $G_1, G_2, \ldots, G_n$. It is therefore itself a closed set. Thus the correspondence $\Phi\colon S \rightrightarrows S$ has closed graph. Moreover S is a non-empty compact convex set, and $\Phi(\mathbf{x})$ is a non-empty convex subset of S for all $\mathbf{x} \in S$. It follows from the Kakutani Fixed Point Theorem (Theorem 5.4) that there exists a fixed point $\mathbf{x}^*$ for the correspondence Φ. This fixed point is strategy profile that satisfies $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$.

Let $\mathbf{x}_i^* = \pi_i(\mathbf{x}^*)$ and $\mathbf{x}_{-i}^* = \pi_{-i}(\mathbf{x}^*)$ for $i = 1, 2, \ldots, n$. Then $\mathbf{x}_i^* \in B_i(\mathbf{x}_{-i}^*)$ for $i = 1, 2, \ldots, n$, because $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$. It follows from the definition of $B_i(\mathbf{x}_{-i}^*$ that

$$u_i(\mathbf{x}_i, \mathbf{x}_{-i}^*) \leq u_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*)$$

for all integers i between 1 and n and for all $\mathbf{x}_i \in S_i$. The strategy profile $(\mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_n^*)$ therefore represents a Nash equilibrium for the game.

Theorem 7.3 (Existence of Nash Equilibria)
Consider an n-person game in which, for each integer i between 1 and n, the strategy set S_i is a compact convex subset of a Euclidean space, and in which the utility function $u_i\colon S_i \times S_{i-1} \to \mathbb{R}$ that determines the utility for the ith player, given a strategy profile $\mathbf{x}_{-i}$ representing strategies chosen by the other players, is a continuous function that, for any fixed $\mathbf{x}_{-i} \in S_{-i}$, determines a quasiconcave function mapping $\mathbf{x}_i$ to $u_i(\mathbf{x}_i, \mathbf{x}_{-i})$ as $\mathbf{x}_i$ varies over the strategy set S_i. Then there exists a Nash equilibrium $(\mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_n^*)$ for the game. Accordingly

$$u_i(\mathbf{x}_i, \mathbf{x}_{-i}^*) \leq u_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*)$$

for all integers i between 1 and n and for all $\mathbf{x}_i \in S_i$.

8 Walrasian Equilibria

8.1 Exchange Economies

We consider an exchange economy consisting of a finite number of commodities and a finite number of households, each provided with an initial endowment of each of the commodities. The commodities are required to be *infinitely divisible*: this means that a household can hold an amount x of that commodity for any non-negative real number x. (Thus salt, for example, could be regarded as an 'infinitely divisible' quantity whereas cars cannot: it makes little sense to talk about a particular household owning 2.637 of a car, for example, though such a household may well own 2.637 kilograms of salt.) Now the households may well wish to exchange commodities with one another so as improve on their initial endowment. They might for example seek to barter commodities with one another: however this method of redistribution would not work very efficiently in a large economy. Alternatively they might attempt to set up a price mechanism to simplify the task of redistributing the commodities. Thus suppose that each commodity is assigned a given price. Then each household could sell its initial endowment to the market, receiving in return the value of its initial endowment at the given prices. The household could then purchase from the market a quantity of each commodity so as to maximize its own preference, subject to the constraint that the total value of the commodities purchased by any household cannot exceed the value of its initial endowment at the given prices. The problem of redistribution then becomes one of fixing prices so that there is exactly enough of each commodity to go around: if the price of any commodity is too low then the demand for that commodity is likely to outstrip supply, whereas if the price is too high then supply will exceed demand. A *Walras equilibrium* is achieved if prices can be found so that the supply of each commodity matches its demand. We shall use *Berge's Maximum Theorem* and the *Kakutani fixed point theorem* to prove the existence of a Walras equilibrium in this idealized economy.

Let our exchange economy consist of n commodities and m households. We suppose that household h is provided with an initial endowment $\overline{x}_{hi}$ of commodity i, where $\overline{x}_{hi} \geq 0$. Thus the initial endowment of household h can be represented by a vector $\overline{\mathbf{x}}_h$ in $\mathbb{R}^n$ whose ith component is $\overline{x}_{hi}$. The prices of the commodities are given by a price vector $\mathbf{p}$ whose ith component p_i specifies the price of a unit of the ith commodity: a price vector $\mathbf{p}$ is required to satisfy $p_i \geq 0$ for all i. Then the value of the initial endowment of household h at the given prices is $\mathbf{p} \, . \, \overline{\mathbf{x}}_h$. This quantity represents the *wealth* of household h at prices $\mathbf{p}$.

Definition For each positive integer n, the *positive orthant* $\mathbb{R}^n_+$ is the subset of $\mathbb{R}^n$ defined so that

$$\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0}\}.$$

In particular $\mathbb{R}_+ = \{t \in \mathbb{R} : t \geq 0\}$.

Definition A real-valued function $u\colon X \to \mathbb{R}$ defined over a subset X of $\mathbb{R}^n$ is said to be *strictly increasing* on X if $u(\mathbf{x}) < u(\mathbf{x}')$ for all $\mathbf{x}, \mathbf{x}'$ in X satisfying $\mathbf{x} \leq \mathbf{x}'$ and $\mathbf{x} \neq \mathbf{x}'$.

8.2 The Budget Correspondence

We now discuss basic properties of the *budget correspondence.*

The budget correspondence is defined on the set of pairs. A *price-wealth pair* is an ordered pair $(\mathbf{p}, w)$, where $\mathbf{p} \in \mathbb{R}^n$, w is a non-negative real number and $\mathbf{p} \geq 0$. The budget correspondence assigns to each price-wealth pair the bundles of commodities that an economic agent with the specified wealth can afford to purchase at the specified prices.

More formally, the definition of the budget correspondence may be given as follows.

Definition In a model of an exchange economy with n commodities, The *budget correspondence* $B\colon \mathbb{R}^n_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^n$ assigns to each price-wealth pair $(\mathbf{p}, w)$ in $\mathbb{R}^n_+ \times \mathbb{R}_+$ the subset $B(\mathbf{p}, w)$ of $\mathbb{R}^n_+$ defined such that

$$B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}.$$

Example Consider the case of two commodities. The budget correspondence $B\colon \mathbb{R}^2_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^2$ is defined so that

$$B(\mathbf{p}, w) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0,\ \ x_2 \geq 0 \text{ and } p_1 x_1 + p_2 x_2 \leq w\}$$

for all $\mathbf{p} \in \mathbb{R}^2_+$ and $w \in \mathbb{R}_+$, where $\mathbf{p} = (p_1, p_2)$.

Let $\mathbf{p}_0$ be the vector in $\mathbb{R}^2_+$ with $\mathbf{p}_0 = (1, 0)$, and let V be the open set in $\mathbb{R}^3$ defined so that

$$V = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 < 1 + \frac{1}{1 + x_2^2} \right\}.$$

Now

$$B(\mathbf{p}_0, w) = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq w \text{ and } x_2 \geq 0\}$$

for all $w > 0$. It follows that $B(\mathbf{p}_0, 1) \subset V$, but $B(\mathbf{p}_0, w) \not\subset V$ for all $w > 1$. Indeed if $w > 1$ then t can be chosen large enough to ensure that

$$w > 1 + \frac{1}{1 + t^2}.$$

But then $(w, t) \in B(\mathbf{p}_0, w)$, but $(w, t) \notin V$. This example demonstrates that the budget correspondence $B\colon \mathbb{R}^2_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^2$ is not upper hemicontinuous at $(\mathbf{p}_0, 1)$, where $\mathbf{p}_0 = (1, 0)$.

Note also that $B(\mathbf{p}, w) = B(w^{-1}\mathbf{p}, 1)$ for all $(\mathbf{p}, w) \in \mathbb{R}^2 \times \mathbb{R}_+$ satisfying $w > 0$. It follows that the budget correspondence $\mathbf{p} \mapsto B(\mathbf{p}, 1)$ is not upper hemicontinuous on $\mathbb{R}^2_+$ at $\mathbf{p}_0$. Now let $\mathbf{p}_0 = (1, 0)$ as before, and let

$$V = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 1\}.$$

Now

$$B(\mathbf{p}_0, 0) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0 \text{ and } x_2 \geq 0\}.$$

It follows that $B(\mathbf{p}_0, 0) \cap V \neq \emptyset$. But if $\mathbf{p} >> \mathbf{0}$ then $B(\mathbf{p}, 0) = \{(0, 0)\}$. Thus $B(\mathbf{p}, 0) \cap V = \emptyset$ whenever $\mathbf{p} \geq \mathbf{0}$. It follows that the budget correspondence B is not lower hemicontinuous at $(\mathbf{p}_0, 0)$.

Proposition 8.1 *Let n be a positive integer, let $\mathbf{c}$ be an element of $\mathbb{R}^n$ satisfying $\mathbf{c} >> \mathbf{0}$, and let $B_\mathbf{c}\colon \mathbb{R}^n_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^n$ be the correspondence that assigns to each price-wealth pair $(\mathbf{p}, w)$ in $\mathbb{R}^n_+ \times \mathbb{R}_+$ the subset $B_\mathbf{c}(\mathbf{p}, w)$ of $\mathbb{R}^n_+$ defined such that*

$$B_\mathbf{c}(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{0} \leq \mathbf{x} \leq \mathbf{c} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}.$$

Then the correspondence $B_\mathbf{c}\colon \mathbb{R}^n_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^n$ is upper hemicontinuous on $\mathbb{R}^n_+ \times \mathbb{R}$ and lower hemicontinuous on

$$\{(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R} : w > 0\}.$$

Moreover $B_\mathbf{c}(\mathbf{p}, w)$ of $\mathbb{R}^n_+$ is non-empty, compact and convex for all $(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R}$.

Proof The set $B_\mathbf{c}(\mathbf{p}, w)$ is a non-empty closed bounded convex subset of $\mathbb{R}^n_+$ for $i = 1, 2, \ldots, n$. Any closed bounded subset of $\mathbb{R}^n$ is compact. It follows that The set $B_\mathbf{c}(\mathbf{p}, w)$ is non-empty, compact convex for all $(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R}_+$.

Next we show that the correspondence $B_\mathbf{c}$ is upper hemicontinuous on $\mathbb{R}^n_+ \times \mathbb{R}_+$. Let $(\mathbf{p}_0, w_0) \subset \mathbb{R}^n_+ \times \mathbb{R}_+$, and let V be an open set in $\mathbb{R}^n$ for which $B_\mathbf{c}(\mathbf{p}_0, w_0) \subset V$. We will show that there exists an open set N in $\mathbb{R}^n_+ \times \mathbb{R}_+$ such that $(\mathbf{p}_0, w_0) \in N$ and $B_\mathbf{c}(\mathbf{p}, w) \subset V$ for all $(\mathbf{p}, w) \in N$.

Now $B_{\mathbf{c}}(\mathbf{p}, w) \subset C$ for all $(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R}_+$, where

$$C = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{0} \leq \mathbf{x} \leq \mathbf{c}\}.$$

It follows that if $C \subset V$ then $B_{\mathbf{c}}(\mathbf{p}, w) \subset V$ for all $(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R}_+$. We may therefore take $N = \mathbb{R}^n_+ \times \mathbb{R}_+$ in the case where $C \subset V$.

In the case where C is not contained in V let $F = C \setminus V$. Then F is a non-empty closed subset of C. If $\mathbf{x} \in C$ and $\mathbf{p}_0 \,.\, \mathbf{x} \leq w_0$ then $\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}_0, w_0)$, and therefore $\mathbf{x} \in V$, because $B_{\mathbf{c}}(\mathbf{p}_0, w_0) \subset V$, and thus $\mathbf{x} \notin F$. It follows that $\mathbf{p}_0 \,.\, \mathbf{x} > w_0$ for all $\mathbf{x} \in F$. It then follows from the Extreme Value Theorem that the continuous function sending each point $\mathbf{x}$ of F to $\mathbf{p}_0 \,.\, \mathbf{x}$ attains a minimum value at some point of the set F, and therefore there exists a point $\mathbf{x}_1$ of F and a real number w_1 such that $\mathbf{p}_0 \,.\, \mathbf{x}_1 = w_1$ and $\mathbf{p}_0 \,.\, \mathbf{x} \geq w_1$ for all $\mathbf{x} \in F$. Then $w_1 > w_0$. It follows that $\mathbf{p}_0 \,.\, \mathbf{x} \geq w_1$ for all $\mathbf{x} \in F$, and therefore $B_{\mathbf{c}}(\mathbf{p}_0, w_1) \cap F = \emptyset$. But $B_{\mathbf{c}}(\mathbf{p}_0, w_1) \subset C$ and $F = C \setminus V$. It follows that $B_{\mathbf{c}}(\mathbf{p}_0, w_1) \subset V$.

Now let N be the subset of $\mathbb{R}^n_+ \times \mathbb{R}^+$ consisting of those price-wealth pairs $(\mathbf{p}, w)$ with the property that

$$(\mathbf{p})_i > \frac{w}{w_1} (\mathbf{p}_0)_i$$

for those integers i between 1 and n for which $(\mathbf{p}_0)_i > 0$. Then N is open in $\mathbb{R}^n_+ \times \mathbb{R}_+$. Moreover the definition of N and the inequality $w_0 < w_1$ together ensure that $(\mathbf{p}_0, w_0) \in N$.

Care needs to be exercised in cases where $w = 0$. Suppose that $\mathbf{p} \geq 0$ and $(\mathbf{p}, 0) \in N$. Then $(\mathbf{p})_i > 0$ for all integers i between 1 and n for which $(\mathbf{p}_0)_i > 0$. It follows that if $\mathbf{x} \in \mathbb{R}^n$ satisfies $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{p} \,.\, \mathbf{x} = 0$ then $(\mathbf{p})_i = 0$ for those integers i between 1 and n for which $(\mathbf{x})_i > 0$. But then $(\mathbf{p}_0)_i = 0$ for those integers i between 1 and n for which $(\mathbf{x})_i > 0$, and therefore $\mathbf{p}_0 \,.\, \mathbf{x} = 0$. We conclude from this that if $(\mathbf{p}, 0) \in N$ and $\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, 0)$ then $\mathbf{p}_0 \,.\, \mathbf{x} = 0$, and therefore $\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}_0, 0)$. But

$$B_{\mathbf{c}}(\mathbf{p}_0, 0) \subset B_{\mathbf{c}}(\mathbf{p}_0, w_0) \subset V.$$

We conclude therefore that if $(\mathbf{p}, 0) \in N$ then $B_{\mathbf{c}}(\mathbf{p}, 0) \subset V$.

Now let $(\mathbf{p}, w) \in N$, where $w > 0$, and let $\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, w)$. Then $\mathbf{x} \geq 0$ and $\mathbf{p} \,.\, \mathbf{x} \leq w$. Then

$$\mathbf{p}_0 \,.\, \mathbf{x} = \sum_{i=1}^{n} (\mathbf{p}_0)_i (\mathbf{x})_i \leq \frac{w_1}{w} \sum_{i=1}^{n} (\mathbf{p})_i (\mathbf{x})_i = \frac{w_1}{w} \, \mathbf{p} \,.\, \mathbf{x} \leq w_1,$$

and therefore $\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}_0, w_1)$. It follows that if $(\mathbf{p}, w) \in N$ and $w > 0$ then

$$B_{\mathbf{c}}(\mathbf{p}, w) \subset B_{\mathbf{c}}(\mathbf{p}_0, w_1) \subset V.$$

We conclude therefore that $B_{\mathbf{c}}(\mathbf{p}, w) \subset V$ for all $(\mathbf{p}, w) \in N$. The results we have so far obtained combine to show that the correspondence $B_{\mathbf{c}}$ is upper hemicontinuous on $\mathbb{R}^n_+ \times \mathbb{R}_+$.

Now let $(\mathbf{p}_0, w_0) \in \mathbb{R}^n_+ \times \mathbb{R}_+$ satisfy $w_0 > 0$, and let V be an open set in $\mathbb{R}^n$ that satisfies $V \cap B_{\mathbf{c}}(\mathbf{p}_0, w_0) \neq \emptyset$. The constraint $w_0 > 0$ ensures that any open ball of positive radius centred on a point of $B_{\mathbf{c}}(\mathbf{p}_0, w_0)$ intersects the interior of that set. It follows that the open set V must intersect the interior of the set $B_{\mathbf{c}}(\mathbf{p}_0, w_0)$, and therefore there exists $\mathbf{x}_0 \in V$ for which $\mathbf{0} \leq \mathbf{x}_0 \leq \mathbf{c}$ and $\mathbf{p}_0 \,.\, \mathbf{x}_0 < w_0$. Let

$$N = \{(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R}_+ : w - \mathbf{p} \,.\, \mathbf{x}_0 > 0\}.$$

Then N is open in $\mathbb{R}^n$, $(\mathbf{p}_0, \mathbf{w}_0) \in N$, and $\mathbf{x}_0 \in V \cap B_{\mathbf{c}}(\mathbf{p}, w)$ for all $(\mathbf{p}, w) \in N$. We conclude from this that the correspondence $B_{\mathbf{c}}$ is lower hemicontinuous on the set $\mathbb{R}^n_+ \times \mathbb{R}_+$. This completes the proof. ∎

Proposition 8.2 *Let n be a positive integer, and let $B\colon \mathbb{R}^n_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^n$ be the budget correspondence that assigns to each price-wealth pair $(\mathbf{p}, w)$ in $\mathbb{R}^n_+ \times \mathbb{R}_+$ the subset $B(\mathbf{p}, w)$ of $\mathbb{R}^n_+$ defined such that*

$$B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}.$$

Then the budget correspondence $B\colon \mathbb{R}^n_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^n$ is both upper hemicontinuous and lower hemicontinuous on the set Γ^n, where

$$\Gamma^n = \{(\mathbf{p}, w) \in \mathbb{R}^n \times \mathbb{R} : \mathbf{p} >> \mathbf{0} \text{ and } w > 0\}.$$

Moreover $B(\mathbf{p}, w)$ of $\mathbb{R}^n_+$ is non-empty, compact and convex for all $(\mathbf{p}, w) \in \Gamma^n$.

Proof Let $(\mathbf{p}_0, w_0)$ be a price-wealth pair for which $\mathbf{p}_0 >> \mathbf{0}$ and $w_0 > 0$. Then $(\mathbf{p})_i > 0$ for $i = 1, 2, \ldots, n$. Let a positive vector $\mathbf{c}$ be chosen so that

$$(\mathbf{c})_i > \frac{w_0}{(\mathbf{p}_0)_i}$$

for $i = 1, 2, \ldots, n$. Let

$$N = \{(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R}_+ : w > 0 \text{ and } (\mathbf{p})_i > \frac{w}{(\mathbf{c})_i} \text{ for } i = 1, 2, \ldots, n\}.$$

Then N is an open subset of $\mathbb{R}^n_+ \times \mathbb{R}_+$, $(\mathbf{p}_0, w_0) \in N$. Moreover if $(\mathbf{p}, w) \in N$, and if $\mathbf{x} \in B(\mathbf{p}, w)$, then $\mathbf{x} \geq 0$, $\mathbf{p} \,.\, \mathbf{x} \leq w$ and $w > 0$ But then $(\mathbf{p})_i > 0$ and

$$(\mathbf{p})_i (\mathbf{x})_i \leq w < (\mathbf{p})_i (\mathbf{c})_i$$

for $i = 1, 2, \ldots, n$, and therefore $\mathbf{x} \leq \mathbf{c}$. It follows that $B(\mathbf{p}, w) = B_{\mathbf{c}}(\mathbf{p}, w)$ for all $(\mathbf{p}, w) \in N$, where

$$B_{\mathbf{c}}(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{0} \leq \mathbf{x} \leq \mathbf{c} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}.$$

Now the correspondence $B_{\mathbf{c}}$ defined in this fashion is both upper hemicontinuous and lower hemicontinuous on the set of all price-wealth pairs $(\mathbf{p}, w)$ for which $w > 0$. (Proposition 8.1). It follows that, because $w > 0$ and $B(\mathbf{p}, w) = B_{\mathbf{c}}(\mathbf{p}, w)$ for all $(\mathbf{p}, w) \in N$, the budget correspondence B is both upper hemicontinuous and lower hemicontinuous on the open subset N of the set of price-wealth pairs, and is therefore both upper and lower hemicontinuous around the price-wealth pair $(\mathbf{p}_0, w_0)$. The result follows. ∎

8.3 Maximizing Normalized Commodity Prices

Proposition 8.3 *Let n be a positive integer, let*

$$\Delta = \left\{ \mathbf{p} \in \mathbb{R}^n : \mathbf{p} \geq \mathbf{0} \text{ and } \sum_{i=1}^{n} (\mathbf{p})_i = 1 \right\}.$$

Let $\gamma\colon \mathbb{R}^n \to \mathbb{R}$ be the function defined so that, for each $\mathbf{x} \in \mathbb{R}^n$, $\gamma(\mathbf{x})$ is the maximum of the components of $\mathbf{x}$, and let $\mu\colon \mathbb{R}^n \rightrightarrows \Delta$ be the correspondence defined such that

$$\mu(\mathbf{x}) = \{\mathbf{p} \in \Delta : \mathbf{p} \,.\, \mathbf{x} = \gamma(\mathbf{x})\}.$$

Then the correspondence $\mu\colon \mathbb{R}^n \rightrightarrows \Delta$ is upper hemicontinuous, and $\mu(\mathbf{x})$ is a non-empty compact convex subset of Δ for all $\mathbf{x} \in \mathbb{R}^n$. Also $\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p}' \,.\, \mathbf{x} = \gamma(\mathbf{x})$ for all $\mathbf{p} \in \Delta$ and $\mathbf{p}' \in \mu(\mathbf{x})$.

Proof Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{p} \in \Delta$, and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. Then $p_i \geq 0$ for $i = 1, 2, \ldots, n$, and

$$\gamma(\mathbf{x}) = \max(x_1, x_2, \ldots, x_n).$$

Let $I(\mathbf{x})$ denote those integers i between 1 and n for which $x_i = \gamma(\mathbf{x})$. Now $0 \leq p_i \leq 1$ for $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} p_i = 1$. It follows that

$$\mathbf{p} \,.\, \mathbf{x} = \sum_{i=1}^{n} p_i x_i \leq \gamma(\mathbf{x}) \sum_{i=1}^{n} p_i = \gamma(\mathbf{x}).$$

Moreover if $x_i < \gamma(\mathbf{x})$ and $p_i > 0$ for some integer i between 1 and n then $\mathbf{p} \,.\, \mathbf{x} < \mu(\mathbf{x})$. It follows that $\mathbf{p} \,.\, \mathbf{x} \leq \gamma(\mathbf{x})$ for all $\mathbf{p} \in \Delta$, and $\mathbf{p} \,.\, \mathbf{x} = \gamma(\mathbf{x})$ if and only if $p_i = 0$ for those integers i between 1 and n for which $x_i < \gamma(\mathbf{x})$. It follows that $\mathbf{p} \,.\, \mathbf{x} = \gamma(\mathbf{x})$ if and only if $p_i = 0$ for those integers i between 1 and n for which $i \notin I(\mathbf{x})$. Therefore

$$\begin{aligned} \mu(\mathbf{x}) &= \{(p_1, p_2, \ldots, p_n) \in \Delta : p_i = 0 \text{ whenever } (\mathbf{x})_i < \gamma(\mathbf{x})\} \\ &= \{(p_1, p_2, \ldots, p_n) \in \Delta : p_i = 0 \text{ whenever } i \notin I(\mathbf{x})\}. \end{aligned}$$

It follows that, for all $\mathbf{x} \in \mathbb{R}$, the set $\mu(\mathbf{x})$ is a closed subset of the simplex Δ, and is therefore a compact set. It is clearly non-empty and convex. Also

$$\mathbf{p} \,.\, \mathbf{x} \leq \mu(\mathbf{x}) = \mathbf{p}' \,.\, \mathbf{x}$$

for all $\mathbf{p} \in \Delta$ and $\mathbf{p}' \in \mu(\mathbf{x})$.

Let $\mathbf{x}' \in \mathbb{R}^n$, and let $\mathbf{x}' = (x'_1, x'_2, \ldots, x'_n)$. If $i \in I(\mathbf{x}')$ then $x'_i = \gamma(\mathbf{x}')$, and if $i \notin I(\mathbf{x}')$ then $x'_i < \gamma(\mathbf{x}')$. There then exists a real number θ such that $\theta < \gamma(\mathbf{x}')$ and $x'_i < \theta$ whenever $i \notin I(\mathbf{x}')$. Let N be the subset of $\mathbb{R}^n$ consisting of those elements $(x_1, x_2, \ldots, x_n)$ of $\mathbb{R}^n$ satisfying the following two conditions:

- $x_i > \theta$ if $i \in I(\mathbf{x}')$;
- $x_i < \theta$ if $i \notin I(\mathbf{x}')$.

Then N is open in $\mathbb{R}^n$ and $\mathbf{x}' \in N$. Moreover $I(\mathbf{x}) \subset I(\mathbf{x}')$ for all $\mathbf{x} \in N$, and therefore $\mu(\mathbf{x}) \subset \mu(\mathbf{x}')$ for all $\mathbf{x} \in N$. Thus if V is open in $\mathbb{R}^n$ and if $\mu(\mathbf{x}') \subset V$ then $\mu(\mathbf{x}) \subset V$ for all $\mathbf{x} \in N$. We conclude from this that the correspondence $\mu\colon \mathbb{R}^n \to \Delta$ is upper hemicontinuous on $\mathbb{R}$. This completes the proof. ∎

Remark Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is the standard basis of $\mathbb{R}^n$, defined so that, for each integer i between 1 and n, the ith component of $\mathbf{e}_i$ is equal to 1 and the other components are zero. Then the simplex Δ is an $(n-1)$-dimensional simplex with vertices $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, and, for each $\mathbf{x} \in \mathbb{R}^n$, the subset $\mu(\mathbf{x})$ of Δ is the face of the simplex Δ spanned by those vertices $\mathbf{e}_i$ of Δ for which $(\mathbf{x})_i = \gamma(\mathbf{x})$, where $\gamma(\mathbf{x})$ denotes the maximum value of the components of the vector $\mathbf{x}$.

8.4 Consumer Preferences

We next discuss how each household sets out to determine its purchase requirements.

We suppose that the preferences of household h are represented by a *utility function* $u_h\colon \mathbb{R}^n_+ \to \mathbb{R}$ that is continuous, strictly increasing and quasiconcave. Such a utility function therefore satisfies the following conditions:

- the function $u\colon \mathbb{R}^n_+ \to \mathbb{R}$ is *continuous*;
- the function $u\colon \mathbb{R}^n_+ \to \mathbb{R}$ is *strictly increasing*, and thus if $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n_+$ satisfy $\mathbf{x} \leq \mathbf{x}'$ and $\mathbf{x} \neq \mathbf{x}'$ then $u(\mathbf{x}) < u(\mathbf{x}')$;
- the function $u\colon \mathbb{R}^n_+ \to \mathbb{R}$ is *quasiconcave*, and thus

$$u((1-t)\mathbf{x} + t\mathbf{x}') \geq \min\Big(u(\mathbf{x}), u(\mathbf{x}')\Big)$$

  for all $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n_+$ and $t \in [0, 1]$.

Proposition 8.4 *Let $u\colon X \to \mathbb{R}$ be a function defined on a closed convex subset X of $\mathbb{R}^n$ that is continuous, strictly increasing and quasiconcave, let $\mathbf{p}$ be a non-zero non-negative price vector in $\mathbb{R}^n$, let w be a positive real number, let*

$$B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}$$

and let $\mathbf{x}^* \in B(\mathbf{p}, w)$. Suppose that there exists some open neighbourhood N of $\mathbf{x}^*$ in $\mathbb{R}^n_+$ with the property that $u(\mathbf{x}) \leq u(\mathbf{x}^*)$ for all $\mathbf{x} \in B(\mathbf{p}, w) \cap N$. Then $\mathbf{p} \,.\, \mathbf{x}^* = w$ and $u(\mathbf{x}) \leq u(\mathbf{x}^*)$ for all $\mathbf{x} \in B(\mathbf{p}, w)$.

Proof Suppose that it were the case that $\mathbf{p} \,.\, \mathbf{x}^* < w$. Then it would be possible to find $\mathbf{x} \in N$ satisfying $\mathbf{x} >> \mathbf{x}^*$ and $\mathbf{p} \,.\, \mathbf{x} < w$. Then $\mathbf{x} \in B(\mathbf{p}, w) \cap N$. The strictly increasing property of the utility function u would then ensure that $u(\mathbf{x}) > u(\mathbf{x}^*)$. But this would contradict that assumption that the maximum of the utility function u on $B(\mathbf{p}, w) \cap N$ is attained at $\mathbf{x}^*$.

Next suppose that there were to exist in the set $B(\mathbf{p}, w)$ a commodity bundle $\mathbf{x}'$ for which $u(\mathbf{x}') > u(\mathbf{x}^*)$. It would then follow from the continuity of the utility function u that the value of utility function u would exceed $u(\mathbf{x}^*)$ throughout some open ball of positive radius centred on $\mathbf{x}'$. Now $w > 0$, and therefore $B(\mathbf{p}, w)$ has non-empty interior. Moreover every open ball of positive radius about an element of $B(\mathbf{p}, w)$ would intersect the interior of this set. It follows that there would exist a commodity bundle $\mathbf{x}''$ in the interior of $B(\mathbf{p}, w)$ lying sufficiently close to $\mathbf{x}'$ to ensure that $u(\mathbf{x}'') > u(\mathbf{x}^*)$ and $\mathbf{p} \,.\, \mathbf{x}'' < w$. The quasiconcavity of the utility function would ensure that the utility function u would take values no less than $u(\mathbf{x}^*)$ along the line segment joining the commodity bundles $\mathbf{x}^*$ and $\mathbf{x}''$. Moreover this line segment would be wholly contained within the convex set $B(\mathbf{p}, w)$.

Now $\mathbf{x}^* \in N$. Therefore there would then exist a commodity bundle $\mathbf{x}'''$ on the line segment joining $\mathbf{x}^*$ and $\mathbf{x}''$ that was distinct from $\mathbf{x}^*$ but was close enough to $\mathbf{x}^*$ to ensure that $\mathbf{x}''' \in N$. Then $u(\mathbf{x}''') \geq u(\mathbf{x}^*)$ and $\mathbf{p} \,.\, \mathbf{x}''' < w$.

There would then exist a commodity bundle $\mathbf{x}$ satisfying $\mathbf{x} \geq \mathbf{x}'''$ and $\mathbf{x} \neq \mathbf{x}'''$ for which $\mathbf{x} \in N$ and $\mathbf{p} \,.\, \mathbf{x} < w$. Then $\mathbf{x} \in B(\mathbf{p}, w) \cap N$ and

$$u(\mathbf{x}) > u(\mathbf{x}''') \geq u(\mathbf{x}^*),$$

contradicting the fact that the function u achieves is maximum value on $B(\mathbf{p}, w) \cap N$ at $\mathbf{x}^*$. We conclude therefore that the maximum value of the utility function u on $B(\mathbf{p}, w)$ is attained at the point $\mathbf{x}^*$, as required. ∎

8.5 Indirect Utility and Consumer Demand

Let Γ^n be the set of price-wealth pairs $(\mathbf{p}, w)$ for which $\mathbf{p} >> \mathbf{0}$ and $w > 0$, so that

$$\Gamma^n = \{(\mathbf{p}, w) \in \mathbb{R}^n \times \mathbb{R} : \mathbf{p} >> \mathbf{0} \text{ and } w > 0\}.$$

Then the closure $\overline{\Gamma}^n$ of Γ^n in $\mathbb{R}^n \times \mathbb{R}$ satisfies

$$\overline{\Gamma}^n = \mathbb{R}^n_+ \times \mathbb{R}_+ = \{(\mathbf{p}, w) \in \mathbb{R}^n \times \mathbb{R} : \mathbf{p} \geq \mathbf{0} \text{ and } w \geq 0\}.$$

Let $B\colon \overline{\Gamma}^n \rightrightarrows \mathbb{R}^n$ denote the budget correspondence on $\overline{\Gamma}^n$, where

$$B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}$$

for all $(\mathbf{p}, w) \in \overline{\Gamma}^n$.

Let $u\colon \overline{\Gamma}^n \to \mathbb{R}$ be a utility function for a given consumer, defined over $\overline{\Gamma}^n$, that is continuous, strictly increasing and quasiconcave. Then the utility function u and the budget correspondence B together determine a single valued function $V\colon \Gamma^n \to \mathbb{R}$ and a correspondence $\xi\colon \Gamma^n \rightrightarrows \mathbb{R}^n_+$, where

$$V(\mathbf{p}, w) = \sup\{u(\mathbf{x}) : \mathbf{x} \in B(\mathbf{p}, w)\}$$

and

$$\xi(\mathbf{p}, w) = \sup\{\mathbf{x} \in B(\mathbf{p}, w) : u(\mathbf{x}) = V(\mathbf{p}, w)\}.$$

The function $V\colon \Gamma^n \to \mathbb{R}$ is referred to as the *indirect utility function* for the given consumer, and the correspondence $\xi\colon \Gamma^n \rightrightarrows \mathbb{R}^n_+$ is referred to as the *demand correspondence* for that consumer. The value of $V(\mathbf{p}, w)$ is the maximum utility that the consumer by purchasing a bundle of commodities that is affordable for that consumer when the commodity prices are given by the price vector $\mathbf{p}$ and the wealth of the consumer is represented by the non-negative real number w. The demand correspondence $\xi\colon \Gamma^n \rightrightarrows \mathbb{R}^n_+$ associates to a price-wealth pair $(\mathbf{p}, w)$ the set consisting of those bundles of commodities that are most desirable for the consumer with wealth w, subject to being affordable at prices $\mathbf{p}$.

Proposition 8.5 *In an exchange economy with n commodities, suppose that the preferences of a given consumer are represented by a utility function $u\colon \overline{\Gamma}^n \to \mathbb{R}$, defined over the line $\overline{\Gamma}^n$ of price-wealth pairs, that is continuous, strictly increasing and quasiconcave. Then the resulting indirect utility function $V\colon \Gamma^n \to \mathbb{R}$ is continuous on the set Γ^n of price-wealth pairs $(\mathbf{p}, w)$ for which $\mathbf{p} >> \mathbf{0}$ and $w > 0$, and the demand correspondence $\xi\colon \Gamma^n \rightrightarrows \mathbb{R}^n_+$ is upper hemicontinuous and maps each price-wealth pair $(\mathbf{p}, w)$ in Γ^n to a non-empty compact convex subset of $\mathbb{R}^n_+$.*

Proof Proposition 8.2 ensures that the budget correspondence $\xi\colon \Gamma^n \rightrightarrows \mathbb{R}^n_+$ is both upper hemicontinuous and lower hemicontinuous on Γ^n. Moreover $\xi(\mathbf{p}, w)$ is a non-empty compact subset of $\mathbb{R}^n_+$ for all $(\mathbf{p}, w) \in \Gamma^n$. It follows from a direct application of Berge's Maximum Theorem (Theorem 2.23) that the indirect utility function is continuous and the demand correspondence is upper hemicontinuous and maps each price-wealth pair in Γ^n to a non-empty compact subset of $\mathbb{R}^n_+$. The convexity of $B(\mathbf{p}, w)$ and the quasiconcavity of the utility function u then ensure that $\xi(\mathbf{p}, w)$ is convex for all price-wealth pairs $(\mathbf{p}, w)$ in Γ^n. ∎

Let $\mathbf{c}$ be an element of $\mathbb{R}^n$ satisfying $\mathbf{c} >> \mathbf{0}$. In what follows we restrict consumer choice to those bundles of commodities that, for a particular price-wealth pair $(\mathbf{p}, w)$, are both affordable and subject to the availability constraint $\mathbf{0} \leq \mathbf{x} \leq \mathbf{c}$. Thus let $B_{\mathbf{c}}\colon \overline{\Gamma}^n \rightrightarrows \mathbb{R}^n$ denote the budget correspondence on $\overline{\Gamma}^n$ when availability is constrained in this fashion, so that

$$B_{\mathbf{c}}(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{0} \leq \mathbf{x} \leq \mathbf{c} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}$$

for all $(\mathbf{p}, w) \in \overline{\Gamma}^n$. It follows from Proposition 8.1 that the correspondence $B_{\mathbf{c}}\colon \overline{\Gamma}^n \rightrightarrows \mathbb{R}^n$ is both upper hemicontinuous and lower hemicontinuous throughout the set $\hat{\Gamma}^n$ defined so that

$$\hat{\Gamma}^n = \{(\mathbf{p}, w) \in \mathbb{R}^n \times \mathbb{R} : \mathbf{p} \geq \mathbf{0} \text{ and } w > 0\}.$$

We still require the utility function $u\colon \overline{\Gamma}^n \to \mathbb{R}$ for the given consumer to be continuous, strictly increasing and quasiconcave. Then the utility function u and the modified budget correspondence $B_{\mathbf{c}}$ together determine a single valued function $\hat{V}_{\mathbf{c}}\colon \Gamma^n \to \mathbb{R}$ and a correspondence $\hat{\xi}_{\mathbf{c}}\colon \Gamma^n \rightrightarrows \mathbb{R}^n_+$, where

$$\hat{V}_{\mathbf{c}}(\mathbf{p}, w) = \sup\{u(\mathbf{x}) : \mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, w)\}$$

and

$$\hat{\xi}_{\mathbf{c}}(\mathbf{p}, w) = \sup\{\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, w) : u(\mathbf{x}) = \hat{V}_{\mathbf{c}}\mathbf{p}, w)\}.$$

Proposition 8.6 *In an exchange economy with n commodities, suppose that the preferences of a given consumer are represented by a utility function $u\colon \overline{\Gamma}^n \to \mathbb{R}$, defined over the line $\overline{\Gamma}^n$ of price-wealth pairs, that is continuous, strictly increasing and quasiconcave. Let $\mathbf{c} \in \mathbb{R}^n$ satisfy $\mathbf{c} >> \mathbf{0}$, and let the consumer be required to select from bundles $\mathbf{x}$ of commodities, represented by non-negative n-dimensional vectors, that, for prices and wealth given by the price-wealth pair $(\mathbf{p}, w)$, satisfy both the budget constraint $\mathbf{p} \,.\, \mathbf{x} \leq w$ and the availability constraint $\mathbf{0} \leq \mathbf{x} \leq \mathbf{c}$. Then the resulting indirect utility function $\hat{V}_{\mathbf{c}}\colon \hat{\Gamma}^n \to \mathbb{R}$ is continuous on the set $\hat{\Gamma}^n$ of price-wealth pairs $(\mathbf{p}, w)$ for which $w > 0$, and the demand correspondence $\hat{\xi}_{\mathbf{c}}\colon \hat{\Gamma}^n \rightrightarrows \mathbb{R}^n_+$ is upper hemicontinuous and maps each price-wealth pair $(\mathbf{p}, w)$ in $\hat{\Gamma}^n$ to a non-empty compact convex subset of $\mathbb{R}^n_+$.*

Proof Proposition 8.1 ensures that the budget correspondence $\hat{\xi}_{\mathbf{c}}\colon \hat{\Gamma}^n \rightrightarrows \mathbb{R}^n_+$ is both upper hemicontinuous and lower hemicontinuous on $\hat{\Gamma}^n$. Moreover $\hat{\xi}_{\mathbf{c}}(\mathbf{p}, w)$ is a non-empty compact subset of $\mathbb{R}^n_+$ for all $(\mathbf{p}, w) \in \hat{\Gamma}^n$. It follows from a direct application of Berge's Maximum Theorem (Theorem 2.23) that the indirect utility function is continuous and the demand correspondence is upper hemicontinuous and maps each price-wealth pair in $\hat{\Gamma}^n$ to a non-empty compact subset of $\mathbb{R}^n_+$. The convexity of $B_{\mathbf{c}}(\mathbf{p}, w)$ and the quasiconcavity of the utility function u then ensure that $\hat{\xi}_{\mathbf{c}}(\mathbf{p}, w)$ is convex for all price-wealth pairs $(\mathbf{p}, w)$ in $\hat{\Gamma}^n$. ∎

8.6 Addition of Compact-Valued Correspondences

We discuss now the addition of vector-valued correspondences.

Suppose that we have m correspondences $\xi_1, \xi_2, \ldots, \xi_m$ defined over some subset Ω of a Euclidean space, and mapping points of Ω to subsets of a Euclidean space $\mathbb{R}^n$. Let $\sum_{h=1}^{n} \xi_h$ denote the correspondence ξ defined such that

$$\xi(\mathbf{p}) = \left\{ \sum_{h=1}^{m} \mathbf{x}_h : \mathbf{x}_h \in \xi_h(\mathbf{p}) \right\}.$$

Proposition 8.7 *Let $\xi_1, \xi_2, \ldots, \xi_m$ be correspondences defined over some subset Ω of a Euclidean space, and mapping points of that space to non-empty compact subsets of the n-dimensional Euclidean space $\mathbb{R}^n$. Suppose that these correspondences are upper hemicontinuous. Then the sum $\sum_{h=1}^{m} \xi_h$ of those correspondences is an upper hemicontinuous correspondence mapping points of Ω to non-empty compact subsets of $\mathbb{R}^n$.*

Proof Let $\xi\colon \Omega \rightrightarrows \mathbb{R}^n$ be the correspondence that is the sum $\sum_{h=1}^{m} \xi_h$ of the correspondences $\xi_1, \xi_2, \ldots, \xi_m$. Now, for each $\mathbf{p} \in \Omega$, the set $\xi(\mathbf{p})$ is the image of the Cartesian product

$$\xi_1(\mathbf{p}) \times \xi_2(\mathbf{p}) \times \cdots \times \xi_m(\mathbf{p})$$

under the continuous function that maps each m-tuple of vectors in $\mathbb{R}^n$ to the sum of its components. Moreover $\xi_h(\mathbf{p})$ is, by assumption, a non-empty compact subset of $\mathbb{R}^n$, and any Cartesian product of non-empty compact sets is non-empty and compact, and the image of a non-empty compact set under a continuous map is non-empty and compact. We conclude therefore that $\xi(\mathbf{p})$ is a non-empty compact subset of $\mathbb{R}^n$ for all $\mathbf{p} \in \Omega$.

We can therefore apply the "ϵ–δ" criterion for upper hemicontinuity of compact-valued correspondences established by Proposition 2.16. Given any subset K of $\mathbb{R}^n$, and given any positive real number r, we denote by $B(K, r)$ the subset of $\mathbb{R}^n$ that lie within a distance less than r of a point of K.

Let $\mathbf{p} \in \Omega$, and let some strictly positive real number ε be given. It follows from Proposition 2.16 that, for each integer h between 1 and m, there exists some open neighbourhood N_h of $\mathbf{p}$ in Ω such that $\xi_h(\mathbf{p}') \subset B(\xi_h(\mathbf{p}), \varepsilon/m)$ for all $\mathbf{p}' \in N_h$. Let N be the open neighbourhood of $\mathbf{p}$ in Ω that is the intersection of $N_1, N_2, \ldots, N_h$. Then a straightforward application of the triangle inequality ensures that $\xi(\mathbf{p}') \subset B(\xi(\mathbf{p}), \varepsilon)$ for all $\mathbf{p}' \in N$. It then follows from Proposition 2.16 that the correspondence $\xi\colon \Omega \rightrightarrows \mathbb{R}^n$ is upper hemicontinuous at $\mathbf{p}$. Its values are non-empty compact subsets of $\mathbb{R}^n$. The result follows. ∎

8.7 Aggregate Supply and Demand in an Exchange Economy

We now consider the properties of aggregate supply and demand in a pure exchange economy, or market, in which n commodities are traded between m households. Each household is provided with an initial endowment of commodities. The initial endowment of household h is then represented by an n-dimensional vector $\overline{\mathbf{x}}_h$ whose ith component specifies the initial endowment (relative to some appropriate unit) of the ith commodity traded in the market. The *aggregate supply* is then represented by a vector $\mathbf{s}$ that is the sum of the initial endowment vectors of all households. Thus

$$\mathbf{s} = \sum_{h=1}^{m} \overline{\mathbf{x}}_h.$$

We restrict our attention to the situation in which $\overline{\mathbf{x}}_h >> \mathbf{0}$ for $h = 1, 2, \ldots, m$. This restriction requires that each household be given an initial endowment of every commodity traded in the market. This ensures that, provided all commodity prices are non-negative, and at least one commodity price is strictly positive, then initial endowment of each household has strictly positive value, and thus each household has wealth to enable it to trade in the market. Within the mathematical model, this ensures that the demand correspondences of each household are lower hemicontinuous (see Proposition 8.6). The requirement that $\overline{\mathbf{x}}_h >> \mathbf{0}$ for all households h also ensures that $\mathbf{s} >> \mathbf{0}$.

The prices of the commodities are encoded in a price vector $\mathbf{p}$ whose components are non-negative real numbers. The ith component of this price vector $\mathbf{p}$ specifies the price of a unit of the ith commodity. We suppose that the price of at least one commodity is non-zero.

Each household seeks to trade its initial endowment for the bundle of commodities that provides it with maximum utility within the budget constraint that requires the value of purchased commodities to be less than or equal to the value of the initial endowment traded in. A consequence of this is that the demand of the ith consumers at prices $\lambda\mathbf{p}$ is identical to the demand at prices $\mathbf{p}$ for all positive real numbers λ. Indeed the bundles of commodities available to household h at prices $\mathbf{p}$ are those represented by vectors $\mathbf{x}_h$ satisfying the budget constraint

$$\mathbf{p} \,.\, \mathbf{x}_h \leq \mathbf{p} \,.\, \overline{\mathbf{x}_h}.$$

It follows that the price vector $\mathbf{p}$ may be replaced by the scalar multiple $\lambda\mathbf{p}$ for any positive real number λ without altering the set of bundles of commodities that the households individually can afford.

It is appropriate therefore to normalize prices in some fashion so that all non-zero non-negative price vectors can be expressed uniquely as a scalar multiple of a normalized price vector. We adopt the normalization scheme in which the sum of the prices of the commodities is required to be equal to one.

Definition A price vector $\mathbf{p}$ (with non-negative components) is said to be *normalized* if $\sum_{i=1}^{n} (\mathbf{p})_i = 1$.

Normalized price vectors are therefore represented by the points of the *price simplex* Δ, where

$$\Delta = \left\{ \mathbf{p} \in \mathbb{R}^n : \mathbf{p} \geq \mathbf{0} \text{ and } \sum_{i=1}^{n} (\mathbf{p})_i = 1 \right\}.$$

We suppose that the demand for each household is determined by the appropriate budget constraint and by a utility function that is continuous, strictly increasing and quasiconcave. This being the case, if the price of the ith commodity is zero, with the result that the commodity is free, then every household can afford to acquire unlimited quantities of it, and because the utility functions are required to be strictly increasing, demand for that commodity cannot be satisfied: the households have an *insatiable* appetite for free commodities.

This might suggest constraining price variation to price vectors whose components are strictly positive. However the fixed point theorems that are used to prove the existence of equilibria in which supply balances demand apply to functions or correspondences defined on compact sets. Therefore the correspondences that specify the demands of the consumers as prices vary should assign a non-empty compact set not only to the normalized price vectors in the interior of the price simplex Δ but also to the price vectors on the boundary of the price simplex.

Accordingly we impose an additional constraint on the purchases of the households. In addition to the budget constraint, we place limits on the amount of each commodity in the bundles available to the households. These limits may be specified by a fixed positive vector $\mathbf{c}$. Accordingly we require that $\mathbf{c} >> \mathbf{0}$ and that, for each integer h between 1 and m, household h selects a bundle at prices $\mathbf{p}$ to maximize utility amongst bundles $\mathbf{x}$ that satisfy both the budget constraint

$$\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p} \,.\, \overline{\mathbf{x}_h}$$

and the additional constraint $\mathbf{0} \leq \mathbf{x} \leq \mathbf{c}$.

We denote by $B_{\mathbf{c},h}(\mathbf{p})$ the set of bundles of commodities from which household $\mathbf{x}$ makes its selection. Accordingly, with this additional constraint, for each price vector $\mathbf{p}$ belonging to the price simplex Δ, household h selects the bundle of commodities that maximizes its utility function u_h over the non-empty compact set $B_{\mathbf{c},h}(\mathbf{p})$, where

$$B_{\mathbf{c},h}(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{0} \leq \mathbf{x} \leq \mathbf{c} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p} \,.\, \overline{\mathbf{x}}_h\}.$$

We denote the set of bundles of commodities that maximizes utility for household h under these constraints by $\hat{\xi}_{\mathbf{c},h}(\mathbf{p})$. We obtain in this fashion a correspondence $\hat{\xi}_{\mathbf{c},h} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ that determines the set of bundles maximizing utility for household h at prices $\mathbf{p}$, subject to the budget constraint and the additional constraint that available bundles of commodities by bounded above by the positive vector $\mathbf{c}$.

Proposition 8.8 *Suppose that, in a model of an exchange economy with n goods and m households, every household receives a strictly positive initial endowment of every commodity, so that the initial endowment vector $\overline{\mathbf{x}}_h$ of household h satisfies $\overline{\mathbf{x}}_h >> \mathbf{0}$ for $h = 1, 2, \ldots, m$. Suppose also that the preferences of household h are determined by a utility function u_h that is continuous, strictly increasing and quasiconcave. Then, for each household, and for each $\mathbf{c} \in \mathbb{R}^n$ satisfying $\mathbf{c} >> \mathbf{0}$ the demand correspondences $\hat{\xi}_{\mathbf{c},h} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ is upper hemicontinuous on the set Δ of normalized price vectors, and maps each normalized price vector $\mathbf{p}$ to a non-empty compact convex subset $\hat{\xi}_{\mathbf{c},h}(\mathbf{p})$ of $\mathbb{R}^n_+$ that consists of those bundles $\mathbf{x}$ of commodities that maximize utility for household h at prices $\mathbf{p}$ subject to both the affordability constraint $\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p} \,.\, \overline{\mathbf{x}}$ and the constraint $\mathbf{0} \leq \mathbf{x} \leq \mathbf{c}$.*

Proof Let

$$\hat{\Gamma}^n = \{(\mathbf{p}, w) \in \mathbb{R}^n \times \mathbb{R} : \mathbf{p} \geq \mathbf{0} \text{ and } w > 0\},$$

and, for all $(\mathbf{p}, w) \in \hat{\Gamma}^n$, let us denote by $\hat{\xi}'_{\mathbf{c},h}(\mathbf{p}, w)$ the demand of household h at prices $\mathbf{p}$, when the household has wealth h, where $\hat{\xi}'_{\mathbf{c},h}(\mathbf{p}, w)$ is the set of bundles $\mathbf{x}$ of commodities maximizing utility for household h at prices $\mathbf{p}$ subject to the constraints $\mathbf{p} \,.\, \mathbf{x} \leq w$ and $\mathbf{0} \leq \mathbf{x} \leq \mathbf{c}$. It follows from Proposition 8.6 that this correspondence $\hat{\xi}'_{\mathbf{c},h}$ is upper hemicontinuous on $\hat{\Gamma}^n$, and moreover it maps each price-wealth pair in $\hat{\Gamma}^n$ to a non-empty compact convex subset of $\mathbb{R}^n$. Let $\psi_h \colon \Delta \to \hat{\Gamma}^n$ be the continuous mapping that sends $\mathbf{p} \in \Delta$ to $(\mathbf{p}, \mathbf{p} \,.\, \overline{\mathbf{x}}_h)$. Then $\hat{\xi}_{\mathbf{c},h} = \hat{\xi}'_{\mathbf{c},h} \circ \psi_h$: in other words,

$$\hat{\xi}_{\mathbf{c},h}(\mathbf{p}) = \hat{\xi}'_{\mathbf{c},h}(\mathbf{p}, \mathbf{p} \,.\, \overline{\mathbf{x}}_h) = \hat{\xi}'_{\mathbf{c},h}(\psi_h(\mathbf{p})).$$

It follows that the correspondence $\hat{\xi}_{\mathbf{c},h} \colon \Delta \to \mathbb{R}^n_+$ is the composition of a continuous mapping followed by an upper hemicontinuous correspondence. Any correspondence of this type must itself be an upper hemicontinuous correspondence. Moreover the images of normalized price vectors in Δ are subsets of $\mathbb{R}^n_+$ that have the required properties. ∎

Now, because the demand correspondences $\hat{\xi}_{\mathbf{c},h} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ for the individual households assign to each normalized price vector ı in the price simplex a non-empty compact subset of $\mathbb{R}^n_+$, these demand correspondences may be added together to obtain an correspondence $\hat{\xi}_{\mathbf{c}} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ that represents aggregate demand from the entire economy for each normalized price vector belonging to the price simplex Δ.

An immediate application of Proposition 8.7 yields the following result.

Corollary 8.9 *Suppose that, in a model of an exchange economy with n goods and m households, every household receives a strictly positive initial endowment of every commodity, so that the initial endowment vector $\overline{\mathbf{x}}_h$ of household h satisfies $\overline{\mathbf{x}}_h >> \mathbf{0}$ for $h = 1, 2, \ldots, m$. Suppose also that the preferences of household h are determined by a utility function u_h that is continuous, strictly increasing and quasiconcave. Let Δ denote the simplex whose elements are the normalized price vectors, and, for each $\mathbf{c} \in \mathbb{R}^n$ satisfying $\mathbf{c} >> \mathbf{0}$, let the demand correspondence $\hat{\xi}_{\mathbf{c},h} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ be defined as specified in the statement of Proposition 8.8, let $\mathbf{s} = \sum_{h=1}^{m} \overline{\mathbf{x}}_h$, and let $\hat{\xi}_{\mathbf{c}} = \sum_{h=1}^{m} \hat{\xi}_{\mathbf{c},h}$. Then the the aggregate demand correspondence $\hat{\xi}_{\mathbf{c}} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ is upper hemicontinuous on Δ, and maps each element of Δ to a non-empty compact convex subset of $\mathbb{R}^n_+$. Moreover $\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p} \,.\, \mathbf{s}$ for all $\mathbf{p} \in \Delta$ and $\mathbf{x} \in \hat{\xi}_{\mathbf{c}}(\mathbf{p})$.*

8.8 Walrasian Equilibria in Exchange Economies

Theorem 8.10 *Let n be a positive integer, let*

$$\Delta = \left\{ \mathbf{p} \in \mathbb{R}^n : \mathbf{p} \geq \mathbf{0} \text{ and } \sum_{i=1}^{n} (\mathbf{p})_i = 1 \right\},$$

let K be a compact subset of $\mathbb{R}^n$, and let $\zeta \colon \Delta \rightrightarrows K$ be an upper hemicontinuous correspondence mapping points of the simplex Δ to non-empty closed convex subsets of K. Suppose that $\mathbf{p} \,.\, \mathbf{z} \leq 0$ for all $\mathbf{p} \in \Delta$ and $\mathbf{z} \in \zeta(\mathbf{p})$. Then there exist $\mathbf{p}^* \in \Delta$ and $\mathbf{z}^* \in \zeta(\mathbf{p}^*)$ for which $\mathbf{z}^* \leq \mathbf{0}$.

Proof The set K is clearly non-empty. We may assume, without loss of generality, that the set K is both compact and convex, because if K were not convex, then it could be replaced by a compact convex set containing it.

Let $\gamma \colon \mathbb{R}^n \to \mathbb{R}$ be the function defined so that, for each $\mathbf{x} \in \mathbb{R}^n$, $\gamma(\mathbf{x})$ is the maximum of the components of $\mathbf{x}$, and let $\mu \colon \mathbb{R}^n \rightrightarrows \Delta$ be the correspondence defined such that

$$\mu(\mathbf{x}) = \{\mathbf{p} \in \Delta : \mathbf{p} \,.\, \mathbf{z} = \gamma(\mathbf{z})\}.$$

It was shown in Proposition 8.3 that the correspondence $\mu \colon \mathbb{R}^n \rightrightarrows \Delta$ is upper hemicontinuous, and $\mu(\mathbf{x})$ is a non-empty compact convex subset of Δ for all $\mathbf{x} \in \mathbb{R}^n$. Moreover $\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p}' \,.\, \mathbf{x} = \gamma(\mathbf{x})$ for all $\mathbf{p} \in \Delta$ and $\mathbf{p}' \in \mu(\mathbf{x})$. (The upper hemicontinuity of μ also follows directly on applying Berge's Maximum Theorem, which is Theorem 2.23 above.)

Let $\Phi \colon \Delta \times K \rightrightarrows \Delta \times K$ be the correspondence defined such that

$$\Phi(\mathbf{p}, \mathbf{z}) = (\mu(\mathbf{z}), \zeta(\mathbf{p}))$$

for all $\mathbf{p} \in \Delta$ and $\mathbf{z} \in K$. The correspondences μ and ζ are upper hemicontinuous and closed-valued, and every upper hemicontinuous closed-valued correspondence has a closed graph (Proposition 2.11). It follows that the correspondence Φ has closed graph. Moreover $\Phi(\mathbf{p}, \mathbf{z})$ is a non-empty closed convex subset of the compact convex set $\Delta \times K$ for all $\mathbf{p} \in \Delta$ and $\mathbf{z} \in K$. It follows from the Kakutani Fixed Point Theorem (Theorem 5.4) that there exists $(\mathbf{p}^*, \mathbf{z}^*) \in \Delta \times K$ for which $(\mathbf{p}^*, \mathbf{z}^*) \in \Phi(\mathbf{p}^*, \mathbf{z}^*)$. Then $\mathbf{p}^* \in \mu(\mathbf{z}^*)$ and $\mathbf{z}^* \in \zeta(\mathbf{p}^*)$.

Now the conditions of the theorem require that $\mathbf{p}^* . \mathbf{z} \leq 0$ for all $\mathbf{z} \in \zeta(\mathbf{p}^*)$. Combining this inequality with the definition of the correspondence μ, and noting that $\mathbf{p}^* \in \mu(\mathbf{z}^*)$ and $\mathbf{z}^* \in \zeta(\mathbf{p}^*)$, we find that

$$\mathbf{p} \,.\, \mathbf{z}^* \leq \mathbf{p}^* \,.\, \mathbf{z}^* \leq 0$$

for all $\mathbf{p} \in \Delta$. Applying this result when $\mathbf{p}$ is the vertex of Δ whose ith component is equal to 1 and whose other components are zero, we find that $(\mathbf{z}^*)_i \leq 0$ for $i = 1, 2, \ldots, n$, and thus $\mathbf{z}^* \leq \mathbf{0}$, as required. ▮

Remark For Theorem 8.10, and its proof, see Gérard Debreu, *Theory of Value* (Cowles Foundation Monograph 17, 1959), Section 5.6. In his notes on Chapter 5 of that monograph, Debreu notes that the result was obtained and published independently by D. Gale (published 1955) and H. Nikaido (published 1956). Debreu also thanks A. Borel, P. Samuel and A. Weil for conversations that he had with them on an early formulation of the result.

Theorem 8.11 *Suppose that, in a model of an exchange economy with n goods and m households, every household receives a strictly positive initial endowment of every commodity, so that the initial endowment vector $\overline{\mathbf{x}}_h$ of household h satisfies $\overline{\mathbf{x}}_h >> \mathbf{0}$ for $h = 1, 2, \ldots, m$. Suppose also that the preferences of household h are determined by a utility function u_h that is continuous, strictly increasing and quasiconcave. Then there exists a normalized price vector $\mathbf{p}^*$ satisfying $\mathbf{p}^* >> \mathbf{0}$ and, for each household h, a corresponding bundle $\mathbf{x}_h^*$ of commodities that maximizes utility for that household subject to the affordability constraint $\mathbf{p} \,.\, \mathbf{x}_h^* \leq \mathbf{p} \,.\, \overline{\mathbf{x}}_h$, so that the total supply is redistributed amongst the households, and thus*

$$\sum_{h=1}^{m} \mathbf{x}_h^* = \sum_{h=1}^{m} \overline{\mathbf{x}}_h.$$

Proof Let $\mathbf{s} = \sum_{h=1}^{m} \mathbf{x}_h$, and let $\mathbf{c} \in \mathbb{R}^n$ be chosen so that $\mathbf{c} >> \mathbf{s}$. Let

$$\Delta = \left\{ \mathbf{p} \in \mathbb{R}^n : \mathbf{p} \geq \mathbf{0} \text{ and } \sum_{i=1}^{n} (\mathbf{p})_i = 1 \right\},$$

and, for each household, let $\hat{\xi}_{\mathbf{c},h}\colon \Delta \rightrightarrows \mathbb{R}^n_+$ be the demand correspondence that sends each normalized price vector $\mathbf{p}$ in Δ to the set $\hat{\xi}_{\mathbf{c},h}(\mathbf{p})$ of bundles of commodities that maximize utility for household h subject to the affordability constraint $\mathbf{p}^* \,.\, \mathbf{x}_h \leq \mathbf{p}^* \,.\, \overline{\mathbf{x}}_h$, and the additional constraint $\mathbf{0} \leq \mathbf{x} \leq \mathbf{c}$. Let the correspondence $\hat{\xi}_{\mathbf{c}}\colon \Delta \rightrightarrows \mathbb{R}^n_+$ be defined so that $\hat{\xi}_{\mathbf{c}} = \sum_{h=1}^{m} \hat{\xi}_{\mathbf{c},h}$. Then the correspondence $\hat{\xi}_{\mathbf{c}}$ is upper hemicontinuous and maps each normalized price vector in Δ to a non-empty compact convex subset of $\mathbb{R}^n_+$ whose elements $\mathbf{x}$ satisfy $\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p} \,.\, \mathbf{s}$ (see Corollary 8.9).

Let the correspondence $\zeta_{\mathbf{c}}\colon \Delta \to \mathbb{R}^n$ be defined so that

$$\zeta_{\mathbf{c}} = \{\mathbf{x} - \mathbf{s} : \mathbf{x} \in \hat{\xi}_{\mathbf{c}}(\mathbf{p})\}$$

for all $\mathbf{p} \in \Delta$. Then $\mathbf{p} \,.\, \mathbf{z} \leq 0$ for all $\mathbf{p} \in \Delta$ and $\mathbf{z} \in \zeta(\mathbf{p})$. Moreover $\zeta_{\mathbf{c}}$ maps Δ into the compact set

$$\{\mathbf{z} \in \mathbb{R}^n : -\mathbf{s} \leq \mathbf{z} \leq \mathbf{c} - \mathbf{s}\}.$$

It then follows from Theorem 8.10 that there exist $\mathbf{p}^* \in \Delta$ and $\mathbf{z}^* \in \zeta_{\mathbf{c}}(\mathbf{p}^*)$ for which $\mathbf{z}^* \leq \mathbf{0}$.

Now $\mathbf{z}^* + \mathbf{s} \in \hat{\xi}_{\mathbf{c}}(\mathbf{p}^*)$. It follows from the definition of $\hat{\xi}_{\mathbf{c}}(\mathbf{p}^*)$. that there exist $\mathbf{x}^*_h \in \hat{\xi}_{\mathbf{c},h}(\mathbf{p}^*)$ for $h = 1, 2, \ldots, n$ for which $\sum_{h=1}^{m} \mathbf{x}^*_h = \mathbf{z}^* + \mathbf{s}$. Then $\sum_{h=1}^{m} \mathbf{x}^*_h \leq \mathbf{s}$, because $\mathbf{z}^* \leq \mathbf{0}$. Now $\mathbf{x}^*_h \geq \mathbf{0}$ for $h = 1, 2, \ldots, m$. It follows that $\mathbf{0} \leq \mathbf{x}^*_h \leq \mathbf{s}$ and therefore $\mathbf{x}^*_h << \mathbf{c}$ for $h = 1, 2, \ldots, m$.

Now $\mathbf{x}^*_h$ maximizes the utility function u_h on the set $B_{\mathbf{c},h}(\mathbf{p}^*)$, where

$$B_{\mathbf{c},h}(\mathbf{p}^*) = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{0} \leq \mathbf{x} \leq \mathbf{c} \text{ and } \mathbf{p}^* \,.\, \mathbf{x} \leq \mathbf{p}^* \,.\, \overline{\mathbf{x}}_h\}.$$

Let

$$B_h(\mathbf{p}^*) = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{p}^* \,.\, \mathbf{x} \leq \mathbf{p}^* \,.\, \overline{\mathbf{x}}_h\}.$$

and let

$$N = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} << \mathbf{c}\}.$$

Then the set N is open in $\mathbb{R}^n$, $\mathbf{x}^*_h \in N$ and the maximum value of the utility function u_h for household h on $B_h(\mathbf{p}^*) \cap N$ is achieved at $\mathbf{x}^*_h$. It follows directly from Proposition 8.4 that

$$\mathbf{p}^* \,.\, \mathbf{x}^*_h = \mathbf{p}^* \,.\, \overline{\mathbf{x}}_h,$$

and moreover the maximum value of the utility function u_h for household h on $B_h(\mathbf{p}^*)$ is achieved at $\mathbf{x}^*_h$.

Next we note that were it the case that $(\mathbf{p}^*)_i = 0$ for some index i between 1 and n then the amount of the ith commodity in the bundle $\mathbf{x}_h^*$ could be increased to obtain a bundle $\mathbf{x}$ for which $\mathbf{x} \neq \mathbf{x}_h^*$, $\mathbf{x} >> \mathbf{x}_h$ and $\mathbf{p}^* . \mathbf{x} = \mathbf{p}^* . \mathbf{x}_h^*$. But then $u_h(\mathbf{x}) > u_h(\mathbf{x}_h^*)$, because the utility function u_h is strictly increasing, and thus $\mathbf{x}_h^*$ would not maximize utility for for household h subject to the affordability constraint. We conclude therefore that $\mathbf{p}^* >> 0$.

Finally we note that

$$\mathbf{s} - \sum_{h=1}^{m} \mathbf{x}_h^* \geq \mathbf{0}$$

and

$$\mathbf{p}^* . \left(\mathbf{s} - \sum_{h=1}^{m} \mathbf{x}_h^* \right) = \sum_{h=1}^{m} \mathbf{p}^* . (\overline{\mathbf{x}}_h - \mathbf{x}_h^*) = 0.$$

It follows that

$$\mathbf{s} = \sum_{h=1}^{m} \mathbf{x}_h^*.$$

This completes the proof. ∎

8.9 Walras's Law

In the exchange economy model under discussion, let $\mathbf{p}$ be a price vector satisfying $\mathbf{p} >> 0$, and let $\xi_h(\mathbf{p})$ be the set of bundles of commodities maximizing utility for household h, subject only to the budget constraint requiring that $\mathbf{p} . \mathbf{x} \leq \mathbf{p} . \overline{\mathbf{x}}_h$ for all bundles $\mathbf{x}$ available to household h. Then $\mathbf{p} . \mathbf{x} = \mathbf{p} . \overline{\mathbf{x}}_h$ for all $\mathbf{x} \in \xi_h(\mathbf{p})$. Summing over all households, we find that $\mathbf{p} . \mathbf{x} = \mathbf{p} . \mathbf{s}$, for all $\mathbf{x} \in \xi(\mathbf{p})$, where $\mathbf{s}$ denotes the aggregate supply, defined so that $\mathbf{s} = \sum_{h=1}^{m} \overline{\mathbf{x}}_h$, and $\xi(\mathbf{p})$ denotes the value of the aggregate demand correspondence at prices p, defined so that $\xi = \sum_{h=1}^{m} \xi_h$. It follows that $\mathbf{p} . \mathbf{z} = 0$ for all $\mathbf{z} \in \zeta(\mathbf{p})$, where ζ denotes the *excess demand correspondence*, defined such that

$$\zeta(\mathbf{p}) = \{\mathbf{x} - \mathbf{s} : \mathbf{x} \in \xi(\mathbf{p})\}$$

for all $\mathbf{p} \in \Delta$ satisfying $\mathbf{p} >> \mathbf{0}$. This property of the excess demand correspondence is often referred to as *Walras's Law*.

8.10 Walrasian Equilibria with Strictly Quasiconcave Utility

We consider an exchange economy with n commodities and m households, retaining the notation of the previous discussion. We now consider the situation in which the utility function of each household is strictly quasiconcave.

Definition A real-valued function $u\colon X \to \mathbb{R}$ defined on a convex subset X of $\mathbb{R}^n$ is said to be *strictly quasiconcave* on X if

$$u((1-t)\mathbf{x} + t\mathbf{x}') > \min\Bigl(u(\mathbf{x}), u(\mathbf{x}')\Bigr)$$

for all distinct points $\mathbf{x}$ and $\mathbf{x}'$ of X and for all real numbers t satisfying $0 < t < 1$.

Suppose that, in the exchange economy, the utility function u_h of household h is continuous, strictly increasing and strictly quasiconcave for $h = 1, 2, \ldots, m$. The utility function of household h cannot then be maximized at two distinct points of any non-empty compact convex set. Let $\mathbf{c}$ be an n-dimensional vector satisfying $\mathbf{c} >> 0$. Then, given any normalized price vector $\mathbf{p}$, and given an initial endowment $\overline{x}_h$ for the ith household, there is a unique bundle of commodities $\hat{\mathbf{x}}_{\mathbf{c},h}(\mathbf{p})$ satisfying the budget constraint $\mathbf{p} \,.\, \hat{\mathbf{x}}_{\mathbf{c},h}(\mathbf{p}) \leq \mathbf{p} \,.\, \overline{\mathbf{x}}_h$ and the total availability constraint $\hat{\mathbf{x}}_{\mathbf{c},h}(\mathbf{p}) \leq \mathbf{c}$ which maximizes the utility function for household h for all bundles of commodities that satisfy the budget constraint and the total availability constraint. Moreover if $\hat{\mathbf{x}}_{\mathbf{c},h}(\mathbf{p}) << \mathbf{c}$ then $\mathbf{p} \,.\, \hat{\mathbf{x}}_{\mathbf{c},h}(\mathbf{p}) = \mathbf{p} \,.\, \overline{\mathbf{x}}_h$.

The preferences of household h, given normalized prices, given its initial endowment, and given the upper bounds on the availability of each commodity specified by the components of the vector $\mathbf{c}$, therefore determine a *demand function* $\hat{\mathbf{x}}_{\mathbf{c},h}\colon \Delta \to \mathbb{R}^+_-$ on the price simplex Δ, where

$$\Delta \;=\; \{(p_1, p_2, \ldots, p_n) \in \mathbb{R}^n : p_i \geq 0 \text{ for } i = 1, 2, \ldots, n, \text{ and } \sum_{i=1}^{n} p_i = 1\}.$$

The results obtained in more generality for demand correspondences, using Berge's Maximum Theorem, ensure that this demand function $\hat{\mathbf{x}}_{\mathbf{c},h}$ is continuous on Δ.

Summing the demand functions for the households, and subtracting the initial endowments, we obtain an *excess demand function* $\hat{\mathbf{z}}_{\mathbf{c}}\colon \Delta \to \mathbb{R}^n$ on the price simplex Δ whose value at normalized prices $\mathbf{p}$ specifies the excess demand for the commodities traded, when each household seeks to purchase commodities to maximize its utility, subject to the budget constraint determined by the prices and its initial endowment, and subject to the availability constraint that no household can purchase an amount of the ith commodity exceeding in amount the ith component of the vector $\mathbf{c}$. This excess demand function on the price simplex Δ is continuous, and satisfies $\mathbf{p} \,.\, \hat{z}_{\mathbf{c}}(\mathbf{p}) \leq 0$ for all $\mathbf{p} \in \Delta$.

The existence of Walrasian equilibria at which supply at least matches demand can then be established on the basis of the following proposition, whose proof makes use of the Brouwer Fixed Point Theorem.

Proposition 8.12 *Let*

$$\Delta = \{(p_1, p_2, \ldots, p_n) \in \mathbb{R}^n : p_i \geq 0 \textit{ for } i = 1, 2, \ldots, n, \textit{ and } \sum_{i=1}^{n} p_i = 1\},$$

let $\mathbf{z}\colon \Delta \to \mathbb{R}^n$ *be a continuous function mapping* Δ *into* $\mathbb{R}^n$*, and let*

$$\mathbf{z}(\mathbf{p}) = (z_1(\mathbf{p}), z_2(\mathbf{p}), \ldots, z_n(\mathbf{p}))$$

for all $\mathbf{p} \in \Delta$*. Suppose that* $\mathbf{p} \,.\, \mathbf{z}(\mathbf{p}) \leq 0$ *for all* $\mathbf{p} \in \Delta$*. Then there exists* $\mathbf{p}^* \in \Delta$ *such that* $z_i(\mathbf{p}^*) \leq 0$ *for* $i = 1, 2, \ldots, n$*.*

Proof Let $\mathbf{v}\colon \Delta \to \mathbb{R}^n$ be the function with ith component v_i given by

$$v_i(\mathbf{p}) = \begin{cases} p_i + z_i(\mathbf{p}) & \text{if } z_i(\mathbf{p}) > 0; \\ p_i & \text{if } z_i(\mathbf{p}) \leq 0. \end{cases}$$

Note that $\mathbf{v}(\mathbf{p}) \neq \mathbf{0}$ and the components of $\mathbf{v}(\mathbf{p})$ are non-negative for all $\mathbf{p} \in \Delta$. It follows that there is a well-defined map $\varphi\colon \Delta \to \Delta$ given by

$$\varphi(\mathbf{p}) = \frac{1}{\sum\limits_{i=1}^{n} v_i(\mathbf{p})} \mathbf{v}(\mathbf{p}),$$

The Brouwer Fixed Point Theorem (Theorem 5.3) ensures that there exists $\mathbf{p}^* \in \Delta$ satisfying $\varphi(\mathbf{p}^*) = \mathbf{p}^*$. Then $\mathbf{v}(\mathbf{p}^*) = \lambda \mathbf{p}^*$ for some $\lambda \geq 1$. We claim that $\lambda = 1$.

Suppose that it were the case that $\lambda > 1$. Then $v_i(\mathbf{p}^*) > p_i^*$, and thus $z_i(\mathbf{p}^*) > 0$ whenever $p_i^* > 0$. But $p_i^* \geq 0$ for all i, and $p_i^* > 0$ for at least one value of i, since $\mathbf{p}^* \in \Delta$. It would follow that $\mathbf{p}^* \,.\, \mathbf{z}(\mathbf{p}^*) > 0$, contradicting the requirement that $\mathbf{p} \,.\, \mathbf{z}(\mathbf{p}) \leq 0$ for all $\mathbf{p} \in \Delta$. We conclude that $\lambda = 1$, and thus $v_i = p_i^*$ and $z_i(\mathbf{p}^*) \leq 0$ for all i, as required. ∎

8.11 Historical Note

The proof of the existence of Walrasian equilibria in exchange economies can be generalized to *Arrow-Debreu* models where economic activity is carried out by both households and firms. The problem of existence of equilibria was studied by L. Walras in the 1870s, though a rigorous proof of the existence of equilibria was not found till the 1930s, when A. Wald proved existence for a limited range of economic models. Proofs of existence using topological fixed point theorems such as the Brouwer Fixed Point Theorem or the Kakutani Fixed Point Theorem were first published in 1954 by K. J. Arrow and G. Debreu and by L. McKenzie. Subsequent research has centred on problems of uniqueness and stability, and the existence theorems have been generalized to economies with an infinite number of commodities and economic agents (households and firms). An alternative approach to the existence theorems using techniques of differential topology was pioneered by G. Debreu and by S. Smale.

More detailed accounts of the theory of 'general equilibrium' can be found in, for example, *The theory of value*, by G. Debreu, *General competitive analysis*, by K. J. Arrow and F. H. Hahn, or *Economics for mathematicians* by J. W. S. Cassels.