Module MA3486: Fixed Point Theorems and Economic Equilibria
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Part II (Sections 3 to 5)

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Contents

3 Simplices and Convexity

3.1 Affine Independence

Definition Points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ in some Euclidean space $\mathbb{R}^k$ are said to be *affinely independent* (or *geometrically independent*) if the only solution of the linear system

$$\left\{ \begin{array}{rcl} \sum_{j=0}^{q} s_j \mathbf{v}_j & = & \mathbf{0}, \\ \sum_{j=0}^{q} s_j & = & 0 \end{array} \right.$$

is the trivial solution $s_0 = s_1 = \cdots = s_q = 0$.

Lemma 3.1 *Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be points of Euclidean space $\mathbb{R}^k$ of dimension k. Then the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent if and only if the displacement vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent.*

Proof Suppose that the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent. Let $s_1, s_2, \ldots, s_q$ be real numbers which satisfy the equation

$$\sum_{j=1}^{q} s_j (\mathbf{v}_j - \mathbf{v}_0) = \mathbf{0}.$$

Then $\sum_{j=0}^{q} s_j \mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^{q} s_j = 0$, where $s_0 = -\sum_{j=1}^{q} s_j$, and therefore

$$s_0 = s_1 = \cdots = s_q = 0.$$

It follows that the displacement vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent.

Conversely, suppose that these displacement vectors are linearly independent. Let $s_0, s_1, s_2, \ldots, s_q$ be real numbers which satisfy the equations $\sum_{j=0}^{q} s_j \mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^{q} s_j = 0$. Then $s_0 = -\sum_{j=1}^{q} s_j$, and therefore

$$\mathbf{0} = \sum_{j=0}^{q} s_j \mathbf{v}_j = s_0 \mathbf{v}_0 + \sum_{j=1}^{q} s_j \mathbf{v}_j = \sum_{j=1}^{q} s_j (\mathbf{v}_j - \mathbf{v}_0).$$

It follows from the linear independence of the displacement vectors $\mathbf{v}_j - \mathbf{v}_0$ for $j = 1, 2, \ldots, q$ that

$$s_1 = s_2 = \cdots = s_q = 0.$$

But then $s_0 = 0$ also, because $s_0 = -\sum_{j=1}^{q} s_j$. It follows that the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent, as required. ∎

It follows from Lemma 3.1 that any set of affinely independent points in $\mathbb{R}^k$ has at most $k+1$ elements. Moreover if a set consists of affinely independent points in $\mathbb{R}^k$, then so does every subset of that set.

3.2 Simplices in Euclidean Spaces

Definition A *q-simplex* in $\mathbb{R}^k$ is defined to be a set of the form

$$\left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j : 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^{q} t_j = 1 \right\},$$

where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent points of $\mathbb{R}^k$. These points are referred to as the *vertices* of the simplex. The non-negative integer q is referred to as the *dimension* of the simplex. (Thus a simplex of dimension q has $q+1$ vertices.)

Example A 0-simplex in a Euclidean space $\mathbb{R}^k$ is a single point of that space.

Example A 1-simplex in a Euclidean space $\mathbb{R}^k$ of dimension at least one is a line segment in that space. Indeed let λ be a 1-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}$ and $\mathbf{w}$. Then

$$\begin{aligned} \lambda &= \{s\,\mathbf{v} + t\,\mathbf{w} : 0 \leq s \leq 1,\ 0 \leq t \leq 1 \text{ and } s + t = 1\} \\ &= \{(1-t)\mathbf{v} + t\,\mathbf{w} : 0 \leq t \leq 1\}, \end{aligned}$$

and thus λ is a line segment in $\mathbb{R}^k$ with endpoints $\mathbf{v}$ and $\mathbf{w}$.

Example A 2-simplex in a Euclidean space $\mathbb{R}^k$ of dimension at least two is a triangle in that space. Indeed let τ be a 2-simplex in $\mathbb{R}^k$ with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Then

$$\tau = \{r\,\mathbf{u} + s\,\mathbf{v} + t\,\mathbf{w} : 0 \leq r, s, t \leq 1 \text{ and } r + s + t = 1\}.$$

Let $\mathbf{x} \in \tau$. Then there exist $r, s, t \in [0, 1]$ such that $\mathbf{x} = r\,\mathbf{u} + s\,\mathbf{v} + t\,\mathbf{w}$ and $r + s + t = 1$. If $r = 1$ then $\mathbf{x} = \mathbf{u}$. Suppose that $r < 1$. Then

$$\mathbf{x} = r\,\mathbf{u} + (1-r)\Big((1-p)\mathbf{v} + p\mathbf{w}\Big)$$

where $p = \dfrac{t}{1-r}$. Moreover $0 \leq r < 1$ and $0 \leq p \leq 1$. Also the above formula determines a point of the 2-simplex τ for each pair of real numbers r and p satisfying $0 \leq r \leq 1$ and $0 \leq p \leq 1$. Thus

$$\tau = \left\{ r\,\mathbf{u} + (1-r)\Big((1-p)\mathbf{v} + p\mathbf{w}\Big) : 0 \leq p, r \leq 1. \right\}.$$

Now the point $(1-p)\mathbf{v} + p\mathbf{w}$ traverses the line segment $\mathbf{v}\,\mathbf{w}$ from $\mathbf{v}$ to $\mathbf{w}$ as p increases from 0 to 1. It follows that τ is the set of points that lie on line segments with one endpoint at $\mathbf{u}$ and the other at some point of the line segment $\mathbf{v}\,\mathbf{w}$. This set of points is thus a triangle with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Example A 3-simplex in a Euclidean space $\mathbb{R}^k$ of dimension at least three is a tetrahedron on that space. Indeed let $\mathbf{x}$ be a point of a 3-simplex σ in $\mathbb{R}^3$ with vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$. Then there exist non-negative real numbers s, t, u and v such that

$$\mathbf{x} = s\,\mathbf{a} + t\,\mathbf{b} + u\,\mathbf{c} + v\,\mathbf{d},$$

and $s+t+u+v = 1$. These real numbers s, t, u and v all have values between 0 and 1, and moreover $0 \leq t \leq 1-s$, $0 \leq u \leq 1-s$ and $0 \leq v \leq 1-s$. Suppose that $\mathbf{x} \neq \mathbf{a}$. Then $0 \leq s < 1$ and $\mathbf{x} = s\,\mathbf{a} + (1-s)\mathbf{y}$, where

$$\mathbf{y} = \frac{t}{1-s}\mathbf{b} + \frac{u}{1-s}\mathbf{c} + \frac{v}{1-s}\mathbf{d}.$$

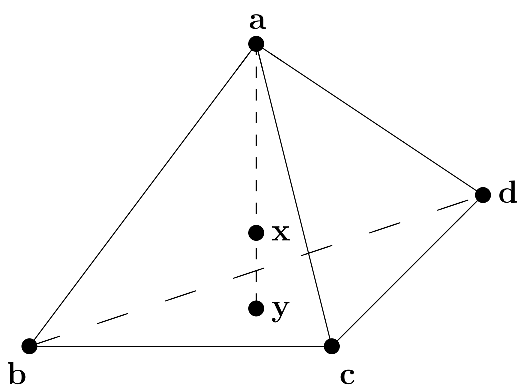
Moreover $\mathbf{y}$ is a point of the triangle $\mathbf{b}\,\mathbf{c}\,\mathbf{d}$, because

$$0 \leq \frac{t}{1-s} \leq 1, \quad 0 \leq \frac{u}{1-s} \leq 1, \quad 0 \leq \frac{v}{1-s} \leq 1$$

and

$$\frac{t}{1-s} + \frac{u}{1-s} + \frac{v}{1-s} = 1.$$

It follows that the point $\mathbf{x}$ lies on a line segment with one endpoint at the vertex $\mathbf{a}$ of the 3-simplex and the other at some point $\mathbf{y}$ of the triangle $\mathbf{b}\,\mathbf{c}\,\mathbf{d}$. Thus the 3-simplex σ has the form of a tetrahedron (i.e., it has the form of a pyramid on a triangular base $\mathbf{b}\,\mathbf{c}\,\mathbf{d}$ with apex $\mathbf{a}$).



A simplex of dimension q in $\mathbb{R}^k$ determines a subset of $\mathbb{R}^k$ that is a translate of a q-dimensional vector subspace of $\mathbb{R}^k$. Indeed let the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of a q-dimensional simplex σ in $\mathbb{R}^k$. Then these points are affinely independent. It follows from Lemma 3.1 that the displacement vectors

$$\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$$

are linearly independent. These vectors therefore span a q-dimensional vector subspace V of $\mathbb{R}^k$. Now, given any point $\mathbf{x}$ of σ, there exist real numbers $t_0, t_1, \ldots, t_q$ such that $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$, $\sum_{j=0}^{q} t_j = 1$ and $\mathbf{x} = \sum_{j=0}^{q} t_j \mathbf{v}_j$. Then

$$\mathbf{x} = \left(\sum_{j=0}^{q} t_j \right) \mathbf{v}_0 + \sum_{j=1}^{q} t_j (\mathbf{v}_j - \mathbf{v}_0) = \mathbf{v}_0 + \sum_{j=1}^{q} t_j (\mathbf{v}_j - \mathbf{v}_0).$$

It follows that

$$\sigma = \left\{ \mathbf{v}_0 + \sum_{j=1}^{q} t_j (\mathbf{v}_j - \mathbf{v}_0) : 0 \leq t_j \leq 1 \text{ for } j = 1, 2, \ldots, q \right. \\ \left. \text{and } \sum_{j=1}^{q} t_j \leq 1 \right\},$$

and therefore $\sigma \subset \mathbf{v_0} + V$. Moreover the q-dimensional vector subspace V of $\mathbb{R}^k$ is the unique q-dimensional vector subspace of $\mathbb{R}^k$ that contains the displacement vectors between each pair of points belonging to the simplex σ.

3.3 Faces of Simplices

Definition Let σ and τ be simplices in $\mathbb{R}^k$. We say that τ is a *face* of σ if the set of vertices of τ is a subset of the set of vertices of σ. A face of σ is said to be a *proper face* if it is not equal to σ itself. An r-dimensional face of σ is referred to as an *r-face* of σ. A 1-dimensional face of σ is referred to as an *edge* of σ.

Note that any simplex is a face of itself. Also the vertices and edges of any simplex are by definition faces of the simplex.

3.4 Barycentric Coordinates on a Simplex

Let σ be a q-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. If $\mathbf{x}$ is a point of σ then there exist real numbers $t_0, t_1, \ldots, t_q$ such that

$$\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{x}, \quad \sum_{j=0}^{q} t_j = 1 \text{ and } 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q.$$

Moreover $t_0, t_1, \ldots, t_q$ are uniquely determined: if $\sum_{j=0}^{q} s_j \mathbf{v}_j = \sum_{j=0}^{q} t_j \mathbf{v}_j$ and $\sum_{j=0}^{q} s_j = \sum_{j=0}^{q} t_j = 1$, then $\sum_{j=0}^{q} (t_j - s_j)\mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^{q} (t_j - s_j) = 0$, and therefore $t_j - s_j = 0$ for $j = 0, 1, \ldots, q$, because the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent.

Definition Let σ be a q-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$, and let $\mathbf{x} \in \sigma$. The *barycentric coordinates* of the point $\mathbf{x}$ (with respect to the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$) are the unique real numbers $t_0, t_1, \ldots, t_q$ for which

$$\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{x} \quad \text{and} \quad \sum_{j=0}^{q} t_j = 1.$$

The barycentric coordinates $t_0, t_1, \ldots, t_q$ of a point of a q-simplex satisfy the inequalities $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$.

Example Consider the triangle τ in $\mathbb{R}^3$ with vertices at $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, where

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0) \quad \text{and} \quad \mathbf{k} = (0, 0, 1).$$

Then

$$\tau = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x, y, z \leq 1 \text{ and } x + y + z = 1\}.$$

The barycentric coordinates on this triangle τ then coincide with the Cartesian coordinates x, y and z, because

$$(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\mathbf{k}$$

for all $(x, y, z) \in \tau$.

Example Consider the triangle in $\mathbb{R}^2$ with vertices at $(0, 0)$, $(1, 0)$ and $(0, 1)$. This triangle is the set

$$\{(x, y) \in \mathbb{R}^2 : x \geq 0,\ \ y \geq 0 \text{ and } x + y \leq 1.\}.$$

The barycentric coordinates of a point (x, y) of this triangle are t_0, t_1 and t_2, where

$$t_0 = 1 - x - y, \quad t_1 = x \quad \text{and} \quad t_2 = y.$$

Example Consider the triangle in $\mathbb{R}^2$ with vertices at $(1, 2)$, $(3, 3)$ and $(4, 5)$. Let t_0, t_1 and t_2 be the barycentric coordinates of a point (x, y) of this triangle. Then t_0, t_1, t_2 are non-negative real numbers, and $t_0 + t_1 + t_2 = 1$. Moreover

$$(x, y) = (1 - t_1 - t_2)(1, 2) + t_1(3, 3) + t_2(4, 5),$$

and thus

$$x = 1 + 2t_1 + 3t_2 \quad \text{and} \quad y = 2 + t_1 + 3t_2.$$

It follows that

$$t_1 = x - y + 1 \quad \text{and} \quad t_2 = \tfrac{1}{3}(x - 1 - 2t_1) = \tfrac{2}{3}y - \tfrac{1}{3}x - 1,$$

and therefore

$$t_0 = 1 - t_1 - t_2 = \tfrac{1}{3}y - \tfrac{2}{3}x + 1.$$

In order to verify these formulae it suffices to note that $(t_0, t_1, t_2) = (1, 0, 0)$ when $(x, y) = (1, 2)$, $(t_0, t_1, t_2) = (0, 1, 0)$ when $(x, y) = (3, 3)$ and $(t_0, t_1, t_2) = (0, 0, 1)$ when $(x, y) = (4, 5)$.

3.5 The Interior of a Simplex

Definition The *interior* of a simplex σ is defined to be the set consisting of all points of σ that do not belong to any proper face of σ.

Lemma 3.2 *Let σ be a q-simplex in some Euclidean space with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. Let $\mathbf{x}$ be a point of σ, and let $t_0, t_1, \ldots, t_q$ be the barycentric coordinates of the point $\mathbf{x}$ with respect to $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$, so that $t_j \geq 0$ for $j = 0, 1, \ldots, q$, $\mathbf{x} = \sum_{j=0}^{q} t_j \mathbf{v}_j$, and $\sum_{j=0}^{q} t_j = 1$. Then the point $\mathbf{x}$ belongs to the interior of σ if and only if $t_j > 0$ for $j = 0, 1, \ldots, q$.*

Proof The point $\mathbf{x}$ belongs to the face of σ spanned by vertices

$$\mathbf{v}_{j_0}, \mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_r},$$

where $0 \leq j_0 < j_1 < \cdots < j_r \leq q$, if and only if $t_j = 0$ for all integers j between 0 and q that do not belong to the set $\{j_0, j_1, \ldots, j_r\}$. Thus the point $\mathbf{x}$ belongs to a proper face of the simplex σ if and only if at least one of the barycentric coordinates t_j of that point is equal to zero. The result follows. ∎

Example A 0-simplex consists of a single vertex $\mathbf{v}$. The interior of that 0-simplex is the vertex $\mathbf{v}$ itself.

Example A 1-simplex is a line segment. The interior of a line segment in a Euclidean space $\mathbb{R}^k$ with endpoints $\mathbf{v}$ and $\mathbf{w}$ is

$$\{(1-t)\,\mathbf{v} + t\,\mathbf{w} : 0 < t < 1\}.$$

Thus the interior of the line segment consists of all points of the line segement that are not endpoints of the line segment.

Example A 2-simplex is a triangle. The interior of a triangle with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is the set

$$\{r\,\mathbf{u} + s\,\mathbf{v} + t\,\mathbf{w} : 0 < r, s, t < 1 \text{ and } r + s + t = 1\}.$$

The interior of this triangle consists of all points of the triangle that do not lie on any edge of the triangle.

Remark Let σ be a q-dimensional simplex in some Euclidean space $\mathbb{R}^k$, where $k \geq q$. If $k > q$ then the interior of the simplex (defined according to the definition given above) will not coincide with the topological interior determined by the usual topology on $\mathbb{R}^k$. Consider for example a triangle embedded in three-dimensional Euclidean space $\mathbb{R}^3$. The interior of the triangle (defined according to the definition given above) consists of all points of the triangle that do not lie on any edge of the triangle. But of course no three-dimensional ball of positive radius centred on any point of that triangle is wholly contained within the triangle. It follows that the topological interior of the triangle is the empty set when that triangle is considered as a subset of three-dimensional space $\mathbb{R}^3$.

Lemma 3.3 *Any point of a simplex belongs to the interior of a unique face of that simplex.*

Proof let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of a simplex σ, and let $\mathbf{x} \in \sigma$. Then $\mathbf{x} = \sum_{j=0}^{q} t_j \mathbf{v}_j$, where $t_0, t_1, \ldots, t_q$ are the barycentric coordinates of the point $\mathbf{x}$. Moreover $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$. The unique face of σ containing $\mathbf{x}$ in its interior is then the face spanned by those vertices $\mathbf{v}_j$ for which $t_j > 0$. ∎

3.6 Convex Subsets of Euclidean Spaces

Definition A subset X of n-dimensional Euclidean space $\mathbb{R}^n$ is said to be *convex* if $(1-t)\mathbf{u} + t\mathbf{v} \in X$ for all points $\mathbf{u}$ and $\mathbf{v}$ of X and for all real numbers t satisfying $0 \leq t \leq 1$.

It follows from the above definition that a subset X of $\mathbb{R}^{>}$ is a convex subset of $\mathbb{R}^m$ if and only if, given any two points of X, the line segment joining those two points is wholly contained in X.

Lemma 3.4 *An simplex in a Euclidean space is a convex subset of that Euclidean space.*

Proof Let σ be a q-simplex in n-dimensional Euclidean space with vertices $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$, and let $\mathbf{u}$ and $\mathbf{v}$ be points of σ. Then there exist non-negative real numbers $y_0, y_1, \ldots, y_q$ and $z_0, z_1, \ldots, z_q$, where $\sum_{i=0}^{q} y_i = 1$ and $\sum_{i=0}^{q} z_i = 1$, such that

$$\mathbf{u} = \sum_{i=0}^{q} y_i \mathbf{w}_i, \quad \mathbf{v} = \sum_{i=0}^{q} z_i \mathbf{w}_i.$$

Then

$$(1-t)\mathbf{u} + t\mathbf{v} = \sum_{i=0}^{q} ((1-t)y_i + tz_i)\mathbf{w}_i.$$

Moreover $(1-t)y_i + tz_i \geq 0$ for $i = 0, 1, \ldots, q$ and for all real numbers t satisfying $0 \leq t \leq 1$. Also

$$\sum_{i=0}^{q} ((1-t)y_i + tz_i) = (1-t)\sum_{i=0}^{q} y_i + t\sum_{i=0}^{q} z_i = 1.$$

It follows that $(1-t)\mathbf{u} + t\mathbf{v} \in \sigma$. Thus σ is a convex subset of $\mathbb{R}^n$. ∎

Lemma 3.5 *Let X be a convex subset of n-dimensional Euclidean space $\mathbb{R}^n$, and let σ be a simplex contained in $\mathbb{R}^n$. Suppose that the vertices of σ belong to X. Then $\sigma \subset X$.*

Proof We prove the result by induction on the dimension q of the simplex σ. The result is clearly true when $q = 0$, because in that case the simplex σ consists of a single point which is the unique vertex of the simplex. Thus let σ be a q-dimensional simplex, and suppose that the result is true for all $(q-1)$-dimensional simplices whose vertices belong to the convex set X. Let $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$ be the vertices of σ. Let $\mathbf{x}$ be a point of σ. Then there exist non-negative real numbers $t_0, t_1, \ldots, t_q$ satisfying $\sum_{i=0}^{q} t_i = 1$ such that $\mathbf{x} = \sum_{i=0}^{q} t_i \mathbf{w}_i$. If $t_0 = 1$ then $\mathbf{x} = \mathbf{w}_0$, and therefore $\mathbf{x} \in X$.

It remains to consider the case when $t_0 < 1$. In that case let $s_i = t_i/(1-t_0)$ for $i = 1, 2, \ldots, q$, and let

$$\mathbf{v} = \sum_{i=1}^{q} s_i \mathbf{w}_i.$$

Now $s_i \geq 0$ for $i = 1, 2, \ldots, q$, and

$$\sum_{i=1}^{q} s_i = \frac{1}{1-t_0} \sum_{i=1}^{q} t_i = \frac{1}{1-t_0} \left(\sum_{i=0}^{q} t_i - t_0 \right) = 1,$$

It follows that $\mathbf{v}$ belongs to the proper face of σ that is spanned by the vertices $\mathbf{w}_1, \ldots, \mathbf{w}_q$. The induction hypothesis then ensures that $\mathbf{v} \in X$. But then

$$\mathbf{x} = t_0 \mathbf{w}_0 + (1 - t_0)\mathbf{v},$$

where $\mathbf{w}_0 \in X$ and $\mathbf{v} \in X$ and $0 \leq t_0 \leq 1$. It follows from the convexity of X that $\mathbf{x} \in X$, as required. ∎

Let X be a convex set in n-dimensional Euclidean space $\mathbb{R}^{\ltimes}$. A point $\mathbf{x}$ of X is said to belong to the *topological interior* of X if there exists some $\delta > 0$ such that $B(\mathbf{x}, \delta) \subset X$, where

$$B(\mathbf{x}, \delta) = \{\mathbf{x}' \in \mathbb{R}^n : |\mathbf{x}' - \mathbf{x}| < \delta\}.$$

Lemma 3.6 *Let X be a convex set in n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathbf{x} = (1-t)\mathbf{u} + t\mathbf{v}$ where $\mathbf{u}, \mathbf{v} \in X$ and $0 < t < 1$. Suppose that either $\mathbf{u}$ or $\mathbf{v}$ belongs to the topological interior of X. Then $\mathbf{x}$ belongs to the topological interior of X.*

Proof Suppose that $\mathbf{v}$ belongs to the topological interior of X. Then there exists $\delta > 0$ such that $B(\mathbf{v}, \delta) \subset X$, where

$$B(\mathbf{v}, \delta) = \{\mathbf{x}' \in \mathbb{R}^n : |\mathbf{x}' - \mathbf{v}| < \delta\}.$$

We claim that $B(\mathbf{x}, t\delta) \subset X$.

Let $\mathbf{x}' \in B(\mathbf{x}, t\delta)$, and let

$$\mathbf{z} = \frac{1}{t}(\mathbf{x}' - \mathbf{x}).$$

Then $\mathbf{v} + \mathbf{z} \in B(\mathbf{v}, \delta)$ and

$$\mathbf{x}' = (1 - t)\mathbf{u} + t(\mathbf{v} + \mathbf{z}),$$

and therefore $\mathbf{x}' \in X$. This proves the result when $\mathbf{v}$ belongs to the topological interior of X. The result when $\mathbf{u}$ belongs to the topological interior of X then follows on interchanging $\mathbf{u}$ and $\mathbf{v}$ and replacing t by $1 - t$. The result follows. ∎

Proposition 3.7 *Let X be a closed bounded convex subset of n-dimensional Euclidean space $\mathbb{R}^n$ whose topological interior contains the origin, let S^{n-1} be the unit sphere in $\mathbb{R}^n$, defined such that*

$$S^{n-1} = \{\mathbf{u} \in \mathbb{R}^n : |\mathbf{u}| = 1\},$$

and let $\lambda\colon S^{n-1} \to \mathbb{R}$ be the real-valued function on S^{n-1} defined such that

$$\lambda(\mathbf{u}) = \sup\{t \in \mathbf{R} : t\mathbf{u} \in X\}$$

for all $\mathbf{u} \in S^{n-1}$. Then the function $\lambda\colon S^{n-1} \to \mathbb{R}$ is continuous.

Proof Let $\mathbf{u}_0 \in S^{n-1}$, let $t_0 = \lambda(\mathbf{u}_0)$, and let some positive real number ε be given, where $0 < \varepsilon < t_0$. It follows from Lemma 3.6 that $(t_0 - \varepsilon)\mathbf{u}$ belongs to the topological interior of X. It then follows from the continuity of the function sending $\mathbf{u} \in S^{n-1}$ to $(t_0 - \varepsilon)\mathbf{u}$ that there exists some positive real number δ_1 such that $(t_0 - \varepsilon)\mathbf{u} \in X$ for all $\mathbf{u} \in S^{n-1}$ satisfying $|\mathbf{u} - \mathbf{u}_0| < \delta_1$. Therefore $\lambda(\mathbf{u}) \geq t_0 - \varepsilon$ whenever $|\mathbf{u} - \mathbf{u}_0| < \delta_1$. Next we note that $(t_0 + \varepsilon)\mathbf{u}_0 \notin X$. Now X is closed in $\mathbb{R}^n$, and therefore the complement $\mathbb{R}^n \setminus X$ of X in $\mathbb{R}^n$ is open. It follows that there exists an open ball of positive radius about the point $(t_0 + \varepsilon)\mathbf{u}_0$ that is wholly contained in the complement of X. It then follows from the continuity of the function sending $\mathbf{u} \in S^{n-1}$ to $(t_0 + \varepsilon)\mathbf{u}$ that there exists some positive real number δ_2 such that $(t_0 + \varepsilon)\mathbf{u} \notin X$ for all $\mathbf{u} \in S^{n-1}$ satisfying $|\mathbf{u} - \mathbf{u}_0| < \delta_2$. It then follows from the convexity of X that $t\mathbf{u} \notin X$ for all positive real numbers t satisfying $t \geq t_0 + \varepsilon$. Therefore $\lambda(\mathbf{u}) \leq t_0 + \varepsilon$ whenever $|\mathbf{u} - \mathbf{u}_0| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and

$$\lambda(\mathbf{u}_0) - \varepsilon \leq \lambda(\mathbf{u}) \leq \lambda(\mathbf{u}_0) + \varepsilon$$

for all $\mathbf{u} \in S^{n-1}$ satisfying $|\mathbf{u} - \mathbf{u}_0| < \delta$. The result follows. ∎

Proposition 3.8 *Let X be a closed bounded convex subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then there exists a continuous map $r\colon \mathbb{R}^n \to X$ such that $r(\mathbb{R}^n) = X$ and $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$.*

Proof We first prove the result in the special case in which the convex set X has non-empty topological interior. Without loss of generality, we may assume that the origin of $\mathbb{R}^n$ belongs to the topological interior of X. Let

$$S^{n-1} = \{\mathbf{u} \in \mathbb{R}^n : |\mathbf{u}| = 1\},$$

and let $\lambda\colon S^{n-1} \to \mathbb{R}$ be the real-valued function on S^{n-1} defined such that

$$\lambda(\mathbf{u}) = \sup\{t \in \mathbf{R} : t\mathbf{u} \in X\}$$

for all $\mathbf{u} \in S^{n-1}$. Then the function $\lambda\colon S^{n-1} \to \mathbb{R}$ is continuous (Proposition 3.7). We may therefore define a function $r\colon \mathbb{R}^n \to X$ such that

$$r(\mathbf{x}) = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \in X; \\ |\mathbf{x}|^{-1}\lambda(|\mathbf{x}|^{-1}\mathbf{x})\mathbf{x} & \text{if } \mathbf{x} \notin X. \end{cases}$$

Let $\mathbf{x} \in X$ and let $\mathbf{u} = |\mathbf{x}|^{-1}\mathbf{x}$. Then $\mathbf{x} = |\mathbf{x}|\,\mathbf{u}$, $|\mathbf{x}| \leq \lambda(\mathbf{u})$ and $\lambda(\mathbf{u})\mathbf{u} \in X$. It follows from Lemma 3.6 that if $|\mathbf{x}| < \lambda(\mathbf{u})$ then the point $\mathbf{x}$ belongs to the topological interior of $\mathbf{u}$. Thus if the point $\mathbf{x}$ of X belongs to the closure of the complement $\mathbb{R}^n \setminus X$ of X then it does not belong to the topological interior of X, and therefore $|\mathbf{x}| = \lambda(|\mathbf{x}|^{-1}\mathbf{x})$, and therefore

$$\mathbf{x} = |\mathbf{x}|^{-1}\lambda(|\mathbf{x}|^{-1}\mathbf{x})\mathbf{x}.$$

The function r defined above is therefore continuous on the closure of $\mathbb{R}^n \setminus X$. It is obviously continuous on X itself. It follows that $r\colon \mathbb{R}^n \to X$ is continuous. This proves the result in the case when the topological interior of the set X is non-empty.

We now extend the result to the case where the topological interior of X is empty. Now the number of points in an affinely independent list of points of $\mathbb{R}^n$ cannot exceed $n+1$. It follows that there exists an integer q not exceeding n such that the convex set X contains a $q+1$ affinely independent points but does not contain $q+1$ affinely independent points. Let $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$ be affinely independent points of X. Let V be the q-dimensional subspace of $\mathbb{R}^n$ spanned by the vectors

$$\mathbf{w}_1 - \mathbf{w}_0, \mathbf{w}_2 - \mathbf{w}_0, \ldots, \mathbf{w}_q - \mathbf{w}_0.$$

Now if there were to exist a point $\mathbf{x}$ of X for which $\mathbf{x} - \mathbf{w}_0 \notin V$ then the points $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q, \mathbf{x}$ would be affinely independent. The definition of q ensures that this is not the case. Thus if

$$X_V = \{\mathbf{x} - \mathbf{w}_0 : \mathbf{x} \in X\}.$$

then $X_V \subset V$. Moreover X_V is a closed convex subset of V. Now it follows from Lemma 3.5 that the convex set X_V contains the q-simplex with vertices

$$\mathbf{0},\ \mathbf{w}_1 - \mathbf{w}_0,\ \mathbf{w}_2 - \mathbf{w}_0, \ldots\ \mathbf{w}_q - \mathbf{w}_0.$$

This q-simplex has non-empty topological interior with respect to the vector space V. It follows that X_V has non-empty topological interior with respect to V. It therefore follows from the result already proved that there exists a continuous function $r_V\colon V \to X_V$ that satisfies $r_V(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X_V$. Basic linear algebra ensures the existence of a linear transformation $T\colon \mathbb{R}^n \to V$ satisfying $T(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in V$. Let

$$r(\mathbf{x}) = r_V(T(\mathbf{x} - \mathbf{w}_0)) + \mathbf{w}_0$$

for all $\mathbf{x} \in \mathbb{R}^n$. Then the function $r\colon \mathbb{R}^n \to X$ is continuous, and $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$, as required. ∎

3.7 Convex Sets and Supporting Hyperplanes

Lemma 3.9 *Let m be a positive integer, let F be a non-empty closed set in $\mathbb{R}^m$, and let $\mathbf{b}$ be a vector in $\mathbb{R}^m$. Then there exists an element $\mathbf{g}$ of F such that $|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$ for all $\mathbf{x} \in F$.*

Proof Let R be a positive real number chosen large enough to ensure that the set F_0 is non-empty, where

$$F_0 = F \cap \{\mathbf{x} \in \mathbb{R}^m : |\mathbf{x} - \mathbf{b}| \leq R\}.$$

Then F_0 is a closed bounded subset of $\mathbb{R}^m$. Let $f\colon F_0 \to \mathbb{R}$ be defined such that $f(\mathbf{x}) = |\mathbf{x} - \mathbf{b}|$ for all $\mathbf{x} \in F$. Then $f\colon F_0 \to \mathbb{R}$ is a continuous function on F_0. Now it is a standard result of real analysis that any continuous real-valued function on a closed bounded subset of a finite-dimensional Euclidean space attains a minimum value at some point of that set. It follows that there exists an element $\mathbf{g}$ of F_0 such that

$$|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$$

for all $\mathbf{x} \in F_0$. If $\mathbf{x} \in F \setminus F_0$ then

$$|\mathbf{x} - \mathbf{b}| \geq R \geq |\mathbf{g} - \mathbf{b}|.$$

It follows that

$$|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$$

for all $\mathbf{x} \in F$, as required. ∎

3.8 A Separating Hyperplane Theorem

Theorem 3.10 *Let m be a positive integer, let X be a closed convex set in $\mathbb{R}^m$, and let $\mathbf{b}$ be point of $\mathbb{R}^m$, where $\mathbf{b} \notin X$. Then there exists a linear functional $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ and a real number c such that $\varphi(\mathbf{x}) > c$ for all $\mathbf{x} \in X$ and $\varphi(\mathbf{b}) < c$.*

Proof It follows from Lemma 3.9 that there exists a point $\mathbf{g}$ of X such that $|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$ for all $\mathbf{x} \in X$. Let $\mathbf{x} \in X$. Then $(1-t)\mathbf{g} + t\mathbf{x} \in X$ for all real numbers t satisfying $0 \leq t \leq 1$, because the set X is convex, and therefore

$$|(1-t)\mathbf{g} + t\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$$

for all real numbers t satisfying $0 \leq t \leq 1$. Now

$$(1-t)\mathbf{g} + t\mathbf{x} - \mathbf{b} = \mathbf{g} - \mathbf{b} + t(\mathbf{x} - \mathbf{g}).$$

It follows by a straightforward calculation from the definition of the Euclidean norm that

$$\begin{aligned} |\mathbf{g} - \mathbf{b}|^2 &\leq |(1-t)\mathbf{g} + t\mathbf{x} - \mathbf{b}|^2 \\ &= |\mathbf{g} - \mathbf{b}|^2 + 2t(\mathbf{g} - \mathbf{b}) \,.\, (\mathbf{x} - \mathbf{g}) \\ &\quad + t^2|\mathbf{x} - \mathbf{g}|^2 \end{aligned}$$

for all real numbers t satisfying $0 \leq t \leq 1$. In particular, this inequality holds for all sufficiently small positive values of t, and therefore

$$(\mathbf{g} - \mathbf{b}) \,.\, (\mathbf{x} - \mathbf{g}) \geq 0$$

for all $\mathbf{x} \in X$.

Let

$$\varphi(\mathbf{x}) = (\mathbf{g} - \mathbf{b}) \,.\, \mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^m$. Then $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ is a linear functional on $\mathbb{R}^m$, and $\varphi(\mathbf{x}) \geq \varphi(\mathbf{g})$ for all $\mathbf{x} \in X$. Moreover

$$\varphi(\mathbf{g}) - \varphi(\mathbf{b}) = |\mathbf{g} - \mathbf{b}|^2 > 0,$$

and therefore $\varphi(\mathbf{g}) > \varphi(\mathbf{b})$. It follows that $\varphi(\mathbf{x}) > c$ for all $\mathbf{x} \in X$, where $c = \frac{1}{2}\varphi(\mathbf{b}) + \frac{1}{2}\varphi(\mathbf{g})$, and that $\varphi(\mathbf{b}) < c$. The result follows. ∎

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. A point $\mathbf{b}$ lies on the *boundary* of X if every open ball of positive radius centred on the point $\mathbf{b}$ intersects both the set X itself and the complement $\mathbb{R}^n \setminus X$ of X in $\mathbb{R}^n$.

If a subset X of $\mathbb{R}^n$ is open in $\mathbb{R}^n$ then every point belonging to the boundary of the set X belongs to the complement of X. If the subset X of $\mathbb{R}^m$ is closed in $\mathbb{R}^m$ then every point belonging to the boundary of the set X belongs to the set X itself.

Theorem 3.11 (Supporting Hyperplane Theorem) *Let m be a positive integer, let X be a closed convex set in $\mathbb{R}^m$, and let $\mathbf{b}$ be point of $\mathbb{R}^m$ that belongs to the boundary of the closed convex set X. Then there exists a linear functional $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ and a real number c such that $\varphi(\mathbf{x}) \geq c$ for all $\mathbf{x} \in X$ and $\varphi(\mathbf{b}) = c$.*

Proof We may assume without loss of generality, that $\mathbf{b} = (0, 0, \ldots, 0)$. We must then prove the existence of a linear functional $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ with the property that $\varphi(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in X$.

Now, because the $\mathbf{b}$ is located on the boundary of the set X, there exists an infinite sequence $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \ldots$ of points of the complement $\mathbb{R}^n \setminus X$ of the set X that converges to $\mathbf{b}$. It follows from basic linear algebra that, given any linear functional $\psi\colon \mathbb{R}^n \to \mathbb{R}$ on $\mathbb{R}^n$, there exists a vector $\mathbf{w}$ in $\mathbb{R}^n$ such that $\psi(\mathbf{x}) = \mathbf{w} \mathbin{.} \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. It therefore follows from Theorem 3.10, that there exists an infinite sequence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ of non-zero vectors in $\mathbb{R}^n$ such that $\mathbf{v}_j \mathbin{.} \mathbf{b}_j < 0$ and $\mathbf{v}_j \mathbin{.} \mathbf{x} \geq 0$ for all $\mathbf{x} \in X$. We may assume, without loss of generality, that $|\mathbf{v}_j| = 1$ for all positive integers j. It follows from the Bolzano-Weierstrass Theorem (Theorem 1.4) that the infinite sequence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ has a convergent subsequence $\mathbf{v}_{k_1}, \mathbf{v}_{k_2}, \mathbf{v}_{k_3}, \ldots$, where

$$k_1 < k_2 < k_3 < \cdots .$$

Let $\mathbf{v} = \lim_{j \to +\infty} \mathbf{v}_{k_j}$. Then $|\mathbf{v}| = 1$. Let $\varphi(\mathbf{x}) = \mathbf{v} \mathbin{.} \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Then

$$\varphi(\mathbf{x}) = \lim_{j \to +\infty} \mathbf{v}_{k_j} \mathbin{.} \mathbf{x} \geq 0$$

for all $\mathbf{x} \in X$. The result follows. ∎

4 Simplicial Complexes

4.1 Simplical Complexes in Euclidean Spaces

Definition A finite collection K of simplices in $\mathbb{R}^k$ is said to be a *simplicial complex* if the following two conditions are satisfied:—

- if σ is a simplex belonging to K then every face of σ also belongs to K,
- if σ_1 and σ_2 are simplices belonging to K then either $\sigma_1 \cap \sigma_2 = \emptyset$ or else $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

Definition The *dimension* of a simplicial complex K is the greatest non-negative integer n with the property that K contains an n-simplex.

Definition The *polyhedron* of a simplicial complex K is the union of all the simplices of K.

The polyhedron $|K|$ of a simplicial complex K is a subset of a Euclidean space that is both closed and bounded. It is therefore a compact subset of that Euclidean space.

Example Let K_σ consist of some n-simplex σ together with all of its faces. Then K_σ is a simplicial complex of dimension n, and $|K_\sigma| = \sigma$.

Lemma 4.1 *Let K be a simplicial complex, and let X be a subset of some Euclidean space. A function $f\colon |K| \to X$ is continuous on the polyhedron $|K|$ of K if and only if the restriction of f to each simplex of K is continuous on that simplex.*

Proof Each simplex of the simplicial complex K is a closed subset of the polyhedron $|K|$ of the simplicial complex K. The numbers of simplices belonging to the simplicial complex is finite. The result therefore follows from a straightforward application of Lemma 1.19. ∎

We shall denote by $\operatorname{Vert} K$ the set of vertices of a simplicial complex K (i.e., the set consisting of all vertices of all simplices belonging to K). A collection of vertices of K is said to *span* a simplex of K if these vertices are the vertices of some simplex belonging to K.

Definition Let K be a simplicial complex in $\mathbb{R}^k$. A *subcomplex* of K is a collection L of simplices belonging to K with the following property:—

- if σ is a simplex belonging to L then every face of σ also belongs to L.

Note that every subcomplex of a simplicial complex K is itself a simplicial complex.

Proposition 4.2 *Let K be a finite collection of simplices in some Euclidean space $\mathbb{R}^k$, and let $|K|$ be the union of all the simplices in K. Then K is a simplicial complex (with polyhedron $|K|$) if and only if the following two conditions are satisfied:—*

- *K contains the faces of its simplices,*
- *every point of $|K|$ belongs to the interior of a unique simplex of K.*

Proof Suppose that K is a simplicial complex. Then K contains the faces of its simplices. We must show that every point of $|K|$ belongs to the interior of a unique simplex of K. Let $\mathbf{x} \in |K|$. Then $\mathbf{x} \in \rho$ for some simplex ρ of K. It follows from Lemma 3.3 that there exists a unique face σ of ρ such that the point $\mathbf{x}$ belongs to the interior of σ. But then $\sigma \in K$, because $\rho \in K$ and K contains the faces of all its simplices. Thus $\mathbf{x}$ belongs to the interior of at least one simplex of K.

Suppose that $\mathbf{x}$ were to belong to the interior of two distinct simplices σ and τ of K. Then $\mathbf{x}$ would belong to some common face $\sigma \cap \tau$ of σ and τ (since K is a simplicial complex). But this common face would be a proper face of one or other of the simplices σ and τ (since $\sigma \neq \tau$), contradicting the fact that $\mathbf{x}$ belongs to the interior of both σ and τ. We conclude that the simplex σ of K containing $\mathbf{x}$ in its interior is uniquely determined.

Conversely, we must show that if K is some finite collection of simplices in some Euclidean space, if K contains the faces of all its simplices, and if every point of the union $|K|$ of those simplices belongs the the interior of a unique simplex in the collection, then that collection is a simplicial complex. To achieve this, we must prove that if σ and τ are simplices belonging to the collection K, and if $\sigma \cap \tau \neq \emptyset$, then $\sigma \cap \tau$ is a common face of σ and τ.

Let $\mathbf{x} \in \sigma \cap \tau$. Then $\mathbf{x}$ belongs to the interior of a unique simplex ω belonging to the collection K. However any point of σ or τ belongs to the interior of a unique face of that simplex, and all faces of σ and τ belong to K. It follows that ω is a common face of σ and τ, and thus the vertices of ω are vertices of both σ and τ. It follows that the simplices σ and τ have vertices in common.

Let ρ be the simplex whose vertex set is the intersection of the vertex sets of σ and τ. Then ρ is a common face of both σ and τ, and therefore $\rho \in K$. Moreover if $\mathbf{x} \in \sigma \cap \tau$ and if ω is the unique simplex of K whose interior contains the point $\mathbf{x}$, then (as we have already shown), all vertices of ω are vertices of both σ and τ. But then the vertex set of ω is a subset of the vertex set of ρ, and thus ω is a face of ρ. Thus each point $\mathbf{x}$ of $\sigma \cap \tau$ belongs to ρ, and therefore $\sigma \cap \tau \subset \rho$. But ρ is a common face of σ and τ and therefore $\rho \subset \sigma \cap \tau$. It follows that $\sigma \cap \tau = \rho$, and thus $\sigma \cap \tau$ is a common face of σ and τ. This completes the proof that the collection K of simplices satisfying the given conditions is a simplicial complex. ▮

4.2 Barycentric Subdivision of a Simplicial Complex

Let σ be a q-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. The *barycentre* of σ is defined to be the point

$$\hat{\sigma} = \frac{1}{q+1}(\mathbf{v}_0 + \mathbf{v}_1 + \cdots + \mathbf{v}_q).$$

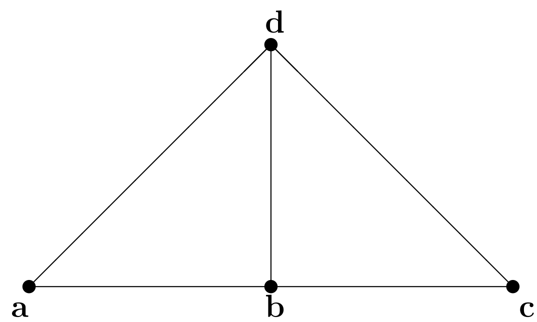
Let σ and τ be simplices in some Euclidean space. If σ is a proper face of τ then we denote this fact by writing $\sigma < \tau$.

A simplicial complex K_1 is said to be a *subdivision* of a simplicial complex K if $|K_1| = |K|$ and each simplex of K_1 is contained in a simplex of K.

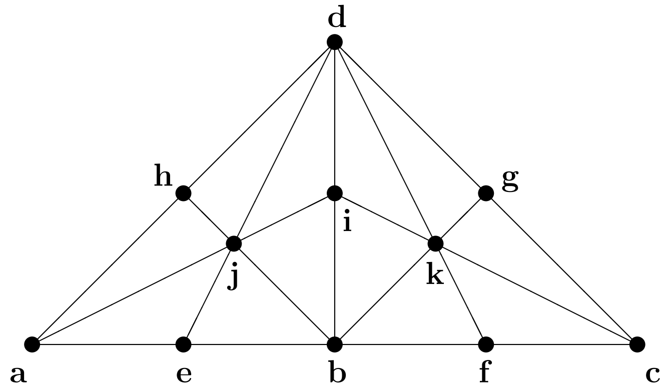
Definition Let K be a simplicial complex in some Euclidean space $\mathbb{R}^k$. The *first barycentric subdivision* K' of K is defined to be the collection of simplices in $\mathbb{R}^k$ whose vertices are $\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_r$ for some sequence $\sigma_0, \sigma_1, \ldots, \sigma_r$ of simplices of K with $\sigma_0 < \sigma_1 < \cdots < \sigma_r$. Thus the set of vertices of K' is the set of all the barycentres of all the simplices of K.

Note that every simplex of K' is contained in a simplex of K. Indeed if $\sigma_0, \sigma_1, \ldots, \sigma_r \in K$ satisfy $\sigma_0 < \sigma_1 < \cdots < \sigma_r$ then the simplex of K' spanned by $\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_r$, is contained in the simplex σ_r of K.

Example Let K be the simplicial complex consisting of two triangles $\mathbf{a}\,\mathbf{b}\,\mathbf{d}$ and $\mathbf{b}\,\mathbf{c}\,\mathbf{d}$ that intersect along a common edge $\mathbf{b}\,\mathbf{d}$, together with all the edges and vertices of the two triangles, as depicted in the following diagram:



The barycentric subdivision K' of this simplicial complex is then as depicted in the following diagram:



We see that K' consists of 12 triangles, together with all the edges and vertices of those triangles. Of the 11 vertices of K', the vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are the vertices of the original complex K, the vertices $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ and $\mathbf{i}$ are the barycentres of the edges $\mathbf{a}\,\mathbf{b}$, $\mathbf{b}\,\mathbf{c}$, $\mathbf{c}\,\mathbf{d}$, $\mathbf{a}\,\mathbf{d}$ and $\mathbf{b}\,\mathbf{d}$ respectively, and are located at the midpoints of those edges, and the vertices $\mathbf{j}$ and $\mathbf{k}$ are the barycentres of the triangles $\mathbf{a}\,\mathbf{b}\,\mathbf{d}$ and $\mathbf{b}\,\mathbf{c}\,\mathbf{d}$ of K. Thus $\mathbf{e} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$, $\mathbf{f} = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c}$, etc., and $\mathbf{j} = \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} + \frac{1}{3}\mathbf{d}$ and $\mathbf{k} = \frac{1}{3}\mathbf{b} + \frac{1}{3}\mathbf{c} + \frac{1}{3}\mathbf{d}$.

Proposition 4.3 *Let K be a simplicial complex in some Euclidean space, and let K' be the first barycentric subdivision of K. Then K' is itself a simplicial complex, and $|K'| = |K|$.*

Proof We prove the result by induction on the number of simplices in K. The result is clear when K consists of a single simplex, since that simplex must then be a point and therefore $K' = K$. We prove the result for a simplicial complex K, assuming that it holds for all complexes with fewer simplices.

It is clear from the definition of the barycentric subdivision K' that any face of a simplex of K' must itself belong to K'. We must verify that any two simplices of K' are disjoint or else intersect in a common face.

Choose a simplex σ of K for which $\dim \sigma = \dim K$, and let $L = K \setminus \{\sigma\}$. Then L is a subcomplex of K, since σ is not a proper face of any simplex of K. Now L has fewer simplices than K. It follows from the induction hypothesis that L' is a simplicial complex and $|L'| = |L|$. Also it follows from the definition of K' that K' consists of the following simplices:—

- the simplices of L',

- the barycentre $\hat{\sigma}$ of σ,
- simplices $\hat{\sigma}\rho$ whose vertex set is obtained by adjoining $\hat{\sigma}$ to the vertex set of some simplex ρ of L', where the vertices of ρ are barycentres of proper faces of σ.

By checking all possible intersections of simplices of the above types, it is easy to verify that any two simplices of K' intersect in a common face. Indeed any two simplices of L' intersect in a common face, since L' is a simplicial complex. If ρ_1 and ρ_2 are simplices of L' whose vertices are barycentres of proper faces of σ, then $\rho_1 \cap \rho_2$ is a common face of ρ_1 and ρ_2 which is of this type, and $\hat{\sigma}\rho_1 \cap \hat{\sigma}\rho_2 = \hat{\sigma}(\rho_1 \cap \rho_2)$. Thus $\hat{\sigma}\rho_1 \cap \hat{\sigma}\rho_2$ is a common face of $\hat{\sigma}\rho_1$ and $\hat{\sigma}\rho_2$. Also any simplex τ of L' is disjoint from the barycentre $\hat{\sigma}$ of σ, and $\hat{\sigma}\rho \cap \tau = \rho \cap \tau$. We conclude that K' is indeed a simplicial complex.

It remains to verify that $|K'| = |K|$. Now $|K'| \subset |K|$, since every simplex of K' is contained in a simplex of K. Let $\mathbf{x}$ be a point of the chosen simplex σ. Then there exists a point $\mathbf{y}$ belonging to a proper face of σ and some $t \in [0,1]$ such that $\mathbf{x} = (1-t)\hat{\sigma} + t\,\mathbf{y}$. But then $\mathbf{y} \in |L|$, and $|L| = |L'|$ by the induction hypothesis. It follows that $\mathbf{y} \in \rho$ for some simplex ρ of L' whose vertices are barycentres of proper faces of σ. But then $\mathbf{x} \in \hat{\sigma}\rho$, and therefore $\mathbf{x} \in |K'|$. Thus $|K| \subset |K'|$, and hence $|K'| = |K|$, as required. ∎

We define (by induction on j) the jth barycentric subdivision $K^{(j)}$ of K to be the first barycentric subdivision of $K^{(j-1)}$ for each $j > 1$.

Lemma 4.4 *Let σ be a q-simplex and let τ be a face of σ. Let $\hat{\sigma}$ and $\hat{\tau}$ be the barycentres of σ and τ respectively. If all the 1-simplices (edges) of σ have length not exceeding d for some $d > 0$ then*

$$|\hat{\sigma} - \hat{\tau}| \leq \frac{qd}{q+1}.$$

Proof Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of σ. Let $\mathbf{x}$ and $\mathbf{y}$ be points of σ. We can write $\mathbf{y} = \sum_{j=0}^{q} t_j \mathbf{v}_j$, where $0 \leq t_i \leq 1$ for $i = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$. Now

$$\begin{aligned} |\mathbf{x} - \mathbf{y}| &= \left| \sum_{i=0}^{q} t_i (\mathbf{x} - \mathbf{v}_i) \right| \leq \sum_{i=0}^{q} t_i |\mathbf{x} - \mathbf{v}_i| \\ &\leq \text{maximum}\left(|\mathbf{x} - \mathbf{v}_0|, |\mathbf{x} - \mathbf{v}_1|, \ldots, |\mathbf{x} - \mathbf{v}_q|\right). \end{aligned}$$

Applying this result with $\mathbf{x} = \hat{\sigma}$ and $\mathbf{y} = \hat{\tau}$, we find that

$$|\hat{\sigma} - \hat{\tau}| \leq \text{maximum}\left(|\hat{\sigma} - \mathbf{v}_0|, |\hat{\sigma} - \mathbf{v}_1|, \ldots, |\hat{\sigma} - \mathbf{v}_q|\right).$$

But

$$\hat{\sigma} = \frac{1}{q+1}\mathbf{v}_i + \frac{q}{q+1}\mathbf{z}_i$$

for $i = 0, 1, \ldots, q$, where $\mathbf{z}_i$ is the barycentre of the $(q-1)$-face of σ opposite to $\mathbf{v}_i$, given by

$$\mathbf{z}_i = \frac{1}{q}\sum_{j \neq i} \mathbf{v}_j.$$

Moreover $\mathbf{z}_i \in \sigma$. It follows that

$$|\hat{\sigma} - \mathbf{v}_i| = \frac{q}{q+1}|\mathbf{z}_i - \mathbf{v}_i| \leq \frac{qd}{q+1}$$

for $i = 1, 2, \ldots, q$, and thus

$$|\hat{\sigma} - \hat{\tau}| \leq \text{maximum}\,(|\hat{\sigma} - \mathbf{v}_0|, |\hat{\sigma} - \mathbf{v}_1|, \ldots, |\hat{\sigma} - \mathbf{v}_q|) \leq \frac{qd}{q+1},$$

as required. ∎

Definition The *mesh* $\mu(K)$ of a simplicial complex K is the length of the longest edge of K.

Lemma 4.5 *Let K be a simplicial complex, and let n be the dimension of K. Let K' be the first barycentric subdivision of K. Then*

$$\mu(K') \leq \frac{n}{n+1}\mu(K).$$

Proof A 1-simplex of K' is of the form $(\hat{\tau}, \hat{\sigma})$, where σ is a q-simplex of K for some $q \leq n$ and τ is a proper face of σ. Then

$$|\hat{\tau} - \hat{\sigma}| \leq \frac{q}{q+1}\mu(K) \leq \frac{n}{n+1}\mu(K)$$

by Lemma 4.4, as required. ∎

Lemma 4.6 *Let K be a simplicial complex, let $K^{(j)}$ be the jth barycentric subdivision of K for all positive integers j, and let $\mu(K^{(j)})$ be the mesh of $K^{(j)}$. Then $\lim_{j \to +\infty} \mu(K^{(j)}) = 0$.*

Proof The dimension of all barycentric subdivisions of a simplicial complex is equal to the dimension of the simplicial complex itself. It therefore follows from Lemma 4.5 that

$$\mu(K^{(j)}) \leq \left(\frac{n}{n+1}\right)^j \mu(K).$$

The result follows. ∎

4.3 Piecewise Linear Maps on Simplicial Complexes

Definition Let K be a simplicial complex in n-dimensional Euclidean space. A function $f\colon |K| \to \mathbb{R}^m$ mapping the polyhedron $|K|$ of K into m-dimensional Euclidean space $\mathbb{R}^m$ is said to be *piecewise linear* on each simplex of K if

$$f\left(\sum_{i=0}^{q} t_i \mathbf{v}_i\right) = \sum_{i=0}^{q} t_i f(\mathbf{v}_i)$$

for all vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of K that span a simplex of K, and for all non-negative real numbers $t_0, t_1, \ldots, t_q$ satisfying $\sum_{i=0}^{q} t_i = 1$.

Lemma 4.7 *Let K be a simplicial complex in n-dimensional Euclidean space, and let $f\colon |K| \to \mathbb{R}^m$ be a function mapping the polyhedron $|K|$ of K into m-dimensional Euclidean space $\mathbb{R}^m$ that is piecewise linear on each simplex of K. Then $f\colon |K| \to \mathbb{R}^m$ is continuous.*

Proof The definition of piecewise linear functions ensures that the restriction of $f\colon |K| \to \mathbb{R}^m$ to each simplex of K is continuous on that simplex. The result therefore follows from Lemma 4.1. ∎

Proposition 4.8 *Let K be a simplicial complex in n-dimensional Euclidean space and let $\alpha\colon \mathrm{Vert}(K) \to \mathbb{R}^m$ be a function mapping the set $\mathrm{Vert}(K)$ of vertices of K into m-dimensional Euclidean space $\mathbb{R}^m$. Then there exists a unique function $f\colon |K| \to \mathbb{R}^m$ defined on the polyhedron $|K|$ of K that is piecewise linear on each simplex of K and satisfies $f(\mathbf{v}) = \alpha(\mathbf{v})$ for all vertices $\mathbf{v}$ of K.*

Proof Given any point $\mathbf{x}$ of K, there exists a unique simplex of K whose interior contains the point $\mathbf{x}$ (Proposition 4.2). Let the vertices of this simplex be $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_p$, where $p \leq n$. Then there exist uniquely-determined strictly positive real numbers $t_0, t_1, \ldots, t_p$ satisfying $\sum_{i=0}^{p} t_i = 1$ for which $\mathbf{x} = \sum_{i=0}^{p} t_i \mathbf{v}_i$. We then define $f(\mathbf{x})$ so that

$$f(\mathbf{x}) = \sum_{i=0}^{p} t_i \alpha(\mathbf{v}_i).$$

Defining $f(\mathbf{x})$ in this fashion at each point $\mathbf{x}$ of $|K|$, we obtain a function $f\colon |K| \to \mathbb{R}^m$ mapping Δ into $\mathbb{R}^m$.

Now let $\mathbf{x} \in \sigma$ for some q-simplex of K. We can order the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of σ so that the point $\mathbf{x}$ belongs to the interior of the face of

σ spanned by $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_p$ where $p \leq q$. Let $t_1, t_2, \ldots, t_q$ be the barycentric coordinates of the point $\mathbf{x}$ with respect to the simplex σ. Then $\mathbf{x} = \sum_{i=0}^{q} t_i \mathbf{v}_i$, where $t_i > 0$ for those integers i satisfying $0 \leq i \leq p$, $t_i = 0$ for those integers i (if any) satisfying $p < i \leq q$, and $\sum_{i=0}^{p} t_i = \sum_{i=0}^{q} t_i = 1$. Then

$$f\left(\sum_{i=0}^{q} t_i \mathbf{v}_i\right) = f(\mathbf{x}) = \sum_{i=0}^{p} t_i \alpha(\mathbf{v}_i) = \sum_{i=0}^{q} t_i f(\mathbf{v}_i).$$

The result follows. ∎

Corollary 4.9 *Let K be a simplicial complex in $\mathbb{R}^n$ and let L be simplicial complexes in $\mathbb{R}^m$, where m and n are positive integers, and let $\varphi\colon \mathrm{Vert}(K) \to \mathrm{Vert}(L)$ be a function mapping vertices of K to vertices of L. Suppose that*

$$\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$$

span a simplex of L for all vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of K that span a simplex of K. Then there exists a unique continuous map $\overline{\varphi}\colon |K| \to |L|$ mapping the polyhedron $|K|$ of K into the polyhedron $|L|$ of L that is piecewise linear on each simplex of K and satisfies $\overline{\varphi}(\mathbf{v}) = \varphi(\mathbf{v})$ for all vertices $\mathbf{v}$ of K. Moreover this function maps the interior of a simplex of K spanned by vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ into the interior of the simplex of L spanned by $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$.

Proof It follows from Proposition 4.8 that there is a unique piecewise linear function $f\colon |K| \to \mathbb{R}^m$ that satisfies $f(\mathbf{v}) = \varphi(\mathbf{v})$ for all $\mathbf{v} \in \mathrm{Vert}(K)$. We show that $f(|K|) \subset |L|$.

Let

$$\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$$

be vertices of a simplex σ of K, and let $t_0, t_1, \ldots, t_q$ be non-negative real numbers satisfying $\sum_{j=0}^{q} t_j = 1$. Then

$$\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$$

span a simplex of L. Let τ be the simplex of L spanned by these vertices of L, and let $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_r$ be the vertices of τ. Then, for each integer j between 1 and r, let u_j be the sum of those t_i for which $\varphi(\mathbf{v}_i) = \mathbf{w}_j$. Then

$$f\left(\sum_{i=0}^{q} t_i \mathbf{v}_i\right) = \sum_{i=0}^{q} t_i \varphi(\mathbf{v}_i) = \sum_{j=0}^{r} u_j \mathbf{w}_j$$

and $\sum_{j=0}^{r} u_j = 1$. It follows that $f(\sigma) \subset \tau$. Moreover, given any integer j between 1 and r, there exists at least one integer i between 1 and q for which $\varphi(\mathbf{v}_i) = \mathbf{w}_j$. It follows that if $t_0, t_1, t_2, \ldots, t_q$ are all strictly positive then $u_0, u_1, \ldots, u_r$ are also all strictly positive. Therefore the piecewise linear function f maps the interior of σ into the interior of τ.

We have already shown that $f\colon |K| \to \mathbb{R}^m$ maps each simplex of K into a simplex of L. Therefore there exists a uniquely-determined linear function $\overline{\varphi}\colon |K| \to |L|$ satisfying $\overline{\varphi}(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in |K|$. The result follows. ∎

4.4 Simplicial Maps

Definition A *simplicial map* $\varphi\colon K \to L$ between simplicial complexes K and L is a function $\varphi\colon \operatorname{Vert} K \to \operatorname{Vert} L$ from the vertex set of K to that of L such that $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$ span a simplex belonging to L whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K.

Note that a simplicial map $\varphi\colon K \to L$ between simplicial complexes K and L can be regarded as a function from K to L: this function sends a simplex σ of K with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ to the simplex $\varphi(\sigma)$ of L spanned by the vertices $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$. It follows from Corollary 4.9 that simplicial map $\varphi\colon K \to L$ also induces in a natural fashion a continuous map $\varphi\colon |K| \to |L|$ between the polyhedra of K and L, where

$$\varphi\left(\sum_{j=0}^{q} t_j \mathbf{v}_j\right) = \sum_{j=0}^{q} t_j \varphi(\mathbf{v}_j)$$

whenever $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$, $\sum_{j=0}^{q} t_j = 1$, and $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. Moreover it also follows from Corollary 4.9 that the interior of a simplex σ of K is mapped into the interior of the simplex $\varphi(\sigma)$ of L.

There are thus three equivalent ways of describing a simplicial map: as a function between the vertex sets of two simplicial complexes, as a function from one simplicial complex to another, and as a continuous map between the polyhedra of two simplicial complexes. In what follows, we shall describe a simplicial map using the representation that is most appropriate in the given context.

4.5 Simplicial Approximations

Definition Let $f\colon |K| \to |L|$ be a continuous map between the polyhedra of simplicial complexes K and L. A simplicial map $s\colon K \to L$ is said to be a *simplicial approximation* to f if, for each $\mathbf{x} \in |K|$, $s(\mathbf{x})$ is an element of the unique simplex of L which contains $f(\mathbf{x})$ in its interior.

Definition Let X and Y be subsets of Euclidean spaces. Continuous maps $f\colon X \to Y$ and $g\colon X \to Y$ from X to Y are said to be *homotopic* if there exists a continuous map $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.

Lemma 4.10 *Let K and L be simplicial complexes, let $f\colon |K| \to |L|$ be a continuous map between the polyhedra of K and L, and let $s\colon K \to L$ be a simplicial approximation to the map f. Then there is a well-defined homotopy $H\colon |K| \times [0,1] \to |L|$, between the maps f and s, where*

$$H(\mathbf{x}, t) = (1-t)f(\mathbf{x}) + ts(\mathbf{x})$$

for all $\mathbf{x} \in |K|$ and $t \in [0,1]$.

Proof Let $\mathbf{x} \in |K|$. Then there is a unique simplex σ of L such that the point $f(\mathbf{x})$ belongs to the interior of σ. Then $s(\mathbf{x}) \in \sigma$. But, given any two points of a simplex embedded in some Euclidean space, the line segment joining those two points is contained within the simplex. It follows that $(1-t)f(\mathbf{x}) + ts(\mathbf{x}) \in |L|$ for all $\mathbf{x} \in K$ and $t \in [0,1]$. Thus the homotopy $H\colon |K| \times [0,1] \to |L|$ is a well-defined map from $|K| \times [0,1]$ to $|L|$. Moreover it follows directly from the definition of this map that $H(\mathbf{x}, 0) = f(\mathbf{x})$ and $H(\mathbf{x}, 1) = s(\mathbf{x})$ for all $\mathbf{x} \in |K|$ and $t \in [0,1]$. The map H is thus a homotopy between the maps f and s, as required. ∎

Definition Let K be a simplicial complex, and let $\mathbf{x} \in |K|$. The *star neighbourhood* $\mathrm{st}_K(\mathbf{x})$ of $\mathbf{x}$ in K is the union of the interiors of all simplices of K that contain the point $\mathbf{x}$.

Lemma 4.11 *Let K be a simplicial complex and let $\mathbf{x} \in |K|$. Then the star neighbourhood $\mathrm{st}_K(\mathbf{x})$ of $\mathbf{x}$ is open in $|K|$, and $\mathbf{x} \in \mathrm{st}_K(\mathbf{x})$.*

Proof Every point of $|K|$ belongs to the interior of a unique simplex of K (Proposition 4.2). It follows that the complement $|K| \setminus \mathrm{st}_K(\mathbf{x})$ of $\mathrm{st}_K(\mathbf{x})$ in $|K|$ is the union of the interiors of those simplices of K that do not contain the point $\mathbf{x}$. But if a simplex of K does not contain the point $\mathbf{x}$, then the same is true of its faces. Moreover the union of the interiors of all the faces of some simplex is the simplex itself. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{x})$ is the union of all simplices of K that do not contain the point $\mathbf{x}$. But each simplex of K is closed in $|K|$. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{x})$ is a finite union of closed sets, and is thus itself closed in $|K|$. We deduce that $\mathrm{st}_K(\mathbf{x})$ is open in $|K|$. Also $\mathbf{x} \in \mathrm{st}_K(\mathbf{x})$, since $\mathbf{x}$ belongs to the interior of at least one simplex of K. ∎

Proposition 4.12 *A function $s\colon \operatorname{Vert} K \to \operatorname{Vert} L$ between the vertex sets of simplicial complexes K and L is a simplicial map, and a simplicial approximation to some continuous map $f\colon |K| \to |L|$, if and only if $f\left(\mathrm{st}_K(\mathbf{v})\right) \subset \mathrm{st}_L\left(s(\mathbf{v})\right)$ for all vertices $\mathbf{v}$ of K.*

Proof Let $s\colon K \to L$ be a simplicial approximation to $f\colon |K| \to |L|$, let $\mathbf{v}$ be a vertex of K, and let $\mathbf{x} \in \mathrm{st}_K(\mathbf{v})$. Then $\mathbf{x}$ and $f(\mathbf{x})$ belong to the interiors of unique simplices $\sigma \in K$ and $\tau \in L$. Moreover $\mathbf{v}$ must be a vertex of σ, by definition of $\mathrm{st}_K(\mathbf{v})$. Now $s(\mathbf{x})$ must belong to τ (since s is a simplicial approximation to the map f), and therefore $s(\mathbf{x})$ must belong to the interior of some face of τ. But $s(\mathbf{x})$ must belong to the interior of $s(\sigma)$, because $\mathbf{x}$ is in the interior of σ (see Corollary 4.9). It follows that $s(\sigma)$ must be a face of τ, and therefore $s(\mathbf{v})$ must be a vertex of τ. Thus $f(\mathbf{x}) \in \mathrm{st}_L(s(\mathbf{v}))$. We conclude that if $s\colon K \to L$ is a simplicial approximation to $f\colon |K| \to |L|$, then $f\left(\mathrm{st}_K(\mathbf{v})\right) \subset \mathrm{st}_L\left(s(\mathbf{v})\right)$.

Conversely let $s\colon \operatorname{Vert} K \to \operatorname{Vert} L$ be a function with the property that $f\left(\mathrm{st}_K(\mathbf{v})\right) \subset \mathrm{st}_L\left(s(\mathbf{v})\right)$ for all vertices $\mathbf{v}$ of K. Let $\mathbf{x}$ be a point in the interior of some simplex of K with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. Then $\mathbf{x} \in \mathrm{st}_K(\mathbf{v}_j)$ and hence $f(\mathbf{x}) \in \mathrm{st}_L\left(s(\mathbf{v}_j)\right)$ for $j = 0, 1, \ldots, q$. It follows that each vertex $s(\mathbf{v}_j)$ must be a vertex of the unique simplex $\tau \in L$ that contains $f(\mathbf{x})$ in its interior. In particular, $s(\mathbf{v}_0), s(\mathbf{v}_1), \ldots, s(\mathbf{v}_q)$ span a face of τ, and $s(\mathbf{x}) \in \tau$. We conclude that the function $s\colon \operatorname{Vert} K \to \operatorname{Vert} L$ represents a simplicial map which is a simplicial approximation to $f\colon |K| \to |L|$, as required. ∎

Corollary 4.13 *If $s\colon K \to L$ and $t\colon L \to M$ are simplicial approximations to continuous maps $f\colon |K| \to |L|$ and $g\colon |L| \to |M|$, where K, L and M are simplicial complexes, then $t \circ s\colon K \to M$ is a simplicial approximation to $g \circ f\colon |K| \to |M|$.*

4.6 The Simplicial Approximation Theorem

Theorem 4.14 (Simplicial Approximation Theorem) *Let K and L be simplicial complexes, and let $f\colon |K| \to |L|$ be a continuous map. Then, for some sufficiently large integer j, there exists a simplicial approximation $s\colon K^{(j)} \to L$ to f defined on the jth barycentric subdivision $K^{(j)}$ of K.*

Proof The collection consisting of the stars $\mathrm{st}_L(\mathbf{w})$ of all vertices $\mathbf{w}$ of L is an open cover of $|L|$, since each star $\mathrm{st}_L(\mathbf{w})$ is open in $|L|$ (Lemma 4.11) and the interior of any simplex of L is contained in $\mathrm{st}_L(\mathbf{w})$ whenever $\mathbf{w}$ is a vertex of that simplex. It follows from the continuity of the map $f\colon |K| \to |L|$ that the collection consisting of the preimages $f^{-1}(\mathrm{st}_L(\mathbf{w}))$ of the stars of all vertices $\mathbf{w}$ of L is an open cover of $|K|$.

Now the set $|K|$ is a closed bounded subset of a Euclidean space. It follows that there exists a Lebesgue number δ_L for the open cover consisting of the preimages of the stars of all the vertices of L (see Proposition 1.21). This Lebesgue number δ_L is a positive real number with the following property: every subset of $|K|$ whose diameter is less than δ_L is contained in the preimage of the star of some vertex $\mathbf{w}$ of L. It follows that every subset of $|K|$ whose diameter is less than δ_L is mapped by f into $\mathrm{st}_L(\mathbf{w})$ for some vertex $\mathbf{w}$ of L.

Now the mesh $\mu(K^{(j)})$ of the jth barycentric subdivision of K tends to zero as $j \to +\infty$ (see Lemma 4.6). Thus we can choose j such that $\mu(K^{(j)}) < \frac{1}{2}\delta_L$. If $\mathbf{v}$ is a vertex of $K^{(j)}$ then each point of $\mathrm{st}_{K^{(j)}}(\mathbf{v})$ is within a distance $\frac{1}{2}\delta_L$ of $\mathbf{v}$, and hence the diameter of $\mathrm{st}_{K^{(j)}}(\mathbf{v})$ is at most δ_L. We can therefore choose, for each vertex $\mathbf{v}$ of $K^{(j)}$ a vertex $s(\mathbf{v})$ of L such that $f(\mathrm{st}_{K^{(j)}}(\mathbf{v})) \subset \mathrm{st}_L(s(\mathbf{v}))$. In this way we obtain a function $s\colon \mathrm{Vert}\, K^{(j)} \to \mathrm{Vert}\, L$ from the vertices of $K^{(j)}$ to the vertices of L. It follows directly from Proposition 4.12 that this is the desired simplicial approximation to f. ∎

5 Fixed Point Theorems

5.1 Sperner's Lemma

Definition Let K be a simplicial complex which is a subdivision of some n-dimensional simplex Δ. We define a *Sperner labelling* of the vertices of K to be a function, labelling each vertex of K with an integer between 0 and n, with the following properties:—

- for each $j \in \{0, 1, \ldots, n\}$, there is exactly one vertex of Δ labelled by j,
- if a vertex $\mathbf{v}$ of K belongs to some face of Δ, then some vertex of that face has the same label as $\mathbf{v}$.

Lemma 5.1 (Sperner's Lemma) *Let K be a simplicial complex which is a subdivision of an n-simplex Δ. Then, for any Sperner labelling of the vertices of K, the number of n-simplices of K whose vertices are labelled by $0, 1, \ldots, n$ is odd.*

Proof Given integers $i_0, i_1, \ldots, i_q$ between 0 and n, let $N(i_0, i_1, \ldots, i_q)$ denote the number of q-simplices of K whose vertices are labelled by $i_0, i_1, \ldots, i_q$ (where an integer occurring k times in the list labels exactly k vertices of the simplex). We must show that $N(0, 1, \ldots, n)$ is odd.

We prove the result by induction on the dimension n of the simplex Δ; it is clearly true when $n = 0$. Suppose that the result holds in dimensions less than n. For each simplex σ of K of dimension n, let $p(\sigma)$ denote the number of $(n-1)$-faces of σ labelled by $0, 1, \ldots, n-1$. If σ is labelled by $0, 1, \ldots, n$ then $p(\sigma) = 1$; if σ is labelled by $0, 1, \ldots, n-1, j$, where $j < n$, then $p(\sigma) = 2$; in all other cases $p(\sigma) = 0$. Therefore

$$\sum_{\substack{\sigma \in K \\ \dim \sigma = n}} p(\sigma) = N(0, 1, \ldots, n) + 2 \sum_{j=0}^{n-1} N(0, 1, \ldots, n-1, j).$$

Now the definition of Sperner labellings ensures that the only $(n-1)$-face of Δ containing simplices of K labelled by $0, 1, \ldots, n-1$ is that with vertices labelled by $0, 1, \ldots, n-1$. Thus if M is the number of $(n-1)$-simplices of K labelled by $0, 1, \ldots, n-1$ that are contained in this face, then $N(0, 1, \ldots, n-1) - M$ is the number of $(n-1)$-simplices labelled by $0, 1, \ldots, n-1$ that intersect the interior of Δ. It follows that

$$\sum_{\substack{\sigma \in K \\ \dim \sigma = n}} p(\sigma) = M + 2\big(N(0, 1, \ldots, n-1) - M\big),$$

since any $(n-1)$-simplex of K that is contained in a proper face of Δ must be a face of exactly one n-simplex of K, and any $(n-1)$-simplex that intersects the interior of Δ must be a face of exactly two n-simplices of K. On combining these equalities, we see that $N(0, 1, \ldots, n) - M$ is an even integer. But the induction hypothesis ensures that Sperner's Lemma holds in dimension $n-1$, and thus M is odd. It follows that $N(0, 1, \ldots, n)$ is odd, as required. ∎

5.2 Proof of Brouwer's Fixed Point Theorem

Proposition 5.2 *Let Δ be an n-simplex with boundary $\partial\Delta$. Then there does not exist any continuous map $r\colon \Delta \to \partial\Delta$ with the property that $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$.*

Proof Suppose that such a map $r\colon \Delta \to \partial\Delta$ were to exist. It would then follow from the Simplicial Approximation Theorem (Theorem 4.14) that there would exist a simplicial approximation $s\colon K \to L$ to the map r, where L is the simplicial complex consisting of all of the proper faces of Δ, and K is the jth barycentric subdivision, for some sufficiently large j, of the simplicial complex consisting of the simplex Δ together with all of its faces.

If $\mathbf{v}$ is a vertex of K belonging to some proper face Σ of Δ then $r(\mathbf{v}) = \mathbf{v}$, and hence $s(\mathbf{v})$ must be a vertex of Σ, since $s\colon K \to L$ is a simplicial approximation to $r\colon \Delta \to \partial\Delta$. In particular $s(\mathbf{v}) = \mathbf{v}$ for all vertices $\mathbf{v}$ of Δ. Thus if $\mathbf{v} \mapsto m(\mathbf{v})$ is a labelling of the vertices of Δ by the integers $0, 1, \ldots, n$, then $\mathbf{v} \mapsto m(s(\mathbf{v}))$ is a Sperner labelling of the vertices of K. Thus Sperner's Lemma (Lemma 5.1) guarantees the existence of at least one n-simplex σ of K labelled by $0, 1, \ldots, n$. But then $s(\sigma) = \Delta$, which is impossible, since Δ is not a simplex of L. We conclude therefore that there cannot exist any continuous map $r\colon \Delta \to \partial\Delta$ satisfying $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$. ∎

Theorem 5.3 (Brouwer Fixed Point Theorem) (Brouwer Fixed Point Theorem) *Let X be a subset of a Euclidean space that is homeomorphic to the closed n-dimensional ball E^n, where*

$$E^n = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq 1\}.$$

Then any continuous function $f\colon X \to X$ mapping the set X into itself has at least one fixed point $\mathbf{x}^*$ for which $f(\mathbf{x}^*) = \mathbf{x}^*$.

Proof The closed n-dimensional ball E^n is itself homeomorphic to an n-dimensional simplex Δ. Therefore there exists a homeomorphism $h\colon X \to \Delta$ mapping the set X onto the simplex Δ. Then the continuous map $f\colon X \to X$ determines a continuous map $g\colon \Delta \to \Delta$, where $g(h(\mathbf{x})) = h(f(\mathbf{x}))$ for all

$\mathbf{x} \in X$. Suppose that it were the case that $f(\mathbf{x}) \neq \mathbf{x}$ for all $\mathbf{x} \in X$. Then $g(\mathbf{z}) \neq \mathbf{z}$ for all $\mathbf{z} \in \Delta$. There would then exist a well-defined continuous map $r\colon \Delta \to \partial\Delta$ mapping each point $\mathbf{z}$ of Δ to the unique point $r(\mathbf{z})$ of the boundary $\partial\Delta$ of Δ at which the half line starting at $g(\mathbf{z})$ and passing through $\mathbf{z}$ intersects $\partial\Delta$. Then $r\colon \Delta \to \partial\Delta$ would be continuous, and $r(\mathbf{z}) = \mathbf{z}$ for all $\mathbf{z} \in \partial\Delta$. However Proposition 5.2 guarantees that there does not exist any continuous map $r\colon \Delta \to \partial\Delta$ with these properties. Therefore the map f must have at least one fixed point, as required. ∎

5.3 The Kakutani Fixed Point Theorem

Theorem 5.4 (Kakutani's Fixed Point Theorem) *Let X be a non-empty, compact and convex subset of n-dimensional Euclidean space $\mathbb{R}^n$, and let $\Phi\colon X \rightrightarrows X$ be a correspondence mapping X into itself. Suppose that the graph of the correspondence Φ is closed and that $\Phi(\mathbf{x})$ is non-empty and convex for all $\mathbf{x} \in X$. Then there exists a point $\mathbf{x}^*$ of X that satisfies $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$.*

Proof There exists a continuous map $r\colon \mathbb{R}^n \to X$ from $\mathbb{R}^n$ to X with the property that $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$. (see Proposition 3.8). Let Δ be an n-dimensional simplex chosen such that $X \subset \Delta$, and let $\Psi(\mathbf{x}) = \Phi(r(\mathbf{x}))$ for all $\mathbf{x} \in \Delta$. If $\mathbf{x}^* \in \Delta$ satisfies $\mathbf{x}^* \in \Psi(\mathbf{x}^*)$ then $\mathbf{x}^* \in X$ and $r(\mathbf{x}^*) = \mathbf{x}^*$, and therefore $\mathbf{x} \in \Phi(\mathbf{x}^*)$. It follows that the result in the general case follows from that in the special case in which the closed bounded convex subset X of $\mathbb{R}^n$ is an n-dimensional simplex.

Thus let Δ be an n-dimensional simplex contained in $\mathbb{R}^n$, and let $\Phi\colon \Delta \rightrightarrows \Delta$ be a correspondence with closed graph, where $\Phi(\mathbf{x})$ is a non-empty closed convex subset of Δ for all $\mathbf{x} \in X$. We must prove that there exists some point $\mathbf{x}^*$ of Δ with the property that $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$.

Let K be the simplicial complex consisting of the n-simplex Δ together with all its faces, and let $K^{(j)}$ be the jth barycentric subdivision of K for all positive integers j. Then $|K^{(j)}| = \Delta$ for all positive integers j. Now $\Phi(\mathbf{v})$ is non-empty for all vertices $\mathbf{v}$ of $K^{(j)}$. Now any function mapping the vertices of a simplicial complex into a Euclidean space extends uniquely to a piecewise linear map defined over the polyhedron of that simplicial complex (Proposition 4.8). Therefore there exists a sequence $f_1, f_2, f_3, \ldots$ of continuous functions mapping the simplex Δ into itself such that, for each positive integer j, the continuous map $f_j\colon \Delta \to \Delta$ is piecewise linear on the simplices of $K^{(j)}$ and satisfies $f_j(\mathbf{v}) \in \Phi(\mathbf{v})$ for all vertices $\mathbf{v}$ of $K^{(j)}$.

Now it follows from the Brouwer Fixed Point Theorem (Theorem 5.3) that, for each positive integer j, there exists $\mathbf{z}_j \in \Delta$ for which $f_j(\mathbf{z}_j) = \mathbf{z}_j$. For each positive integer j, there exist vertices $\mathbf{v}_{0,j}, \mathbf{v}_{1,j}, \ldots, \mathbf{v}_{n,j}$ of $K^{(j)}$ spanning a simplex of K and non-negative real numbers $t_{0,j}, t_{1,j}, \ldots, t_{n,j}$ satisfying $\sum_{i=1}^{n} t_{i,j} = 1$ such that

$$\mathbf{z}_j = \sum_{i=0}^{n} t_{i,j}\mathbf{v}_{i,j}$$

for all positive integers j. Let $\mathbf{y}_{i,j} = f_j(\mathbf{v}_{i,j})$ for $i = 0, 1, \ldots, n$ and for all positive integers j. Then $\mathbf{y}_{i,j} \in \Phi(\mathbf{v}_{i,j})$ for $i = 0, 1, \ldots, n$ and for all positive integers j. The function f_j is piecewise linear on the simplices of $K^{(j)}$. It follows that

$$\begin{aligned} \sum_{i=0}^{n} t_{i,j}\mathbf{v}_{i,j} &= \mathbf{z}_j = f_j(\mathbf{z}_j) = f_j\left(\sum_{i=0}^{n} t_{i,j}\mathbf{v}_{i,j}\right) \\ &= \sum_{i=0}^{n} t_{i,j} f_j(\mathbf{v}_{i,j}) = \sum_{i=0}^{n} t_{i,j}\mathbf{y}_{i,j} \end{aligned}$$

for all positive integers j. Also $|\mathbf{v}_{i,j} - \mathbf{v}_{0,j}| \leq \mu(K^{(j)})$ for $i = 0, 1, \ldots, n$ and for all positive integers j, where $\mu(K^{(j)})$ denotes the mesh of the simplicial complex $K^{(j)}$ (i.e., the length of the longest side of that simplicial complex). Moreover $\mu(K^j) \to 0$ as $j \to +\infty$ (see Lemma 4.6). It follows that

$$\lim_{j\to+\infty} |\mathbf{v}_{i,j} - \mathbf{v}_{0,j}| = 0.$$

Now the multidimensional Bolzano-Weierstrass Theorem (Theorem 1.4) ensures the existence of points

$$\mathbf{x}^*, \mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n$$

of the simplex Δ, non-negative real numbers $t_0, t_1, \ldots, t_n$ and an infinite sequence $m_1, m_2, m_3, \ldots$ of positive integers, where

$$m_1 < m_2 < m_3 < \cdots,$$

such that

$$\begin{aligned} \mathbf{x}^* &= \lim_{j\to+\infty} \mathbf{v}_{0,m_j}, \\ \mathbf{y}_i &= \lim_{j\to+\infty} \mathbf{y}_{i,m_j} \quad (0 \leq i \leq n), \\ t_i &= \lim_{j\to+\infty} t_{i,m_j} \quad (0 \leq i \leq n). \end{aligned}$$

Now

$$|\mathbf{v}_{i,m_j} - \mathbf{x}^*| \leq |\mathbf{v}_{i,m_j} - \mathbf{v}_{0,m_j}| + |\mathbf{v}_{0,m_j} - \mathbf{x}^*|$$

for $i = 0, 1, \ldots, n$ and for all positive integers j. Moreover $\lim_{j \to +\infty} |\mathbf{v}_{i,m_j} - \mathbf{v}_{0,m_j}| = 0$ and $\lim_{j \to +\infty} |\mathbf{v}_{0,m_j} - \mathbf{x}^*| = 0$. It follows that $\lim_{j \to +\infty} \mathbf{v}_{i,m_j} = \mathbf{x}^*$ for $i = 0, 1, \ldots, n$. Also

$$\sum_{i=0}^{n} t_i = \lim_{j \to +\infty} \left(\sum_{i=0}^{n} t_{i,m_j} \right) = 1.$$

It follows that

$$\begin{aligned} \lim_{j \to +\infty} \left(\sum_{i=0}^{n} t_{i,m_j} \mathbf{v}_{i,m_j} \right) &= \sum_{i=0}^{n} \left(\lim_{j \to +\infty} t_{i,m_j} \right) \left(\lim_{j \to +\infty} \mathbf{v}_{i,m_j} \right) \\ &= \sum_{i=0}^{n} t_i \mathbf{x}^* = \mathbf{x}^*. \end{aligned}$$

But we have also shown that $\sum_{i=0}^{n} t_{i,j} \mathbf{y}_{i,j} = \sum_{i=0}^{n} t_{i,j} \mathbf{v}_{i,j}$ for all positive integers j. It follows that

$$\sum_{i=0}^{n} t_i \mathbf{y}_i = \lim_{j \to +\infty} \left(\sum_{i=0}^{n} t_{i,m_j} \mathbf{y}_{i,m_j} \right) = \lim_{j \to +\infty} \left(\sum_{i=0}^{n} t_{i,m_j} \mathbf{v}_{i,m_j} \right) = \mathbf{x}^*.$$

Next we show that $\mathbf{y}_i \in \Phi(\mathbf{x}^*)$ for $i = 0, 1, \ldots, n$. Now

$$(\mathbf{v}_{i,m_j}, \mathbf{y}_{i,m_j}) \in \text{Graph}(\Phi)$$

for all positive integers j, and the graph $\text{Graph}(\Phi)$ of the correspondence Φ is closed. It follows that

$$(\mathbf{x}^*, \mathbf{y}_i) = \lim_{j \to +\infty} (\mathbf{v}_{i,m_j}, \mathbf{y}_{i,m_j}) \in \text{Graph}(\Phi)$$

and thus $\mathbf{y}_i \in \Phi(\mathbf{x}^*)$ for $i = 0, 1, \ldots, m$ (see Proposition 2.6). It follows from the convexity of $\Phi(\mathbf{x}^*)$ that

$$\sum_{i=0}^{n} t_i \mathbf{y}_* \in \Phi(\mathbf{x}^*).$$

(see Lemma 3.5). But $\sum_{i=0}^{n} t_i \mathbf{y}_* = \mathbf{x}^*$. It follows that $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$, as required. ∎