Module MA3486: Fixed Point Theorems and Economic Equilibria
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Part I (Sections 1 to 2)

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Contents

1 Review of Basic Results of Analysis in Euclidean Spaces

1.1 Basic Properties of Vectors and Norms

We denote by $\mathbb{R}^n$ the set consisting of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. The set $\mathbb{R}^n$ represents n-dimensional *Euclidean space* (with respect to the standard Cartesian coordinate system). Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad \mathbf{y} = (y_1, y_2, \ldots, y_n),$$

and let λ be a real number. We define

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\
\mathbf{x} - \mathbf{y} &= (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n), \\
\lambda\mathbf{x} &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n), \\
\mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \\
|\mathbf{x}| &= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.
\end{aligned}$$

The quantity $\mathbf{x} \cdot \mathbf{y}$ is the *scalar product* (or *inner product*) of $\mathbf{x}$ and $\mathbf{y}$, and the quantity $|\mathbf{x}|$ is the *Euclidean norm* of $\mathbf{x}$. Note that $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$. The *Euclidean distance* between two points $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^n$ is defined to be the Euclidean norm $|\mathbf{y} - \mathbf{x}|$ of the vector $\mathbf{y} - \mathbf{x}$.

Let $\mathbf{x}$ and $\mathbf{y}$ be elements in $\mathbb{R}^n$, Let $p(t) = |t\mathbf{x} + \mathbf{y}|^2$ for all real numbers t. Then

$$\begin{aligned}
p(t) &= (t\mathbf{x} + \mathbf{y}) \,.\, (t\mathbf{x} + \mathbf{y}) \\
&= t^2|\mathbf{x}|^2 + 2t\mathbf{x} \,.\, \mathbf{y} + |\mathbf{y}|^2
\end{aligned}$$

for all real numbers t. But $p(t) \geq 0$ for all real numbers t. It follows that $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$. This inquality is known as *Schwarz's Inequality.*

Moreover, given any elements $\mathbf{x}$ and $\mathbf{y}$ of $\mathbf{R}^n$,

$$\begin{aligned}
|\mathbf{x} + \mathbf{y}|^2 &= (\mathbf{x} + \mathbf{y}).(\mathbf{x} + \mathbf{y}) = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x} \cdot \mathbf{y} \\
&\leq |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2|\mathbf{x}||\mathbf{y}| = (|\mathbf{x}| + |\mathbf{y}|)^2.
\end{aligned}$$

It follows that $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$. It follows from this inequality that

$$|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|$$

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$. This identity is known as the *Triangle Inequality.* It expresses the geometric result that the length of any side of a triangle in a Euclidean space of any dimension is the sum of the lengths of the other two sides of that triangle.

Definition A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to *converge* to a point $\mathbf{p}$ if and only if the following criterion is satisfied:—

> given any real number ε satisfying $\varepsilon > 0$ there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$.

We refer to $\mathbf{p}$ as the *limit* $\lim\limits_{j \to +\infty} \mathbf{x}_j$ of the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$.

Lemma 1.1 *Let $\mathbf{p}$ be a point of $\mathbb{R}^n$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. Then a sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to $\mathbf{p}$ if and only if the ith components of the elements of this sequence converge to p_i for $i = 1, 2, \ldots, n$.*

A proof of Lemma 1.1 is to be found in Appendix A.

1.2 The Bolzano-Weierstrass Theorem

An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to be *strictly increasing* if $x_{j+1} > x_j$ for all positive integers j, *strictly decreasing* if $x_{j+1} < x_j$ for all positive integers j, *non-decreasing* if $x_{j+1} \geq x_j$ for all positive integers j, *non-increasing* if $x_{j+1} \leq x_j$ for all positive integers j. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

Theorem 1.2 *Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.*

A proof of Theorem 1.2 is to be found in Appendix A.

Definition Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. A *subsequence* of this infinite sequence is a sequence of the form $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$ where $j_1, j_2, j_3, \ldots$ is an infinite sequence of positive integers with

$$j_1 < j_2 < j_3 < \cdots .$$

Theorem 1.3 (Bolzano-Weierstrass in One Dimension)
Every bounded sequence of real numbers has a convergent subsequence.

A proof of Theorem 1.3 is to be found in Appendix A.

Theorem 1.4 (Multidimensional Bolzano-Weierstrass Theorem)
Every bounded sequence of points in a Euclidean space has a convergent subsequence.

A proof of Theorem 1.4 is to be found in Appendix A.

Definition Let X be a subset of $\mathbb{R}^n$. Given a point $\mathbf{p}$ of X and a non-negative real number r, the *open ball* $B_X(\mathbf{p}, r)$ in X of *radius* r about $\mathbf{p}$ is defined to be the subset of X defined so that

$$B_X(\mathbf{p}, r) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < r\}.$$

(Thus $B_X(\mathbf{p}, r)$ is the set consisting of all points of X that lie within a sphere of radius r centred on the point $\mathbf{p}$.)

Definition Let X be a subset of $\mathbb{R}^n$. A subset V of X is said to be *open* in X if, given any point $\mathbf{p}$ of V, there exists some strictly positive real number δ such that $B_X(\mathbf{p}, \delta) \subset V$, where $B_X(\mathbf{p}, \delta)$ is the open ball in X of radius δ about on the point $\mathbf{p}$. The empty set $\emptyset$ is also defined to be an open set in X.

Lemma 1.5 *Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any positive real number r, the open ball $B_X(\mathbf{p}, r)$ in X of radius r about $\mathbf{p}$ is open in X.*

A proof of Lemma 1.5 is to be found in Appendix A.

Proposition 1.6 *Let X be a subset of $\mathbb{R}^n$. The collection of open sets in X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are both open in X;

(ii) the union of any collection of open sets in X is itself open in X;

(iii) the intersection of any finite *collection of open sets in X is itself open in X.*

A proof of Proposition 1.6 is to be found in Appendix A.

Proposition 1.7 *Let X be a subset of $\mathbb{R}^n$, and let U be a subset of X. Then U is open in X if and only if there exists some open set V in $\mathbb{R}^n$ for which $U = V \cap X$.*

A proof of Proposition 1.7 is to be found in Appendix A.

Lemma 1.8 *A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to a point $\mathbf{p}$ if and only if, given any open set U which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in U$ for all j satisfying $j \geq N$.*

A proof of Lemma 1.8 is to be found in Appendix A.

Definition Let X be a subset of $\mathbb{R}^n$. A subset F of X is said to be *closed* in X if and only if its complement $X \setminus F$ in X is open in X. (Recall that $X \setminus F = \{\mathbf{x} \in X : \mathbf{x} \notin F\}$.)

Proposition 1.9 *Let X be a subset of $\mathbb{R}^n$. The collection of closed sets in X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are both closed in X;

(ii) the intersection of any collection of closed sets in X is itself closed in X;

(iii) the union of any finite *collection of closed sets in X is itself closed in X.*

A proof of Proposition 1.9 is to be found in Appendix A.

Lemma 1.10 *Let X be a subset of $\mathbb{R}^n$, and let F be a subset of X which is closed in X. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of F which converges to a point $\mathbf{p}$ of X. Then $\mathbf{p} \in F$.*

A proof of Lemma 1.10 is to be found in Appendix A.

Definition Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A function $f\colon X \to Y$ from X to Y is said to be *continuous* at a point $\mathbf{p}$ of X if and only if the following criterion is satisfied:—

> given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$.

The function $f\colon X \to Y$ is said to be continuous on X if and only if it is continuous at every point $\mathbf{p}$ of X.

Lemma 1.11 *Let X, Y and Z be subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions satisfying $f(X) \subset Y$. Suppose that f is continuous at some point $\mathbf{p}$ of X and that g is continuous at $f(\mathbf{p})$. Then the composition function $g \circ f\colon X \to Z$ is continuous at $\mathbf{p}$.*

A proof of Lemma 1.11 is to be found in Appendix A.

Lemma 1.12 *Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f\colon X \to Y$ be a continuous function from X to Y. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of X which converges to some point $\mathbf{p}$ of X. Then the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$.*

A proof of Lemma 1.12 is to be found in Appendix A.

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f\colon X \to Y$ be a function from X to Y. Then

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

for all $\mathbf{x} \in X$, where $f_1, f_2, \ldots, f_n$ are functions from X to $\mathbb{R}$, referred to as the *components* of the function f.

Proposition 1.13 *Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $\mathbf{p} \in X$. A function $f\colon X \to Y$ is continuous at the point $\mathbf{p}$ if and only if its components are all continuous at $\mathbf{p}$.*

A proof of Proposition 1.13 is to be found in Appendix A.

Proposition 1.14 *Let X be a subset of $\mathbb{R}^n$, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be continuous functions from X to $\mathbb{R}$. Then the functions $f+g$, $f-g$ and $f \cdot g$ are continuous. If in addition $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ then the quotient function f/g is continuous.*

A proof of Proposition 1.14 is to be found in Appendix A.

Lemma 1.15 *Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a continuous function mapping X into $\mathbb{R}^n$, and let $|f|\colon X \to \mathbb{R}$ be defined such that $|f|(\mathbf{x}) = |f(\mathbf{x})|$ for all $\mathbf{x} \in X$. Then the real-valued function $|f|$ is continuous on X.*

A proof of Proposition 1.15 is to be found in Appendix A.

Given any function $f\colon X \to Y$, we denote by $f^{-1}(V)$ the *preimage* of a subset V of Y under the map f, defined by $f^{-1}(V) = \{\mathbf{x} \in X : f(\mathbf{x}) \in V\}$.

Proposition 1.16 *Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f\colon X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(V)$ is open in X for every open subset V of Y.*

A proof of Proposition 1.16 is to be found in Appendix A.

Let X be a subset of $\mathbb{R}^n$, let $f\colon X \to \mathbb{R}$ be continuous, and let c be some real number. Then the sets $\{\mathbf{x} \in X : f(\mathbf{x}) > c\}$ and $\{\mathbf{x} \in X : f(\mathbf{x}) < c\}$ are open in X, and, given real numbers a and b satisfying $a < b$, the set $\{\mathbf{x} \in X : a < f(\mathbf{x}) < b\}$ is open in X.

Corollary 1.17 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a continuous function from X to Y. Then $\varphi^{-1}(F)$ is closed in X for every subset F of Y that is closed in Y.*

Proof Let F be a subset of Y that is closed in Y, and let let $V = Y \setminus F$. Then V is open in Y. It follows from Proposition 1.16 that $\varphi^{-1}(V)$ is open in X. But

$$\varphi^{-1}(V) = \varphi^{-1}(Y \setminus F) = X \setminus \varphi^{-1}(F).$$

Indeed let $\mathbf{x} \in X$. Then

$$\begin{array}{rl} & \mathbf{x} \in \varphi^{-1}(V) \\ \iff & \mathbf{x} \in \varphi^{-1}(Y \setminus F) \\ \iff & \varphi(\mathbf{x}) \in Y \setminus F \\ \iff & \varphi(\mathbf{x}) \notin F \\ \iff & \mathbf{x} \notin \varphi^{-1}(F) \\ \iff & \mathbf{x} \in X \setminus \varphi^{-1}(F). \end{array}$$

It follows that the complement $X \setminus \varphi^{-1}(F)$ of $\varphi^{-1}(F)$ in X is open in X, and therefore $\varphi^{-1}(F)$ itself is closed in X, as required. ∎

Lemma 1.18 *Let X be a closed subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then a subset of X is closed in X if and only if it is closed in $\mathbb{R}^n$.*

Proof Let F be a subset of X. Then F is closed in X if and only if, given any point $\mathbf{p}$ of X for which $\mathbf{p} \notin F$, there exists some strictly positive real number δ such that there is no point of F whose distance from the point $\mathbf{p}$ is less than δ. It follows easily from this that is F is closed in $\mathbb{R}^n$ then F is closed in X.

Conversely suppose that F is closed in X, where X itself is closed in $\mathbb{R}^n$. Let $\mathbf{p}$ be a point of $\mathbb{R}^n$ that satisfies $\mathbf{p} \notin F$. Then either $\mathbf{p} \in X$ or $\mathbf{p} \notin X$.

Suppose that $\mathbf{p} \in X$. Then there exists some strictly positive real number δ such that there is no point of F whose distance from the point $\mathbf{p}$ is less than δ.

Otherwise $\mathbf{p} \notin X$. Then there exists some strictly positive real number δ such that there is no point of X whose distance from the point $\mathbf{p}$ is less than δ, because X is closed in $\mathbb{R}^n$. But $F \subset X$. It follows that there is no point of F whose distance from the point $\mathbf{p}$ is less than δ. We conclude that the set F is closed in $\mathbb{R}^n$, as required. ∎

The following result, together with its generalizations, is sometimes referred to as the *Glueing Lemma*.

Lemma 1.19 (Glueing Lemma) *Let $\varphi\colon X \to \mathbb{R}^n$ be a function mapping a subset X of $\mathbb{R}^m$ into $\mathbb{R}^n$. Let $F_1, F_2, \ldots, F_k$ be a finite collection of subsets of X such that F_i is closed in X for $i = 1, 2, \ldots, k$ and*

$$F_1 \cup F_2 \cup \cdots \cup F_k = X.$$

Then the function φ is continuous on X if and only if the restriction of φ to F_i is continuous on F_i for $i = 1, 2, \ldots, k$.

Proof Suppose that $\varphi\colon X \to \mathbb{R}^n$ is continuous. Then it follows directly from the definition of continuity that the restriction of φ to each subset of X is continuous on that subset. Therefore the restriction of φ to F_i is continuous on F_i for $i = 1, 2, \ldots, k$.

Conversely we must prove that if the restriction of the function φ to F_i is continuous on F_i for $i = 1, 2, \ldots, k$ then the function $\varphi\colon X \to \mathbb{R}^m$ is continuous. Let $\mathbf{p}$ be a point of X, and let some positive real number ε be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots \delta_k$ satisfying the following conditions:—

(i) if $\mathbf{p} \in F_i$, where $1 \leq i \leq k$, and if $\mathbf{x} \in F_i$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_i$ then $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$;

(ii) if $\mathbf{p} \notin F_i$, where $1 \leq i \leq k$, and if $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_i$ then $\mathbf{x} \notin F_i$.

Indeed the continuity of the function φ on each set F_i ensures that δ_i may be chosen to satisfy (i) for each integer i between 1 and k for which $\mathbf{p} \in F_i$. Also the requirement that F_i be closed in X ensures that $X \setminus F_i$ is open in X and therefore δ_i may be chosen to to satisfy (ii) for each integer i between 1 and k for which $\mathbf{p} \notin F_i$.

Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. Let $\mathbf{x} \in X$ satisfy $|\mathbf{x} - \mathbf{p}| < \delta$. If $\mathbf{p} \notin F_i$ then the choice of δ_i ensures that if $\mathbf{x} \notin F_i$. But X is the union of the sets $F_1, F_2, \ldots, F_k$, and therefore there must exist some integer i between 1 and k for which $\mathbf{x} \in F_i$. Then $\mathbf{p} \in F_i$, and the choice of δ_i ensures that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$. We have thus shown that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$ for all points $\mathbf{x}$ of X that satisfy $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $\varphi\colon X \to \mathbb{R}^n$ is continuous, as required. ∎

1.3 The Extreme Value Theorem

We use the Bolzano-Weierstrass Theorem in order to prove the following important result.

Theorem 1.20 (The Multidimensional Extreme Value Theorem) *Let X be a closed bounded set in n-dimensional Euclidean space, and let $f\colon X \to \mathbb{R}$ be a continuous real-valued function defined on X. Then there exist points $\mathbf{u}$ and $\mathbf{v}$ of X such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.*

Proof We first prove that if $f\colon X \to \mathbb{R}$ is a *bounded* continuous real-valued function on X then f attains a maximum and a minimum value on the set X. We then apply this result to show that all continuous real-valued functions on X are bounded. It will then follow that all continuous real-valued functions on X attain a maximum and a minimum value on the set X.

Thus suppose that $f\colon X \to \mathbb{R}$ is a bounded continuous real-valued function on the closed bounded set X. Then the set

$$\{f(\mathbf{x}) : \mathbf{x} \in X\}$$

of values of the function f is a bounded non-empty set and thus has a least upper bound M. There then exists an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X such that $f(\mathbf{x}_j) > M - 1/j$ for all positive integers j. The infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is a bounded sequence, because it is contained in the bounded set X. It follows from the Bolzano-Weierstrass Theorem Theorem 1.4 that the infinite sequence has a subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ that converges to some point $\mathbf{v}$ of $\mathbb{R}^m$. But then $\mathbf{v} \in X$, because the set X is closed (Lemma 1.10). But the continuity of f then ensures that $M = \lim\limits_{j \to +\infty} f(\mathbf{x}_{k_j}) = f(\mathbf{v})$. Therefore $f(\mathbf{x}) \leq f(\mathbf{v})$ for all points $\mathbf{x}$ of X.

Applying this result with f replaced by $-f$, we deduce also that there exists some point $\mathbf{u}$ of X with the property that $f(\mathbf{x}) \geq f(\mathbf{u})$ for all points $\mathbf{x}$ of X. We have thus shown that if $f\colon X \to \mathbb{R}$ is both continuous and bounded, and if the set X is both closed and bounded, then there exist points $\mathbf{u}$ and $\mathbf{v} \in X$ such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.

Now let $f\colon X \to \mathbb{R}$ be any continuous real-valued function on X, and let $g\colon X \to \mathbb{R}$ be defined such that

$$g(\mathbf{x}) = \frac{1}{1 + f(\mathbf{x})^2}$$

for all $\mathbf{x} \in X$. Then the function g is both continuous on X, and $0 < g(\mathbf{x}) \leq 1$ for all $\mathbf{x} \in X$. It follows from the result already obtained that there exists some point $\mathbf{w}$ of X such that $g(\mathbf{x}) \geq g(\mathbf{w})$ for all $\mathbf{x} \in X$. Moreover $g(\mathbf{w}) > 0$. Let K be a positive constant chosen large enough to ensure that $1/K^2 < g(\mathbf{w})$. Then $-K < f(\mathbf{x} < K$ for all points $\mathbf{x}$ of X. The function f is thus bounded in X. The general result therefore follows from the result already proved under the assumption that the function is both continuous and bounded. ∎

1.4 Lebesgue Numbers

Definition Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. A collection of subsets of $\mathbb{R}^n$ is said to *cover* X if and only if every point of X belongs to at least one of these subsets.

Definition Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. An *open cover* of X is a collection of subsets of X that are open in X and cover the set X.

Proposition 1.21 *Let X be a closed bounded set in n-dimensional Euclidean space, and let $\mathcal{V}$ be an open cover of X. Then there exists a positive real number δ_L with the property that, given any point $\mathbf{u}$ of X, there exists a member V of the open cover $\mathcal{V}$ for which*

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta_L\} \subset V.$$

Proof Let

$$B_X(\mathbf{u}, \delta) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\}$$

for all $\mathbf{u} \in X$ and for all positive real numbers δ. Suppose that there did not exist any positive real number δ_L with the stated property.

Then, given any positive number δ, there would exist a point $\mathbf{u}$ of X for which the ball $B_X(\mathbf{u}, \delta)$ would not be wholly contained within any open set V belonging to the open cover $\mathcal{V}$. Then

$$B_X(\mathbf{u}, \delta) \cap (X \setminus V) \neq \emptyset$$

for all members V of the open cover $\mathcal{V}$. There would therefore exist an infinite sequence

$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$$

of points of X with the property that, for all positive integers j, the open ball

$$B_X(\mathbf{u}_j, 1/j) \cap (X \setminus V) \neq \emptyset$$

for all members V of the open cover $\mathcal{V}$.

The sequence

$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$$

would be bounded, because the set X is bounded. It would then follow from the multidimensional Bolzano-Weierstrass Theorem (Theorem 1.4) that there would exist a convergent subsequence

$$\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \mathbf{u}_{j_3}, \ldots$$

of

$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots.$$

Let $\mathbf{p}$ be the limit of this convergent subsequence. Then the point $\mathbf{p}$ would then belong to X, because X is closed (see Lemma 1.10). But then the point $\mathbf{p}$ would belong to an open set V belonging to the open cover $\mathcal{V}$. It would then follow from the definition of open sets that there would exist a positive real number δ for which $B_X(\mathbf{p}, 2\delta) \subset V$. Let $j = j_k$ for a positive integer k large enough to ensure that both $1/j < \delta$ and $\mathbf{u}_j \in B_X(\mathbf{p}, \delta)$. The Triangle Inequality would then ensure that every point of X within a distance $1/j$ of the point $\mathbf{u}_j$ would lie within a distance 2δ of the point $\mathbf{p}$, and therefore

$$B_X(\mathbf{u}_j, 1/j) \subset B_X(\mathbf{p}, 2\delta) \subset V.$$

But $B(\mathbf{u}_j, 1/j) \cap (X \setminus V) \neq \emptyset$ for all members V of the open cover $\mathcal{V}$, and therefore it would not be possible for this open set to be contained in the open set V. Thus the assumption that there is no positive number δ_L with the required property has led to a contradiction. Therefore there must exist some positive number δ_L with the property that, for all $\mathbf{u} \in X$ the open ball $B_X(\mathbf{u}, \delta_L)$ in X is contained wholly within at least one open set belonging to the open cover $\mathcal{V}$, as required. ∎

Definition Let X be a subset of n-dimensional Euclidean space, and let $\mathcal{V}$ be an open cover of X. A positive real number δ_L is said to be a *Lebesgue number* for the open cover $\mathcal{V}$ if, given any point $\mathbf{p}$ of X, there exists some member V of the open cover $\mathcal{V}$ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta_L\} \subset V.$$

Proposition 1.21 ensures that, given any open cover of a closed bounded subset of n-dimensional Euclidean space, there exists a positive real number that is a Lebesgue number for that open cover.

Definition The *diameter* $\operatorname{diam}(A)$ of a bounded subset A of n-dimensional Euclidean space is defined so that

$$\operatorname{diam}(A) = \sup\{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in A\}.$$

It follows from this definition that $\operatorname{diam}(A)$ is the smallest real number K with the property that $|\mathbf{x} - \mathbf{y}| \leq K$ for all $\mathbf{x}, \mathbf{y} \in A$.

Lemma 1.22 *Let X be a bounded subset of n-dimensional Euclidean space, and let δ be a positive real number. Then there exists a finite collection $A_1, A_2, \ldots, A_s$ of subsets of X such that the* $\operatorname{diam}(A_i) < \delta$ *for $i = 1, 2, \ldots, s$ and*

$$X = A_1 \cup A_2 \cup \cdots \cup A_k.$$

Proof Let b be a real number satisfying $0 < \sqrt{n}\, b < \delta$ and, for each n-tuple $(j_1, j_2, \ldots, j_n)$ of integers, let $H_{(j_1, j_2, \ldots, j_n)}$ denote the hypercube in $\mathbb{R}^n$ defined such that

$$\begin{aligned} & H_{(j_1, j_2, \ldots, j_n)} \\ & \qquad = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : j_i b \leq x_i \leq (j_i + 1) b \text{ for } i = 1, 2, \ldots, n\}. \end{aligned}$$

Note that if $\mathbf{u}$ and $\mathbf{v}$ are points of $H_{(j_1, j_2, \ldots, j_n)}$ for some n-tuple $(j_1, j_2, \ldots, j_n)$ of integers then $|u_i - v_i| < b$ for $i = 1, 2, \ldots, n$, and therefore $|\mathbf{u} - \mathbf{v}| \leq \sqrt{n}\, b < \delta$. Therefore the diameter of each hypercube $H_{(j_1, j_2, \ldots, j_n)}$ is less than δ.

The boundedness of the set X ensures that there are only finitely many n-tuples $(j_1, j_2, \ldots, j_n)$ of integers for which $X \cap H_{(j_1, j_2, \ldots, j_n)}$ is non-empty. It follows that X is covered by a finite collection $A_1, A_2, \ldots, A_k$ of subsets of X, where each of these subsets is of the form $X \cap H_{(j_1, j_2, \ldots, j_n)}$ for some n-tuple $(j_1, j_2, \ldots, j_n)$ of integers. These subsets all have diameter less than δ. The result follows. ∎

Definition Let $\mathcal{V}$ and $\mathcal{W}$ be open covers of some subset X of a Euclidean space. Then $\mathcal{W}$ is said to be a *subcover* of $\mathcal{V}$ if and only if every open set belonging to $\mathcal{W}$ also belongs to $\mathcal{V}$.

Definition A subset X of a Euclidean space is said to be *compact* if and only if every open cover of X possesses a finite subcover.

Theorem 1.23 (The Multidimensional Heine-Borel Theorem) *A subset of n-dimensional Euclidean space $\mathbb{R}^n$ is compact if and only if it is both closed and bounded.*

Proof Let X be a compact subset of $\mathbb{R}^n$ and let

$$V_j = \{\mathbf{x} \in X : |\mathbf{x}| < j\}$$

for all positive integers j. Then the sets $V_1, V_2, V_3, \ldots$ constitute an open cover of X. This open cover has a finite subcover, and therefore there exist positive integers $j_1, j_2, \ldots, j_k$ such that

$$X \subset V_{j_1} \cup V_{j_2} \cup \cdots \cup V_{j_k}.$$

Let M be the largest of the positive integers $j_1, j_2, \ldots, j_k$. Then $|\mathbf{x}| \leq M$ for all $\mathbf{x} \in X$. Thus the set X is bounded.

Let $\mathbf{q}$ be a point of $\mathbb{R}^n$ that does not belong to X, and let

$$W_j = \left\{ \mathbf{x} \in X : |\mathbf{x} - \mathbf{q}| > \frac{1}{j} \right\}$$

for all positive integers j. Then the sets $W_1, W_2, W_3, \ldots$ constitute an open cover of X. This open cover has a finite subcover, and therefore there exist positive integers $j_1, j_2, \ldots, j_k$ such that

$$X \subset W_{j_1} \cup W_{j_2} \cup \cdots \cup W_{j_k}.$$

Let $\delta = 1/M$, where M is the largest of the positive integers $j_1, j_2, \ldots, j_k$. Then $|\mathbf{x} - \mathbf{q}| \geq \delta$ for all $\mathbf{x} \in X$ and thus the open ball of radius δ about the point $\mathbf{q}$ does not intersect the set X. We conclude that the set X is closed. We have now shown that every compact subset of $\mathbb{R}^n$ is both closed and bounded.

We now prove the converse. Let X be a closed bounded subset of $\mathbb{R}^n$, and let $\mathcal{V}$ be an open cover of X. It follows from Proposition 1.21 that there exists a Lebesgue number δ_L for the open cover $\mathcal{V}$. It then follows from Lemma 1.22 that there exist subsets $A_1, A_2, \ldots, A_s$ of X such that $\operatorname{diam}(A_i) < \delta_L$ for $i = 1, 2, \ldots, s$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_s.$$

We may suppose that A_i is non-empty for $i = 1, 2, \ldots, s$ (because if $A_i = \emptyset$ then A_i could be deleted from the list). Choose $\mathbf{p}_i \in A_i$ for $i = 1, 2, \ldots, s$. Then $A_i \subset B_X(\mathbf{p}_i, \delta_L)$ for $i = 1, 2, \ldots, s$. The definition of the Lebesgue number δ_L then ensures that there exist members $V_1, V_2, \ldots, V_s$ of the open cover $\mathcal{V}$ such that $B_X(\mathbf{p}_i, \delta_L) \subset V_i$ for $i = 1, 2, \ldots, s$. Then $A_i \subset V_i$ for $i = 1, 2, \ldots, s$, and therefore

$$X \subset V_1 \cup V_2 \cup \cdots \cup V_s.$$

Thus $V_1, V_2, \ldots, V_s$ constitute a finite subcover of the open cover $\mathcal{U}$. We have therefore proved that every closed bounded subset of n-dimensional Euclidean space is compact, as required. ∎

2 Correspondences and Hemicontinuity

2.1 Correspondences

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A *correspondence* $\Phi\colon X \rightrightarrows Y$ assigns to each point $\mathbf{x}$ of X a subset $\Phi(\mathbf{x})$ of Y.

The *power set* $\mathcal{P}(Y)$ of Y is the set whose elements are the subsets of Y. A correspondence $\Phi\colon X \rightrightarrows Y$ may be regarded as a function from X to $\mathcal{P}(Y)$.

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Then the following definitions apply:–

- the correspondence $\Phi\colon X \to Y$ is said to be *non-empty-valued* if $\Phi(\mathbf{x})$ is a non-empty subset of Y for all $\mathbf{x} \in X$;
- the correspondence $\Phi\colon X \to Y$ is said to be *closed-valued* if $\Phi(\mathbf{x})$ is a closed subset of Y for all $\mathbf{x} \in X$;
- the correspondence $\Phi\colon X \to Y$ is said to be *compact-valued* if $\Phi(\mathbf{x})$ is a compact subset of Y for all $\mathbf{x} \in X$.

The multidimensional Heine-Borel Theorem (Theorem 1.23) ensures that the correspondence $\Phi\colon X \to Y$ is compact-valued if and only if $\Phi(\mathbf{x})$ is a closed bounded subset of $\mathbb{R}^m$ for all $\mathbf{x} \in X$.

Definition Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is said to be *upper hemicontinuous* at a point $\mathbf{p}$ of X if, given any set V in Y that is open in Y and satisfies $\Phi(\mathbf{p}) \subset V$, there exists some positive real number δ such that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. The correspondence Φ is upper hemicontinuous on X if it is upper hemicontinuous at each point of X.

Example Let $F\colon \mathbb{R} \rightrightarrows \mathbb{R}$ and $G\colon \mathbb{R} \rightrightarrows \mathbb{R}$ be the correspondences from $\mathbb{R}$ to $\mathbb{R}$ defined such that

$$F(x) = \begin{cases} [1,2] & \text{if } x < 0, \\ [0,3] & \text{if } x \geq 0, \end{cases}$$

and

$$G(x) = \begin{cases} [1,2] & \text{if } x \leq 0, \\ [0,3] & \text{if } x > 0, \end{cases}$$

The correspondences F and G are upper hemicontinuous at x for all non-zero real numbers x. The correspondence F is also upper hemicontinuous at 0, for if V is an open set in $\mathbb{R}$ and if $F(0) \subset V$ then $[0,3] \subset V$ and therefore $F(x) \in V$ for all real numbers x.

However the correspondence G is not upper hemicontinuous at 0. Indeed let

$$V = \{y \in \mathbb{R} : \tfrac{1}{2} < y < \tfrac{5}{2}\}.$$

Then $G(0) \subset V$, but $G(x)$ is not contained in V for any positive real number x. Therefore there cannot exist any positive real number δ such that $G(x) \subset V$ whenever $|x| < \delta$.

Let

$$\mathrm{Graph}(F) = \{(x,y) \in \mathbb{R}^2 : y \in F(x)\}$$

and

$$\mathrm{Graph}(G) = \{(x,y) \in \mathbb{R}^2 : y \in G(x)\}.$$

Then $\mathrm{Graph}(F)$ is a closed subset of $\mathbb{R}^2$ but $\mathrm{Graph}(G)$ is not a closed subset of $\mathbb{R}^2$.

Example Let S^1 be the unit circle in $\mathbb{R}^2$, defined such that

$$S^1 = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 = 1\},$$

let Z be the closed square with corners at $(1,1)$, $(-1,1)$, $(-1,-1)$ and $(1,-1)$, so that

$$Z = \{(x,y) \in \mathbb{R}^2 : -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1\}.$$

Let $g_{(u,v)} \colon \mathbb{R}^2 \to \mathbb{R}$ be defined for all $(u,v) \in S^1$ such that

$$g_{(u,v)}(x,y) = ux + vy,$$

and let $\Phi \colon S^1 \rightrightarrows \mathbb{R}^2$ be defined such that, for all $(u,v) \in S^1$, $\Phi(u,v)$ is the subset of $\mathbb{R}^2$ consisting of the point of points of Z at which the linear functional $g_{(u,v)}$ attains its maximum value on Z. Thus a point (x,y) of Z belongs to $\Phi(u,v)$ if and only if $g_{(u,v)}(x,y) \geq g_{(u,v)}(x',y')$ for all $(x',y') \in Z$. Then

$$\Phi(u,v) = \begin{cases} \{(1,1)\} & \text{if } u > 0 \text{ and } v > 0; \\ \{(x,1) : -1 \leq x \leq 1\} & \text{if } u = 0 \text{ and } v > 0; \\ \{(-1,1)\} & \text{if } u < 0 \text{ and } v > 0; \\ \{(-1,y) : -1 \leq y \leq 1\} & \text{if } u < 0 \text{ and } v = 0; \\ \{(-1,-1)\} & \text{if } u < 0 \text{ and } v < 0; \\ \{(x,-1) : -1 \leq x \leq 1\} & \text{if } u = 0 \text{ and } v < 0; \\ \{(1,-1)\} & \text{if } u > 0 \text{ and } v < 0; \\ \{(1,y) : -1 \leq y \leq 1\} & \text{if } u > 0 \text{ and } v = 0. \end{cases}$$

It is a straightforward exercise to verify that the correspondence $\Phi \colon S^1 \rightrightarrows \mathbb{R}^2$ is upper hemicontinuous.

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence between X and Y. Given any subset V of Y, we denote by $\Phi^+(V)$ the subset of X defined such that

$$\Phi^+(V) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V\}.$$

Lemma 2.1 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous on X if and only if, given any set V in Y that is open in Y, the set $\Phi^+(V)$ is open in X.*

Proof First suppose that $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at each point of X. Let V be an open set in Y and let $\mathbf{p} \in \Phi^+(V)$. Then $\Phi(\mathbf{p}) \subset V$. It then follows from the definition of upper hemicontinuity that there exists some positive real number δ such that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then $\mathbf{x} \in \Phi^+(V)$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $\Phi^+(V)$ is open in X.

Conversely suppose that $\Phi\colon X \rightrightarrows Y$ is a correspondence with the property that, for all subsets V of Y that are open in Y, $\Phi^+(V)$ is open in X. Let $\mathbf{p} \in X$, and let V be an open set in Y satisfying $\Phi(\mathbf{p}) \subset V$. Then $\Phi^+(V)$ is open in X and $\mathbf{p} \in \Phi^+(V)$, and therefore there exists some positive number δ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} \subset \Phi^+(V).$$

Then $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$. The result follows. ∎

Definition Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is said to be *lower hemicontinuous* at a point $\mathbf{p}$ of X if, given any set V in Y that is open in Y and satisfies $\Phi(\mathbf{p}) \cap V \neq \emptyset$, there exists some positive real number δ such that $\Phi(\mathbf{x}) \cap V \neq \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. The correspondence Φ is lower hemicontinuous on X if it is lower hemicontinuous at each point of X.

Example Let $F\colon \mathbb{R} \rightrightarrows \mathbb{R}$ and $G\colon \mathbb{R} \rightrightarrows \mathbb{R}$ be the correspondences from $\mathbb{R}$ to $\mathbb{R}$ defined such that

$$F(x) = \begin{cases} [1,2] & \text{if } x < 0, \\ [0,3] & \text{if } x \geq 0, \end{cases}$$

and

$$G(x) = \begin{cases} [1,2] & \text{if } x \leq 0, \\ [0,3] & \text{if } x > 0, \end{cases}$$

The correspondences F and G are lower hemicontinuous at x for all non-zero real numbers x. The correspondence G is also lower hemicontinuous at 0, for if V is an open set in $\mathbb{R}$ and if $G(0) \cap V \neq \emptyset$ then $[1,2] \cap V \neq \emptyset$ and therefore $G(x) \cap V \neq \emptyset$ for all real numbers x.

However the correspondence F is not lower hemicontinuous at 0. Indeed let

$$V = \{y \in \mathbb{R} : 0 < y < \tfrac{1}{2}\}.$$

Then $F(0) \cap V \neq \emptyset$, but $F(x) \cap V = \emptyset$ for all negative real numbers x. Therefore there cannot exist any positive real number δ such that $F(x) \cap V \neq \emptyset$ whenever $|x| < \delta$.

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence between X and Y. Given any subset V of Y, we denote by $\Phi^-(V)$ the subset of X defined such that

$$\Phi^-(V) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \cap V \neq \emptyset\}.$$

Lemma 2.2 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is lower hemicontinuous on X if and only if, given any set V in Y that is open in Y, the set $\Phi^-(V)$ is open in X.*

Proof First suppose that $\Phi\colon X \rightrightarrows Y$ is lower hemicontinuous at each point of X. Let V be an open set in Y and let $\mathbf{p} \in \Phi^-(V)$. Then $\Phi(\mathbf{p}) \cap V$ is non-empty. It then follows from the definition of lower hemicontinuity that there exists some positive real number δ such that $\Phi(\mathbf{x}) \cap V$ is non-empty for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then $\mathbf{x} \in \Phi^-(V)$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $\Phi^-(V)$ is open in X.

Conversely suppose that $\Phi\colon X \rightrightarrows Y$ is a correspondence with the property that, for all subsets V of Y that are open in Y, $\Phi^-(V)$ is open in X. Let $\mathbf{p} \in X$, and let V be an open set in Y satisfying $\Phi(\mathbf{p}) \cap V \neq \emptyset$. Then $\Phi^-(V)$ is open in X and $\mathbf{p} \in \Phi^-(V)$, and therefore there exists some positive number δ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} \subset \Phi^-(V).$$

Then $\Phi(\mathbf{x}) \cap V \neq \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus $\Phi\colon X \rightrightarrows Y$ is lower hemicontinuous at $\mathbf{p}$. The result follows. ∎

Definition Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is said to be *continuous* at a point $\mathbf{p}$ of X if it is both upper hemicontinuous and lower hemicontinuous at $\mathbf{p}$. The correspondence Φ is continuous on X if it is continuous at each point of X.

Lemma 2.3 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $\varphi\colon X \to Y$ be a function from X to Y, and let $\Phi\colon X \rightrightarrows Y$ be the correspondence defined such that $\Phi(\mathbf{x}) = \{\varphi(\mathbf{x})\}$ for all $\mathbf{x} \in X$. Then $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous if and only if $\varphi\colon X \to Y$ is continuous. Similarly $\Phi\colon X \rightrightarrows Y$ is lower hemicontinuous if and only if $\varphi\colon X \to Y$ is continuous.*

Proof The function $\varphi\colon X \to Y$ is continuous if and only if

$$\{\mathbf{x} \in X : \varphi(\mathbf{x}) \in V\}$$

is open in X for all subsets V of Y that are open in Y (see Proposition 1.16). Let V be a subset of Y that is open in Y. Then $\Phi(\mathbf{x}) \subset V$ if and only if $\varphi(\mathbf{x}) \in V$. Also $\Phi(\mathbf{x}) \cap V \neq \emptyset$ if and only if $\varphi(\mathbf{x}) \in V$. The result therefore follows from the definitions of upper and lower hemicontinuity. ∎

2.2 The Graph of a Correspondence

Let m and n be integers. Then the Cartesian product $\mathbb{R}^n \times \mathbb{R}^m$ of the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ of dimensions n and m is itself a Euclidean space of dimension $n + m$ whose Euclidean norm is characterized by the property that

$$|(\mathbf{x}, \mathbf{y})|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2$$

for all $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$.

Lemma 2.4 *Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ and $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ be infinite sequences of points in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\mathbf{p} \in \mathbb{R}^n$ and $\mathbf{q} \in \mathbb{R}^m$. Then the infinite sequence*

$$(\mathbf{x}_1, \mathbf{y}_1),\ (\mathbf{x}_2, \mathbf{y}_2),\ (\mathbf{x}_3, \mathbf{y}_3), \ldots$$

converges in $\mathbb{R}^n \times \mathbb{R}^m$ to the point $(\mathbf{p}, \mathbf{q})$ if and only if the infinite sequence Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to the point $\mathbf{p}$ and the infinite sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ converges to the point $\mathbf{q}$.

Proof Suppose that the infinite sequence

$$(\mathbf{x}_1, \mathbf{y}_1),\ (\mathbf{x}_2, \mathbf{y}_2),\ (\mathbf{x}_3, \mathbf{y}_3), \ldots$$

converges in $\mathbb{R}^n \times \mathbb{R}^m$ to the point $(\mathbf{p}, \mathbf{q})$. Let some strictly positive real number ε be given. Then there exists some positive integer N such that

$$|\mathbf{x}_j - \mathbf{p}|^2 + |\mathbf{y}_j - \mathbf{q}|^2 < \varepsilon^2$$

whenever $j \geq N$. But then

$$|\mathbf{x}_j - \mathbf{p}| < \varepsilon \quad \text{and} \quad |\mathbf{y}_j - \mathbf{q}| < \varepsilon$$

whenever $j \geq N$. It follows that $\mathbf{x}_j \to \mathbf{p}$ and $\mathbf{y}_j \to \mathbf{q}$ as $j \to +\infty$.

Conversely suppose that $\mathbf{x}_j \to \mathbf{p}$ and $\mathbf{y}_j \to \mathbf{q}$ as $j \to +\infty$. Let some positive real number ε be given. Then there exist positive integers N_1 and N_2 such that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon/\sqrt{2}$ whenever $j \geq N_1$ and $|\mathbf{y}_j - \mathbf{q}| < \varepsilon/\sqrt{2}$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. Then

$$|\mathbf{x}_j - \mathbf{p}|^2 + |\mathbf{y}_j - \mathbf{q}|^2 < \varepsilon^2$$

whenever $j \geq N$. It follows that $(\mathbf{x}_j, \mathbf{y}_j) \to (\mathbf{p}, \mathbf{q})$ as $j \to +\infty$, as required. ∎

Lemma 2.5 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let V be a subset of $X \times Y$. Then V is open in $X \times Y$ if and only if, given any point $(\mathbf{p}, \mathbf{q})$ of V, where $\mathbf{p} \in X$ and $\mathbf{q} \in Y$, there exist subsets W_X and W_Y of X and Y respectively such that $\mathbf{p} \in W_X$, $\mathbf{q} \in W_Y$, W_X is open in X, W_Y is open in Y and $W_X \times W_Y \subset V$.*

Proof Let V be a subset of $X \times Y$ and let $(\mathbf{p}, \mathbf{q}) \in V$, where $\mathbf{p} \in X$ and $\mathbf{q} \in Y$.

Suppose that V is open in $X \times Y$. Then there exists a positive real number δ such that $(\mathbf{x}, \mathbf{y}) \in V$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying

$$|\mathbf{x} - \mathbf{p}|^2 + |\mathbf{y} - \mathbf{q}|^2 < \delta^2.$$

Let

$$W_X = \left\{ \mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \frac{\delta}{\sqrt{2}} \right\}$$

and

$$W_Y = \left\{ \mathbf{y} \in Y : |\mathbf{y} - \mathbf{q}| < \frac{\delta}{\sqrt{2}} \right\}$$

If $\mathbf{x} \in W_X$ and $\mathbf{y} \in W_Y$ then

$$|\mathbf{x} - \mathbf{p}|^2 + |\mathbf{y} - \mathbf{q}|^2 < 2 \left(\frac{\delta}{\sqrt{2}} \right)^2 = \delta^2$$

and therefore $(\mathbf{x}, \mathbf{y}) \in V$. It follows that $W_X \times W_Y \subset V$.

Conversely suppose that there exist open sets W_X and W_Y in X and Y respectively such that $\mathbf{p} \in W_X$, $\mathbf{q} \in W_Y$ and $W_X \times W_Y \subset V$. Then there exists some positive real number δ such that $\mathbf{x} \in W_X$ for all $\mathbf{x} \in X$ satisfying

$|\mathbf{x}-\mathbf{p}| < \delta$ and also $\mathbf{y} \in W_Y$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y}-\mathbf{q}| < \delta$. If $(\mathbf{x}, \mathbf{y})$ is a point of $X \times Y$ that lies within a distance δ of $(\mathbf{p}, \mathbf{q})$ then $|\mathbf{x}-\mathbf{p}| < \delta$ and $|\mathbf{y}-\mathbf{q}| < \delta$, and therefore $(\mathbf{x}, \mathbf{y}) \in W_X \times W_Y$. But $W_X \times W_Y \subset V$. It follows that the open ball of radius δ about the point $(\mathbf{p}, \mathbf{q})$ is wholly contained within the subset V of $X \times Y$. The result follows. ∎

Proposition 2.6 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let G be a subset of $X \times Y$. Then G is closed in $X \times Y$ if and only if*

$$(\lim_{j\to\infty} \mathbf{x}_j, \lim_{j\to\infty} \mathbf{y}_j) \in G$$

for all convergent infinite sequences $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ in X and for all convergent infinite sequences $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ in Y with the property that $(\mathbf{x}_j, \mathbf{y}_j) \in G$ for all positive integers j.

Proof Suppose that G is closed in $X \times Y$. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be an infinite sequence in X converging to some point $\mathbf{p}$ of X and let $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ be an infinite sequence in Y converging to a point $\mathbf{q}$ of Y, where $(\mathbf{x}_j, \mathbf{y}_j) \in G$ for all positive integers j. We must prove that $(\mathbf{p}, \mathbf{q}) \in G$. Now the infinite sequence consisting of the ordered pairs $(\mathbf{x}_j, \mathbf{y}_j)$ converges in $X \times Y$ to $(\mathbf{p}, \mathbf{q})$ (see Lemma 2.4). Now every infinite sequence contained in G that converges to a point of $X \times Y$ must converge to a point of G, because G is closed in $X \times Y$ (see Lemma 1.10). It follows that $(\mathbf{p}, \mathbf{q}) \in G$.

Now suppose that G is not closed in $X \times Y$. Then the complement of G in $X \times Y$ is not open, and therefore there exists a point $(\mathbf{p}, \mathbf{q})$ of $X \times Y$ that does not belong to G though every open ball of positive radius about the point $(\mathbf{p}, \mathbf{q})$ intersects G. It follows that, given any positive integer j, the open ball of radius $1/j$ about the point $(\mathbf{p}, \mathbf{q})$ intersects G and therefore there exist $\mathbf{x}_j \in X$ and $\mathbf{y}_j \in Y$ for which $|\mathbf{x}_j - \mathbf{p}| < 1/j$, $|\mathbf{y}_j - \mathbf{q}| < 1/j$ and $(\mathbf{x}_j, \mathbf{y}_j) \in G$. Then $\lim_{j\to+\infty} \mathbf{x}_j = \mathbf{p}$ and $\lim_{j\to+\infty} \mathbf{y}_j = \mathbf{q}$ and therefore

$$(\lim_{j\to\infty} \mathbf{x}_j, \lim_{j\to\infty} \mathbf{y}_j) \notin G.$$

The result follows. ∎

Definition Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a function from X and Y. The *graph* $\mathrm{Graph}(\varphi)$ of the function φ is the subset of $\mathbb{R}^n \times \mathbb{R}^m$ defined so that

$$\mathrm{Graph}(\varphi) = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \mathbf{x} \in X \text{ and } \mathbf{y} = \varphi(\mathbf{x})\}.$$

Definition Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence between X and Y. The *graph* $\mathrm{Graph}(\Phi)$ of the correspondence Φ is the subset of $\mathbb{R}^n \times \mathbb{R}^m$ defined so that

$$\mathrm{Graph}(\Phi) = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \mathbf{x} \in X \text{ and } \mathbf{y} \in \Phi(\mathbf{x})\}.$$

Lemma 2.7 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a function from X to Y. Suppose that $\varphi\colon X \to Y$ is continuous. Then the graph $\mathrm{Graph}(\varphi)$ of the function φ is closed in $X \times Y$.*

Proof Let $\psi\colon X \times Y \to Y$ be the function defined such that

$$\psi(\mathbf{x}, \mathbf{y}) = \mathbf{y} - \varphi(\mathbf{x})$$

for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$. Then $\mathrm{Graph}(\varphi) = \psi^{-1}(\{\mathbf{0}\})$, and $\{\mathbf{0}\}$ is closed in $\mathbb{R}^m$. It follows that $\mathrm{Graph}(\varphi)$ is closed in $X \times Y$ (see Corollary 1.17). ∎

Example Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined such that

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

Then the graph $\mathrm{Graph}(f)$ of the function f satisfies $\mathrm{Graph}(f) = Z \cup H$, where

$$Z = \{(x, y) \in \mathbb{R}^2 : x \leq 0 \text{ and } y = 0\}$$

and

$$H = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } xy = 1\}.$$

Each of the sets Z and H is a closed set in $\mathbb{R}^2$. It follows that $\mathrm{Graph}(f)$ is a closed set in $\mathbb{R}^2$. However the function $f\colon \mathbb{R} \to \mathbb{R}$ is not continuous at 0.

Lemma 2.8 *Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$, let S be a non-empty subset of X, and let*

$$d(\mathbf{x}, S) = \inf\{|\mathbf{x} - \mathbf{s}| : \mathbf{s} \in S\}$$

for all $\mathbf{x} \in X$. Then the function sending $\mathbf{x}$ to $d(\mathbf{x}, S)$ for all $\mathbf{x} \in X$ is a continuous function on X.

Proof Let $f(\mathbf{x}) = d(\mathbf{x}, S) = \inf\{|\mathbf{x} - \mathbf{s}| : \mathbf{s} \in S\}$ for all $\mathbf{x} \in X$.

Let $\mathbf{x}$ and $\mathbf{x}'$ be points of X. It follows from the Triangle Inequality that

$$f(\mathbf{x}) \leq |\mathbf{x} - \mathbf{s}| \leq |\mathbf{x} - \mathbf{x}'| + |\mathbf{x}' - \mathbf{s}|$$

for all $\mathbf{s} \in S$, and therefore

$$|\mathbf{x}' - \mathbf{s}| \geq f(\mathbf{x}) - |\mathbf{x} - \mathbf{x}'|$$

for all $\mathbf{s} \in S$. Thus $f(\mathbf{x}) - |\mathbf{x} - \mathbf{x}'|$ is a lower bound for the quantities $|\mathbf{x}' - \mathbf{s}|$ as $\mathbf{s}$ ranges over the set S, and therefore cannot exceed the greatest lower bound of these quantities. It follows that

$$f(\mathbf{x}') = \inf\{|\mathbf{x}' - \mathbf{s}| : \mathbf{s} \in S\} \geq f(\mathbf{x}) - |\mathbf{x} - \mathbf{x}'|,$$

and thus

$$f(\mathbf{x}) - f(\mathbf{x}') \leq |\mathbf{x} - \mathbf{x}'|.$$

Interchanging $\mathbf{x}$ and $\mathbf{x}'$, it follows that

$$f(\mathbf{x}') - f(\mathbf{x}) \leq |\mathbf{x} - \mathbf{x}'|.$$

Thus

$$|f(\mathbf{x}) - f(\mathbf{x}')| \leq |\mathbf{x} - \mathbf{x}'|$$

for all $\mathbf{x}, \mathbf{x}' \in X$. It follows that the function $f\colon X \to \mathbb{R}$ is continuous, as required. ∎

The multidimensional Heine-Borel Theorem (Theorem 1.23) ensures that a subset of a Euclidean space is compact if and only if it is both closed and bounded.

Proposition 2.9 *Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$, let V be a subset of X that is open in X, and let K be a compact subset of $\mathbb{R}^n$ satisfying $K \subset V$. Then there exists some positive real number ε with the property that $B_X(K, \varepsilon) \subset V$, where $B_X(K, \varepsilon)$ denotes the subset of X consisting of those points of X that lie within a distance less than ε of some point of K.*

Proof of Proposition 2.9 using the Extreme Value Theorem Let $f\colon K \to \mathbb{R}$ be defined such that

$$f(\mathbf{x}) = \inf\{|\mathbf{z} - \mathbf{x}| : \mathbf{z} \in X \setminus V\}.$$

for all $\mathbf{x} \in K$. It follows from Lemma 2.8 that the function f is continuous on K.

Now $K \subset V$ and therefore, given any point $\mathbf{x} \in K$, there exists some positive real number δ such that the open ball of radius δ about the point $\mathbf{x}$ is contained in V, and therefore $f(\mathbf{x}) \geq \delta$. It follows that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in K$.

It follows from the Extreme Value Theorem for continuous real-valued functions on closed bounded subsets of Euclidean spaces (Theorem 1.20) that the function $f\colon K \to \mathbb{R}$ attains its minimum value at some point of K. Let that minimum value be ε. Then $f(\mathbf{x}) \geq \varepsilon > 0$ for all $\mathbf{x} \in K$, and therefore $|\mathbf{x} - \mathbf{z}| \geq \varepsilon > 0$ for all $\mathbf{x} \in K$ and $\mathbf{z} \in X \setminus V$. It follows that $B_X(K, \varepsilon) \subset V$, as required. ∎

Example Let

$$F = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ \ y \geq 0 \text{ and } xy \geq 1\}.$$

and let

$$V = \{(x, y) \in \mathbb{R}^2 : y > 0\}.$$

Note that if $(x, y) \in F$ then $x > 0$ and $y > 0$, because $xy = 1$. It follows that $F \subset V$. Also F is a closed set in $\mathbb{R}^2$ and V is an open set in $\mathbb{R}^2$. However F is not a compact subset of $\mathbb{R}^2$ because F is not bounded.

We now show that there does not exist any positive real number ε with the property that $B_{\mathbb{R}^2}(F, \varepsilon) \subset V$, where $B_{\mathbb{R}^2}(F, \varepsilon)$ denotes the set of points of $\mathbb{R}^2$ that lie within a distance ε of some point of F. Indeed let some positive real number ε be given, let x be a positive real number satisfying $x > 2\varepsilon^{-1}$, and let $y = x^{-1} - \frac{1}{2}\varepsilon$. Then $y < 0$, and therefore $(x, y) \notin V$. But $(x, y + \frac{1}{2}\varepsilon) \in F$, and therefore $(x, y) \in B_{\mathbb{R}^2}(F, \varepsilon)$. This shows that there does not exist any positive real number ε for which $B_{\mathbb{R}^2}(F, \varepsilon) \subset V$.

Proposition 2.10 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let K be a non-empty compact subset of Y, and let U be an subset in $X \times Y$ that is open in $X \times Y$. Let*

$$d_Y(\mathbf{y}, K) = \inf\{|\mathbf{y} - \mathbf{z}| : \mathbf{z} \in K\}$$

for all $\mathbf{y} \in Y$. Let $\mathbf{p}$ be a point of X with the property that $(\mathbf{p}, \mathbf{z}) \in U$ for all $\mathbf{z} \in K$. Then there exists some positive number δ such that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ and $d(\mathbf{y}, K) < \delta$.

Proof Let

$$\tilde{K} = \{(\mathbf{p}, \mathbf{z}) : \mathbf{z} \in K\}.$$

Then $\tilde{K}$ is a closed bounded subset of $\mathbb{R}^n \times \mathbb{R}^m$. It follows from Proposition 2.9 that there exists some positive real number ε such that

$$B_{X \times Y}(\tilde{K}, \varepsilon) \subset U$$

where $B_{X \times Y}(\tilde{K}, \varepsilon)$ denotes that subset of $X \times Y$ consisting of those points $(\mathbf{x}, \mathbf{y})$ of $X \times Y$ that lie within a distance ε of a point of $\tilde{K}$. Now a point

$(\mathbf{x}, \mathbf{y})$ of $X \times Y$ belongs to $B_{X\times Y}(\tilde{K}, \varepsilon)$ if and only if there exists some point $\mathbf{z}$ of K for which

$$|\mathbf{x} - \mathbf{p}|^2 + |\mathbf{y} - \mathbf{z}|^2 < \varepsilon^2.$$

Let $\delta = \varepsilon/\sqrt{2}$. If $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfy $|\mathbf{x} - \mathbf{p}| < \delta$ and $d_Y(\mathbf{y}, K) < \delta$ then there exists some point $\mathbf{z}$ of K for which $|\mathbf{y} - \mathbf{z}| < \delta$. But then

$$|\mathbf{x} - \mathbf{p}|^2 + |\mathbf{y} - \mathbf{z}|^2 < 2\delta^2 = \varepsilon^2,$$

and therefore $(\mathbf{x}, \mathbf{y}) \in U$, as required. ∎

Proposition 2.11 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that $\Phi(\mathbf{x})$ is closed in Y for every $\mathbf{x} \in X$. Suppose also that $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous. Then the graph* $\mathrm{Graph}(\Phi)$ *of $\Phi\colon X \rightrightarrows Y$ is closed in $X \times Y$.*

Proof Let $(\mathbf{p}, \mathbf{q})$ be a point of the complement $X \times Y \setminus \mathrm{Graph}(\Phi)$ of the graph $\mathrm{Graph}(\Phi)$ of Φ in $X \times Y$. Then $\Phi(\mathbf{p})$ is closed in Y and $\mathbf{q} \notin \Phi(\mathbf{p})$. It follows that there exists some positive real number δ_Y such that $|\mathbf{y} - \mathbf{q}| > \delta_Y$ for all $\mathbf{y} \in \Phi(\mathbf{p})$.

Let

$$V = \{\mathbf{y} \in Y : |\mathbf{y} - \mathbf{q}| > \delta_Y\}$$

and

$$W = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V\}.$$

Then V is open in Y and $\Phi(\mathbf{p}) \subset V$. Now the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous. It therefore follows from the definition of upper hemicontinuity that the subset W of X is open in X. Moreover $\mathbf{p} \in W$. It follows that there exists some positive real number δ_X such that $\mathbf{x} \in W$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta_X$. Then $\Phi(\mathbf{x}) \subset V$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta_X$. Let δ be the minimum of δ_X and δ_Y, and let $(\mathbf{x}, \mathbf{y})$ be a point of $X \times Y$ whose distance from the point $(\mathbf{p}, \mathbf{q})$ is less than δ. Then $|\mathbf{x} - \mathbf{p}| < \delta_X$ and therefore $\Phi(\mathbf{x}) \subset V$. Also $|\mathbf{y} - \mathbf{q}| < \delta_Y$, and therefore $\mathbf{y} \notin V$. It follows that $\mathbf{y} \notin \Phi(\mathbf{x})$, and therefore $(\mathbf{x}, \mathbf{y}) \notin \mathrm{Graph}(\Phi)$. We conclude from this that the complement of $\mathrm{Graph}(\Phi)$ is open in $X \times Y$. It follows that $\mathrm{Graph}(\Phi)$ itself is closed in $X \times Y$, as required. ∎

Proposition 2.12 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that the graph* $\mathrm{Graph}(\Phi)$ *of the correspondence Φ is closed in $X \times Y$. Suppose also that Y is a compact subset of $\mathbb{R}^m$. Then the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous.*

Proof of Proposition 2.12 using Proposition 2.10 Let $\mathbf{p}$ be a point of X, let V be an open set satisfying $\Phi(\mathbf{p}) \subset V$, and let $K = Y \backslash V$. The compact set Y is closed and bounded in $\mathbb{R}^m$. Also K is closed in Y. It follows that K is a closed bounded subset of $\mathbb{R}^m$ (see Lemma 1.18). Let U be the complement of $\mathrm{Graph}(\Phi)$ in $X \times Y$. Then U is open in $X \times Y$, because $\mathrm{Graph}(\Phi)$ is closed in $X \times Y$. Also $(\mathbf{p}, \mathbf{y}) \in U$ for all $\mathbf{y} \in K$, because $\Phi(\mathbf{p}) \cap K = \emptyset$. It follows from Proposition 2.10 that there exists some positive number δ such that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in K$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus if $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then $\mathbf{y} \notin \Phi(\mathbf{x})$ for all $\mathbf{y} \in K$, and therefore $\Phi(\mathbf{x}) \subset V$, where $V = Y \setminus K$. Thus the correspondence Φ is upper hemicontinuous at $\mathbf{p}$, as required. ∎

Corollary 2.13 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a function from X to Y. Suppose that the graph* $\mathrm{Graph}(\varphi)$ *of the function φ is closed in $X \times Y$. Suppose also that Y is a compact subset of $\mathbb{R}^m$. Then the function $\varphi\colon X \to Y$ is continuous.*

Proof Let $\Phi\colon X \rightrightarrows Y$ be the correspondence defined such that $\Phi(\mathbf{x}) = \{\varphi(\mathbf{x})\}$ for all $\mathbf{x} \in X$. Then $\mathrm{Graph}(\Phi) = \mathrm{Graph}(\varphi)$, and therefore $\mathrm{Graph}(\Phi)$ is closed in $X \times Y$. The subset Y of $\mathbb{R}^m$ is compact. It therefore follows from Proposition 2.12 that the correspondence Φ is upper hemicontinuous. It then follows from Lemma 2.3 that the function $\varphi\colon X \to Y$ is continuous, as required. ∎

2.3 Compact-Valued Upper Hemicontinuous Correspondences

Lemma 2.14 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous. Then*

$$\{\mathbf{x} \in X : \Phi(\mathbf{x}) \neq \emptyset\}$$

is closed in X.

Proof Given any open set V in Y, let

$$\Phi^+(V) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V\}.$$

It follows from the upper hemicontinuity of Φ that $\Phi^+(V)$ is open in X for all open sets V in Y (see Lemma 2.1). Now the empty set $\emptyset$ is open in Y. It follows that $\Phi^+(\emptyset)$ is open in X. But

$$\Phi^+(\emptyset) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset \emptyset\} = \{\mathbf{x} \in X : \Phi(\mathbf{x}) = \emptyset\}.$$

It follows that the set of point $\mathbf{x}$ in X for which $\Phi(\mathbf{x}) = \emptyset$ is open in X, and therefore the set of points $\mathbf{x} \in X$ for which $\Phi(\mathbf{x}) \neq \emptyset$ is closed in X, as required. ∎

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Given any subset S of X, we define the *image* $\Phi(S)$ of S under the correspondence Φ to be the subset of Y defined such that

$$\Phi(S) = \bigcup_{\mathbf{x} \in S} \Phi(\mathbf{x})$$

Lemma 2.15 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is compact-valued and upper hemicontinuous. Let K be a compact subset of X. Then $\Phi(K)$ is a compact subset of Y.*

Proof Let $\mathcal{V}$ be collection of open sets in Y that covers $\Phi(K)$. Given any point $\mathbf{p}$ of K, there exists a finite subcollection $\mathcal{W}_{\mathbf{p}}$ of $\mathcal{V}$ that covers the compact set $\Phi(\mathbf{p})$. Let $U_{\mathbf{p}}$ be the union of the open sets belonging to this subcollection $\mathcal{W}_{\mathbf{p}}$. Then $\Phi(\mathbf{p}) \subset U_{\mathbf{p}}$. Now it follows from the upper hemicontinuity of $\Phi\colon X \rightrightarrows Y$ that there exists an open set $N_{\mathbf{p}}$ in X such that $\Phi(\mathbf{x}) \subset U_{\mathbf{p}}$ for all $\mathbf{x} \in N_{\mathbf{p}}$. Moreover, given any $\mathbf{p} \in K$, the finite collection $\mathcal{W}_{\mathbf{p}}$ of open sets in Y covers $\Phi(N_{\mathbf{p}})$. It then follows from the compactness of K that there exist points

$$\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k$$

of K such that

$$K \subset N_{\mathbf{p}_1} \cup N_{\mathbf{p}_2} \cup \cdots \cup N_{\mathbf{p}_k}.$$

Let

$$\mathcal{W} = \mathcal{W}_{\mathbf{p}_1} \cup \mathcal{W}_{\mathbf{p}_2} \cup \cdots \cup \mathcal{W}_{\mathbf{p}_k}.$$

Then $\mathcal{W}$ is a finite subcollection of $\mathcal{V}$ that covers $\Phi(K)$. The result follows. ∎

Proposition 2.16 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a compact-valued correspondence from X to Y. Let $\mathbf{p}$ be a point of X for which $\Phi(\mathbf{p})$ is non-empty. Then the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$ if and only if, given any positive real number ε, there exists some positive real number δ such that*

$$\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, where $B_Y(\Phi(\mathbf{p}), \varepsilon)$ denotes the subset of Y consisting of all points of Y that lie within a distance ε of some point of $\Phi(\mathbf{p})$.

Proof Let $\Phi\colon X \rightrightarrows Y$ is a compact-valued correspondence, and let $\mathbf{p}$ be a point of X for which $\Phi(\mathbf{p}) \neq \emptyset$.

First suppose that, given any positive real number ε, there exists some positive real number δ such that

$$\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. We must prove that $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$.

Let V be an open set in Y that satisfies $\Phi(\mathbf{p}) \subset V$. Now $\Phi(\mathbf{p})$ is a compact subset of Y, because $\Phi\colon X \to Y$ is compact-valued. It follows that there exists some positive real number ε such that $B_Y(\Phi(\mathbf{p}), \varepsilon) \subset V$ (see Proposition 2.9). There then exists some positive number δ such that

$$\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon) \subset V$$

whenever $|\mathbf{x} - \mathbf{p}| < \delta$. Thus $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$.

Conversely suppose that the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at the point $\mathbf{p}$. Now $\Phi(\mathbf{p})$ is a non-empty subset of Y. Let some positive number ε be given. Then $B_Y(\Phi(\mathbf{p}), \varepsilon)$ is open in Y and $\Phi(\mathbf{p}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$. It follows from the upper hemicontinuity of Φ at $\mathbf{p}$ that there exists some positive number δ such that $\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$ whenever $|\mathbf{x} - \mathbf{p}| < \delta$. The result follows. ∎

Proposition 2.17 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Then the correspondence is both compact-valued and upper hemicontinuous at a point $\mathbf{p} \in X$ if and only if, given any infinite sequences*

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

in X and Y respectively, where $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim\limits_{j\to+\infty} \mathbf{x}_j = \mathbf{p}$, there exists a subsequence of

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

which converges to a point of $\Phi(\mathbf{p})$.

Proof Throughout this proof, let us say that the correspondence Φ satisfies the *constrained convergent subsequence criterion* if (and only if), given any infinite sequences

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

in X and Y respectively, where $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$, there exists a subsequence of

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

which converges to a point of $\Phi(\mathbf{p})$.

We must prove that the correspondence $\Phi\colon X \rightrightarrows Y$ satisfies the constrained convergent subsequence criterion if and only if it is compact-valued and upper hemicontinuous.

Suppose first that the correspondence $\Phi\colon X \rightrightarrows Y$ satisfies the constrained convergent subsequence criterion. Applying this criterion when $\mathbf{x}_j = \mathbf{p}$ for all positive integers j, we conclude that every infinite sequence $(\mathbf{y}_j : j \in \mathbb{N})$ of points of $\Phi(\mathbf{p})$ has a convergent subsequence, and therefore $\Phi(\mathbf{x})$ is compact.

Let

$$D = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \neq \emptyset\}.$$

We show that D is closed in X. Let

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

be a sequence of points of D converging to some point of $\mathbf{p}$ of X. Then $\Phi(\mathbf{x}_j)$ is non-empty for all positive integers j, and therefore there exists an infinite sequence

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

of points of Y such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j. The constrained convergent subsequence criterion ensures that this infinite sequence in Y must have a subsequence that converges to some point of $\Phi(\mathbf{p})$. It follows that $\phi(\mathbf{p})$ is non-empty, and thus $\mathbf{p} \in D$.

Let $\mathbf{p}$ be a point of the complement of D. Then $\Phi(\mathbf{p}) = \emptyset$. There then exists $\delta > 0$ such that $\Phi(\mathbf{x}) = \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then $\Phi(\mathbf{x}) \subset V$ for all open sets V in Y. It follows that the correspondence Φ is upper hemicontinuous at those points $\mathbf{p}$ for which $\Phi(\mathbf{p}) = \emptyset$.

Now consider the situation in which $\Phi\colon X \rightrightarrows Y$ satisfies the constrained convergent subsequence criterion and $\mathbf{p}$ is some point of X for which $\Phi(\mathbf{p}) \neq \emptyset$. Let $K = \Phi(\mathbf{p})$. Then K is a compact non-empty subset of Y. Let V be an open set in Y that satisfies $\Phi(\mathbf{p}) \subset V$. Suppose that there did not exist any positive real number δ with the property that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It would then follow that there would exist infinite sequences

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

in X and Y respectively for which $|\mathbf{x}_j - \mathbf{p}| < 1/j$, $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ and $\mathbf{y}_j \not\in V$. Then $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$, and thus the constrained convergent subsequence criterion satisfied by the correspondence Φ would ensure the existence of a subsequence

$$\mathbf{y}_{k_1}, \mathbf{y}_{k_2}, \mathbf{y}_{k_3}, \ldots$$

of $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ converging to some point $\mathbf{q}$ of $\Phi(\mathbf{p})$. But then $\mathbf{q} \not\in V$, because $\mathbf{y}_{k_j} \not\in V$ for all positive integers j, and the complement $Y \setminus V$ of V is closed in Y. But $\Phi(\mathbf{p}) \subset V$, and $\mathbf{q} \in \Phi(\mathbf{p})$, and therefore $\mathbf{q} \in V$. Thus a contradiction would arise were there not to exist a positive real number δ with the property that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus such a real number δ must exist, and thus the constrained convergent subsequence criterion ensures that the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$.

It remains to show that any compact-valued upper hemicontinuous correspondence $\Phi\colon X \rightrightarrows Y$ satisfies the constrained convergent subsequence criterion. Let $\Phi\colon X \rightrightarrows Y$ be compact-valued and upper hemicontinuous. It follows from Lemma 2.14 that

$$\{\mathbf{x} \in X : \Phi(\mathbf{x}) \neq \emptyset\}$$

is closed in X.

Let

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

be infinite sequences in X and Y respectively, where $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$. Then $\Phi(\mathbf{p})$ is non-empty, because

$$\{\mathbf{x} \in X : \Phi(\mathbf{x}) \neq \emptyset\}$$

is closed in X (see Lemma 2.14). Let $K = \Phi(\mathbf{p})$. Then K is compact, because $\Phi\colon X \rightrightarrows Y$ is compact-valued by assumption. For each integer j let $d(\mathbf{y}_j, K)$ denote the greatest lower bound on the distances from $\mathbf{y}_j$ to points of K. There then exists an infinite sequence

$$\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots$$

of points of K such that $|\mathbf{y}_j - \mathbf{z}_j| \leq 2d(\mathbf{y}_j, K)$. for all positive integers j. (Indeed if $d(\mathbf{y}_j, K) = 0$ then $\mathbf{y}_j \in K$, because the compact set K is closed, and in that case we can take $\mathbf{z}_j = \mathbf{y}_j$. Otherwise $2d(\mathbf{y}, K)$ is strictly greater than the greatest lower bound on the distances from $\mathbf{y}_j$ to points of K, and we can therefore find $\mathbf{z}_j \in K$ with $|\mathbf{y}_j - \mathbf{z}_j| \leq 2d(\mathbf{y}_j, K)$.)

Now the upper hemicontinuity of $\Phi\colon X \rightrightarrows Y$ ensures that $d(\mathbf{y}_j, K) \to 0$ as $j \to +\infty$. Indeed, given any positive real number ε, the set $B_Y(K, \varepsilon)$ of points of Y that lie within a distance ε of a point of K is an open set containing $\Phi(\mathbf{p})$. It follows from the upper hemicontinuity of Φ that there exists some positive real number δ such that $\Phi(\mathbf{x}) \subset B_Y(K, \varepsilon)$ whenever $|\mathbf{x} - \mathbf{p}| < \delta$. Now $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$. It follows that there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$. But then $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ and therefore $d(\mathbf{y}_j, K) < \varepsilon$ whenever $j \geq N$. Now the compactness of K ensures that the infinite sequence

$$\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots$$

of points of K has a subsequence

$$\mathbf{z}_{k_1}, \mathbf{z}_{k_2}, \mathbf{z}_{k_3}, \ldots$$

that converges to some point $\mathbf{q}$ of K. Now $|\mathbf{y}_j - \mathbf{z}_j| \leq 2d(\mathbf{y}_j, K)$ for all positive integers j, and $d(\mathbf{y}_j, K) \to 0$ as $j \to +\infty$. It follows that $\mathbf{y}_{k_j} \to \mathbf{q}$ as $j \to +\infty$. Morever $\mathbf{q} \in \Phi(\mathbf{p})$. We have therefore verified that the constrained convergent subsequence criterion is satisfied by any correspondence $\Phi\colon X \rightrightarrows Y$ that is compact-valued and upper hemicontinuous. This completes the proof. ∎

Proposition 2.18 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is both upper hemicontinuous and compact-valued. Let U be an open set in $X \times Y$. Then*

$$\{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \textit{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X.

Proof of Proposition 2.18 using Proposition 2.17 Let

$$W = \{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\},$$

and let $\mathbf{p} \in W$. Suppose that there did not exist any strictly positive real number δ with the property that $\mathbf{x} \in W$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Then, given any positive real number δ, there would exist points $\mathbf{x}$ of X and $\mathbf{y}$ of Y such that $|\mathbf{x} - \mathbf{p}| < \delta$, $\mathbf{y} \in \Phi(\mathbf{x})$ and $(\mathbf{x}, \mathbf{y}) \notin U$. Therefore there would exist infinite sequences

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

in X and Y respectively such that $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ and $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ and $(\mathbf{x}_j, \mathbf{y}_j) \notin U$ for all positive integers j. The correspondence $\Phi\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous. Proposition 2.17 would therefore ensure the existence of a subsequence

$$\mathbf{y}_{k_1}, \mathbf{y}_{k_2}, \mathbf{y}_{k_3}, \ldots$$

of Y converging to some point $\mathbf{q}$ of $\Phi(\mathbf{p})$. Now the complement of U in $X \times Y$ is closed in $X \times Y$, because U is open in $X \times Y$ and $(\mathbf{x}_j, \mathbf{y}_j) \notin U$. It would therefore follow that $(\mathbf{p}, \mathbf{q}) \notin U$ (see Proposition 2.6). But this gives rise to a contradiction, because $\mathbf{q} \in \Phi(\mathbf{p})$ and $(\mathbf{p}, \mathbf{y}) \in U$ for all $\mathbf{y} \in \Phi(\mathbf{p})$. In order to avoid the contradiction, there must exist some positive real number δ with the property that with the property that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ and $\mathbf{y} \in \Phi(\mathbf{x})$. The result follows. ∎

Remark It should be noted that other results proved in this section do not necessarily generalize to correspondences $\Phi\colon X \rightrightarrows Y$ mapping the topological space X into an arbitrary topological space Y. For example all closed-valued upper hemicontinuous correspondences between metric spaces have closed graphs. The appropriate generalization of this result states that any closed-valued upper hemicontinuous correspondence $\Phi\colon X \rightrightarrows Y$ from a topological space X to a regular topological space Y has a closed graph. To interpret this, one needs to know the definition of what is meant by saying that a topological space is *regular*. A topological space Y is said to be *regular* if, given any closed subset F of Y, and given any point p of the complement $Y \setminus F$ of F, there exist open sets V and W in Y such that $F \subset V$, $p \in W$ and $V \cap W = \emptyset$. Metric spaces are regular. Also compact Hausdorff spaces are regular.

2.4 A Criterion characterizing Lower Hemicontinuity

Proposition 2.19 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is lower hemicontinuous at a point $\mathbf{p}$ of X if and only if given any infinite sequence*

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

in X for which $\lim\limits_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$ and given any point $\mathbf{q}$ of $\Phi(\mathbf{p})$, there exists an infinite sequence

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

of points of Y *such that* $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ *for all positive integers* j *and* $\lim\limits_{j \to +\infty} \mathbf{y}_j = \mathbf{q}$.

Proof First suppose that $\Phi\colon X \to Y$ is lower hemicontinuous at some point $\mathbf{p}$ of X. Let $\mathbf{q} \in \Phi(\mathbf{p})$, and let some positive number ε be given. Then the open ball $B_Y(\mathbf{q}, \varepsilon)$ in Y of radius ε centred on the point $\mathbf{q}$ is an open set in Y. It follows from the lower hemicontinuity of $\Phi\colon X \to Y$ that there exists some positive real number δ such that $\Phi(\mathbf{x}) \cap B_Y(\mathbf{q}, \varepsilon)$ is non-empty whenever $|\mathbf{x} - \mathbf{p}| < \delta$. Then, given any point $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ there exists some $\mathbf{y} \in \Phi(\mathbf{x})$ that satisfies $|\mathbf{y} - \mathbf{q}| < \varepsilon$. In particular, given any positive integer s, there exists some positive integer δ_s such that, given any point $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta_s$, there exists some $\mathbf{y} \in \Phi(\mathbf{x})$ that satisfies $|\mathbf{y} - \mathbf{q}| < 1/s$.

Now $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$. It follows that there exist positive integers $k(1), k(2), k(3), \ldots$, where

$$k(1) < k(2) < k(3) < \cdots$$

such that $|\mathbf{x}_j - \mathbf{p}| < \delta_s$ for all positive integers j satisfying $j \geq k(s)$. There then exists an infinite sequence

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $|\mathbf{y}_j - \mathbf{q}| < 1/s$ for all positive integers j and s satisfying $k(s) \leq j < k(s+1)$. Then $\lim\limits_{j \to +\infty} \mathbf{y}_j = \mathbf{q}$. We have thus shown that if $\Phi\colon X \to Y$ is lower hemicontinuous at the point $\mathbf{p}$, if $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is a sequence in X converging to the point $\mathbf{p}$, and if $\mathbf{q} \in \Phi(\mathbf{p})$, then there exists an infinite sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ in Y such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integer j and $\lim\limits_{j \to +\infty} \mathbf{y}_j = \mathbf{q}$.

Next suppose that the correspondence $\Phi\colon X \rightrightarrows Y$ is not lower hemicontinuous at $\mathbf{p}$. Then there exists an open set V in Y such that $\Phi(\mathbf{p}) \cap V$ is non-empty but there does not exist any positive real number δ with the property that $\Phi(\mathbf{x}) \cap V \neq \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{p} - \mathbf{x}| < \delta$. Let $\mathbf{q} \in \Phi(\mathbf{p}) \cap V$. There then exists an infinite sequence

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

converging to the point $\mathbf{p}$ with the property that $\Phi(\mathbf{x}_j) \cap V = \emptyset$ for all positive integers j. It is not then possible to construct an infinite sequence

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim\limits_{j \to +\infty} \mathbf{y}_j = \mathbf{q}$. The result follows. ∎

2.5 Intersections of Correspondences

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ and $\Psi\colon X \to Y$ be correspondences between X and Y. The *intersection* $\Phi \cap \Psi$ of the correspondences Φ and Ψ is defined such that

$$(\Phi \cap \Psi)(\mathbf{x}) = \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$$

for all $\mathbf{x} \in X$.

Proposition 2.20 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $\Phi\colon X \rightrightarrows Y$ and $\Psi\colon X \rightrightarrows Y$ be correspondences from X to Y, where the correspondence $\Phi\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous and the correspondence $\Psi\colon X \rightrightarrows Y$ has closed graph. Let $\Phi \cap \Psi\colon X \rightrightarrows Y$ be the correspondence defined such that*

$$(\Phi \cap \Psi)(\mathbf{x}) = \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$$

for all $\mathbf{x} \in X$. Then the correspondence Let $\Phi \cap \Psi\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous.

Proof Let

$$W = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : \mathbf{y} \notin \Psi(\mathbf{x})\}.$$

Then W is the complement of the graph $\mathrm{Graph}(\Psi)$ of Ψ in $X \times Y$. The graph of Ψ is closed in $X \times Y$, by assumption. It follows that W is open in $X \times Y$.

Let $\mathbf{x} \in X$. The subset $\Psi(\mathbf{x})$ of Y is closed in Y, because the graph of the correspondence Ψ is closed. It follows from the compactness of $\Phi(\mathbf{x})$ that $\Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$ is a closed subset of the compact set $\Phi(\mathbf{x})$, and must therefore be compact. Thus the correspondence $\Phi \cap \Psi$ is compact-valued.

Now let $\mathbf{p}$ be a point of X, and let V be any open set in Y for which $\Phi(\mathbf{p}) \cap \Psi(\mathbf{p}) \subset V$. In order to prove that $\Phi \cap \Psi$ is upper hemicontinuous we must show that there exists some positive real number δ such that $\Phi(\mathbf{x}) \cap \Psi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Let

$$U = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : \text{either } \mathbf{y} \in V \text{ or else } \mathbf{y} \notin \Psi(\mathbf{x})\}.$$

Then U is the union of the subsets $X \times V$ and W of $X \times Y$, where both these subsets are open in $X \times Y$. It follows that U is open in $X \times Y$. Moreover if $\mathbf{y} \in \Phi(\mathbf{p})$ then either $\mathbf{y} \in \Phi(\mathbf{p}) \cap \Psi(\mathbf{p})$, in which case $\mathbf{y} \in V$, or else $\mathbf{y} \notin \Psi(\mathbf{p})$. It follows that $(\mathbf{p}, \mathbf{y}) \in U$ for all $\mathbf{y} \in \Phi(\mathbf{p})$.

Now it follows from Proposition 2.18 that

$$\{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X. Therefore there exists some positive real number δ such that $(\mathbf{x}, \mathbf{y}) \in U$ for all $(\mathbf{x}, \mathbf{y}) \in X \times Y$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ and $\mathbf{y} \in \Phi(\mathbf{x})$. Now if $(\mathbf{x}, \mathbf{y})$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ and $\mathbf{y} \in \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$ then $(\mathbf{x}, \mathbf{y}) \in U$ but $(\mathbf{x}, \mathbf{y}) \notin W$. It follows from the definition of U that $\mathbf{y} \in V$. Thus $\Phi(\mathbf{x}) \cap \Psi(\mathbf{x}) \subset V$ whenever $|\mathbf{x} - \mathbf{p}| < \delta$. The result follows. ∎

2.6 Berge's Maximum Theorem

Lemma 2.21 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is both upper hemicontinuous and compact-valued. Let $f\colon X \times Y \to \mathbb{R}$ be a continuous real-valued function on $X \times Y$, and let c be a real number. Then*

$$\{\mathbf{x} \in X : f(\mathbf{x}, \mathbf{y}) < c \textit{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X.

Proof Let

$$U = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : f(\mathbf{x}, \mathbf{y}) < c\}.$$

It follows from the continuity of the function f that U is open in $X \times Y$. It then follows from Proposition 2.18 that

$$\{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X. The result follows. ∎

Lemma 2.22 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is lower hemicontinuous. Let $f\colon X \times Y \to \mathbb{R}$ be a continuous real-valued function on $X \times Y$, and let c be a real number. Then*

$$\{\mathbf{x} \in X : \textit{ there exists } \mathbf{y} \in \Phi(\mathbf{x}) \textit{ for which } f(\mathbf{x}, \mathbf{y}) > c\}$$

is open in X.

Proof Let

$$U = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : f(\mathbf{x}, \mathbf{y}) > c\},$$

and let

$$W = \{\mathbf{x} \in X : \text{ there exists } \mathbf{y} \in \Phi(\mathbf{x}) \text{ for which } f(\mathbf{x}, \mathbf{y}) > c\},$$

Let $\mathbf{p} \in W$. Then there exists $\mathbf{y} \in \Phi(\mathbf{p})$ for which $(\mathbf{p}, \mathbf{y}) \in U$. There then exist subsets W_X of X and W_Y of Y, where W_X is open in X and W_Y is open in Y, such that $\mathbf{p} \in W_X$, $\mathbf{y} \in W_Y$ and $W_X \times W_Y \subset U$ (see Lemma 2.5). There then exists some positive real number δ_1 such that $\mathbf{x} \in W_X$ whenever $|\mathbf{x} - \mathbf{p}| < \delta_1$.

Now $\Phi(\mathbf{p}) \cap W_Y \neq \emptyset$, because $\mathbf{y} \in \Phi(\mathbf{p}) \cap W_Y$. It follows from the lower hemicontinuity of the correspondence Φ that there exists some positive real number δ_2 such that $\Phi(\mathbf{x}) \cap W_Y \neq \emptyset$ whenever $|\mathbf{x} - \mathbf{p}| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then there exists $\mathbf{y} \in \Phi(\mathbf{x})$ for which $\mathbf{y} \in W_Y$. But then $(\mathbf{x}, \mathbf{y}) \in W_X \times W_Y$ and therefore $(\mathbf{x}, \mathbf{y}) \in U$, and thus $f(\mathbf{x}, \mathbf{y}) > c$. The result follows. ▮

Theorem 2.23 (Berge's Maximum Theorem) *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $f\colon X \times Y \to \mathbb{R}$ be a continuous real-valued function on $X \times Y$, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that $\Phi(\mathbf{x})$ is both non-empty and compact for all $\mathbf{x} \in X$ and that the correspondence $\Phi\colon X \to Y$ is both upper hemicontinuous and lower hemicontinuous. Let*

$$m(\mathbf{x}) = \sup\{f(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in \Phi(\mathbf{x})\}$$

for all $\mathbf{x} \in X$, and let

$$M(\mathbf{x}) = \{\mathbf{y} \in \Phi(\mathbf{x}) : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}$$

for all $\mathbf{x} \in X$. Then $m\colon X \to \mathbb{R}$ is continuous, $M(\mathbf{x})$ is a non-empty compact subset of Y for all $\mathbf{x} \in X$, and the correspondence $M\colon X \rightrightarrows Y$ is upper hemicontinuous.

Proof Let $\mathbf{x} \in X$. Then $\Phi(\mathbf{x})$ is a non-empty compact subset of Y. It is thus a closed bounded subset of $\mathbb{R}^m$. It follows from the Extreme Value Theorem (Theorem 1.20) that there exists at least one point $\mathbf{y}^*$ of $\Phi(\mathbf{x})$ with the property that $f(\mathbf{x}, \mathbf{y}^*) \geq f(\mathbf{x}, \mathbf{y})$ for all $\mathbf{y} \in \Phi(\mathbf{x})$. Then $m(\mathbf{x}) = f(\mathbf{x}, \mathbf{y}^*)$ and $\mathbf{y}^* \in M(\mathbf{x})$. Moreover

$$M(\mathbf{x}) = \{\mathbf{y} \in \Phi(\mathbf{x}) : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}.$$

It follows from the continuity of f that the set $M(\mathbf{x})$ is closed in Y (see Corollary 1.17). It is thus a closed subset of the compact set $\Phi(\mathbf{x})$ and must therefore itself be compact.

Let some positive number ε be given. Then $f(\mathbf{p}, \mathbf{y}) < m(\mathbf{p}) + \varepsilon$ for all $\mathbf{y} \in \Phi(\mathbf{p})$. It follows from Lemma 2.21 that

$$\{\mathbf{x} \in X : f(\mathbf{x}, \mathbf{y}) < m(\mathbf{p}) + \varepsilon \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X, and thus there exists some positive real number δ_1 such that $f(\mathbf{x}, \mathbf{y}) < m(\mathbf{p}) + \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_1$ and $\mathbf{y} \in \Phi(\mathbf{x})$ Then $m(\mathbf{x}) < m(\mathbf{p}) + \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_1$.

The correspondence $\Phi\colon X \rightrightarrows Y$ is also lower hemicontinuous. It therefore follows from Lemma 2.22 that there exists some positive real number δ_2 such that, given any $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_2$, there exists some $\mathbf{y} \in \Phi(\mathbf{x})$ for which $f(\mathbf{x}, \mathbf{y}) > m(\mathbf{p}) - \varepsilon$. It follows that $m(\mathbf{x}) > m(\mathbf{p}) - \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and

$$m(\mathbf{p}) - \varepsilon < m(\mathbf{x}) < m(\mathbf{p}) + \varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$. Thus the function $m\colon X \to \mathbb{R}$ is continuous on X.

It only remains to prove that the correspondence $M\colon X \rightrightarrows Y$ is upper hemicontinuous. Let

$$\Psi(\mathbf{x}) = \{\mathbf{y} \in Y : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}$$

for all $\mathbf{x} \in X$. Then

$$\mathrm{Graph}(\Psi) = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}$$

Thus $\mathrm{Graph}(\Psi)$ is the preimage of zero under the continuous real-valued function that sends $(\mathbf{x}, \mathbf{y}) \in X \times Y$ to $f(\mathbf{x}, \mathbf{y}) - m(\mathbf{x})$. It follows that $\mathrm{Graph}(\Psi)$ is a closed subset of $X \times Y$.

Now $M(\mathbf{x}) = \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$ for all $\mathbf{x} \in X$, where the correspondence Φ is compact-valued and upper hemicontinuous and the correspondence Ψ has closed graph. It follows from Proposition 2.20 that the correspondence M must itself be both compact-valued and upper hemicontinuous. This completes the proof of Berge's Maximum Theorem. ∎