Module MA3486: Fixed Point Theorems and Economic Equilibria
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Appendices

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Contents

A Proofs of Basic Results of Real Analysis

Lemma 1.1 *Let* $\mathbf{p}$ *be a point of* $\mathbb{R}^n$*, where* $\mathbf{p} = (p_1, p_2, \ldots, p_n)$*. Then a sequence* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *of points in* $\mathbb{R}^n$ *converges to* $\mathbf{p}$ *if and only if the* i*th components of the elements of this sequence converge to* p_i *for* $i = 1, 2, \ldots, n$*.*

Proof of Lemma 1.1 Let $(\mathbf{x}_j)_i$ denote the ith components of $\mathbf{x}_j$. Then $|(\mathbf{x}_j)_i - p_i| \leq |\mathbf{x}_j - \mathbf{p}|$ for $i = 1, 2, \ldots, n$ and for all positive integers j. It follows directly from the definition of convergence that if $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ then $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$.

Conversely suppose that, for each integer i between 1 and n, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$. Let $\varepsilon > 0$ be given. Then there exist positive integers $N_1, N_2, \ldots, N_n$ such that $|(\mathbf{x}_j)_i - p_i| < \varepsilon/\sqrt{n}$ whenever $j \geq N_i$. Let N be the maximum of $N_1, N_2, \ldots, N_n$. If $j \geq N$ then $j \geq N_i$ for $i = 1, 2, \ldots, n$, and therefore

$$|\mathbf{x}_j - \mathbf{p}|^2 = \sum_{i=1}^{n} ((\mathbf{x}_j)_i - p_i)^2 < n \left(\frac{\varepsilon}{\sqrt{n}}\right)^2 = \varepsilon^2.$$

Thus $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$, as required. ∎

The real number system satisfies the *Least Upper Bound Principle*:

> Any set of real numbers which is non-empty and bounded above has a least upper bound.

Let S be a set of real numbers which is non-empty and bounded above. The least upper bound, or *supremum*, of the set S is denoted by $\sup S$, and is characterized by the following two properties:

(i) $x \leq \sup S$ for all $x \in S$;

(ii) if u is a real number, and if $x \leq u$ for all $x \in S$, then $\sup S \leq u$.

A straightforward application of the Least Upper Bound guarantees that any set of real numbers that is non-empty and bounded below has a greatest lower bound, or *infimum*. The greatest lower bound of such a set S of real numbers is denoted by $\inf S$.

Theorem 1.2 *Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.*

Proof of Theorem 1.2 Let $x_1, x_2, x_3, \ldots$ be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Principle that there exists a least upper bound p for the set $\{x_j : j \in \mathbb{N}\}$. We claim that the sequence converges to p.

Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j - p| < \varepsilon$ whenever $j \geq N$. Now $p - \varepsilon$ is not an upper bound for the set $\{x_j : j \in \mathbb{N}\}$ (since p is the least upper bound), and therefore there must exist some positive integer N such that $x_N > p - \varepsilon$. But then $p - \varepsilon < x_j \leq p$ whenever $j \geq N$, since the sequence is non-decreasing and bounded above by p. Thus $|x_j - p| < \varepsilon$ whenever $j \geq N$. Therefore $x_j \to p$ as $j \to +\infty$, as required.

If the sequence $x_1, x_2, x_3, \ldots$ is non-increasing and bounded below then the sequence $-x_1, -x_2, -x_3, \ldots$ is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence $x_1, x_2, x_3, \ldots$ is also convergent. ∎

Theorem 1.3 *Every bounded sequence of real numbers has a convergent subsequence.*

Proof of Theorem 1.3 Let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers. We define a *peak index* to be a positive integer j with the property that $a_j \geq a_k$ for all positive integers k satisfying $k \geq j$. Thus a positive integer j is a peak index if and only if the jth member of the infinite sequence $a_1, a_2, a_3, \ldots$ is greater than or equal to all succeeding members of the sequence. Let S be the set of all peak indices. Then

$$S = \{j \in \mathbb{N} : a_j \geq a_k \text{ for all } k \geq j\}.$$

First let us suppose that the set S of peak indices is infinite. Arrange the elements of S in increasing order so that $S = \{j_1, j_2, j_3, j_4, \ldots\}$, where $j_1 < j_2 < j_3 < j_4 < \cdots$. It follows from the definition of peak indices that $a_{j_1} \geq a_{j_2} \geq a_{j_3} \geq a_{j_4} \geq \cdots$. Thus $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a non-increasing subsequence of the original sequence $a_1, a_2, a_3, \ldots$. This subsequence is bounded below (since the original sequence is bounded). It follows from Theorem 1.2 that $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a convergent subsequence of the original sequence.

Now suppose that the set S of peak indices is finite. Choose a positive integer j_1 which is greater than every peak index. Then j_1 is not a peak index. Therefore there must exist some positive integer j_2 satisfying $j_2 > j_1$ such that $a_{j_2} > a_{j_1}$. Moreover j_2 is not a peak index (because j_2 is greater than j_1 and j_1 in turn is greater than every peak index). Therefore there must exist some positive integer j_3 satisfying $j_3 > j_2$ such that $a_{j_3} > a_{j_2}$. We can continue in this way to construct (by induction on j) a strictly increasing subsequence $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ of our original sequence. This increasing subsequence is bounded above (since the original sequence is bounded) and thus is convergent, by Theorem 1.2. This completes the proof of the Bolzano-Weierstrass Theorem. ∎

We introduce some terminology and notation for discussing convergence along subsequences of bounded sequences of points in Euclidean spaces. This will be useful in proving the multi-dimensional version of the Bolzano-Weierstrass Theorem.

Definition Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be an infinite sequence of points in $\mathbb{R}^n$, let J be an infinite subset of the set $\mathbb{N}$ of positive integers, and let $\mathbf{p}$ be a point of $\mathbb{R}^n$. We say that $\mathbf{p}$ is the *limit* of $\mathbf{x}_j$ as j *tends to infinity in the set* J, and write "$\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ in J" if the following criterion is satisfied:—

> given any real number ε satisfying $\varepsilon > 0$ there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \in J$ and $j \geq N$.

The one-dimensional version of the Bolzano-Weierstrass Theorem asserts that every bounded sequence of real numbers has a convergent subsequence. We seek to generalize this result to bounded sequences of points in n-dimensional Euclidean space $\mathbb{R}^n$.

Now the one-dimensional version of the Bolzano-Weierstrass Theorem is equivalent to the following statement:

> Given any bounded infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers, there exists an infinite subset J of the set $\mathbb{N}$ of positive integers and a real number p such that $x_j \to p$ as $j \to +\infty$ in J.

Given an infinite subset J of $\mathbb{N}$, the elements of J can be labelled as $k_1, k_2, k_3, \ldots$, where $k_1 < k_2 < k_3 < \cdots$, so that k_1 is the smallest positive integer belonging of J, k_2 is the next smallest, etc. Therefore any standard result concerning convergence of sequences of points can be applied in the context of the convergence of subsequences of a given sequence of points.

The following result is therefore a direct consequence of the one-dimensional Bolzano-Weierstrass Theorem:

> Given any bounded infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers, and given an infinite subset J of the set $\mathbb{N}$ of positive integers, there exists an infinite subset K of J and a real number p such that $x_j \to p$ as $j \to +\infty$ in K.

The above statement in fact corresponds to the following assertion:—

Given any bounded infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers, and given any subsequence

$$x_{k_1}, x_{k_2}, x_{k_3}, \cdots$$

of the given infinite sequence, there exists a convergent subsequence

$$x_{k_{m_1}}, x_{k_{m_2}}, x_{k_{m_3}}, \ldots$$

of the given subsequence. Moreover this convergent subsequence of the given subsequence is itself a convergent subsequence of the given infinite sequence, and it contains only members of the given subsequence of the given sequence.

The basic principle can be presented purely in words as follows:

Given a bounded sequence of real numbers, and given a subsequence of that original given sequence, there exists a convergent subsequence of the given subsequence. Moreover this subsequence of the subsequence is a convergent subsequence of the original given sequence.

We employ this principle in the following proof of the Multidimensional Bolzano-Weierstrass Theorem.

Theorem 1.4 *Every bounded sequence of points in a Euclidean space has a convergent subsequence.*

Proof of Theorem 1.4 Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a bounded infinite sequence of points in $\mathbb{R}^n$, and, for each positive integer j, and for each integer i between 1 and n, let $(\mathbf{x}_j)_i$ denote the ith component of $\mathbf{x}_j$. Then

$$\mathbf{x}_j = \Big((\mathbf{x}_j)_1, (\mathbf{x}_j)_2, \ldots, (\mathbf{x}_j)_n\Big).$$

for all positive integers j. It follows from the one-dimensional Bolzano-Weierstrass Theorem that there exists an infinite subset J_1 of the set $\mathbb{N}$ of positive integers and a real number p_1 such that $(\mathbf{x}_j)_1 \to p_1$ as $j \to +\infty$ in J_1.

Let k be an integer between 1 and $n-1$. Suppose that there exists an infinite subset J_k of $\mathbb{N}$ and real numbers $p_1, p_2, \ldots, p_k$ such that, for each integer i between 1 and k, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$ in J_k. It then follows from the one-dimensional Bolzano-Weierstrass Theorem that there exists an infinite subset J_{k+1} of J_k and a real number p_{k+1}, such that $(\mathbf{x}_j)_{k+1} \to p_{k+1}$

as $j \to +\infty$ in J_{k+1}. Moreover the requirement that $J_{k+1} \subset J_k$ then ensures that, for each integer i between 1 and $k+1$, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$ in J_{k+1}. Repeated application of this result then ensures the existence of an infinite subset J_n of $\mathbb{N}$ and real numbers $p_1, p_2, \ldots, p_n$ such that, for each integer i between 1 and n, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$ in J_n.

Let

$$J_n = \{k_1, k_2, k_3, \ldots\},$$

where $k_1 < k_2 < k_3 < \cdots$. Then $\lim_{j\to+\infty} (\mathbf{x}_{k_j})_i = p_i$ for $i = 1, 2, \ldots, n$. It then follows from Proposition 1.1 that $\lim_{j\to+\infty} \mathbf{x}_{k_j} = \mathbf{p}$. The result follows. ∎

Lemma 1.5 *Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any positive real number r, the open ball $B_X(\mathbf{p}, r)$ in X of radius r about $\mathbf{p}$ is open in X.*

Proof of Lemma 1.5 Let $\mathbf{x}$ be an element of $B_X(\mathbf{p}, r)$. We must show that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. Let $\delta = r - |\mathbf{x} - \mathbf{p}|$. Then $\delta > 0$, since $|\mathbf{x} - \mathbf{p}| < r$. Moreover if $\mathbf{y} \in B_X(\mathbf{x}, \delta)$ then

$$|\mathbf{y} - \mathbf{p}| \leq |\mathbf{y} - \mathbf{x}| + |\mathbf{x} - \mathbf{p}| < \delta + |\mathbf{x} - \mathbf{p}| = r,$$

by the Triangle Inequality, and hence $\mathbf{y} \in B_X(\mathbf{p}, r)$. Thus $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. This shows that $B_X(\mathbf{p}, r)$ is an open set, as required. ∎

Proposition 1.6 *Let X be a subset of $\mathbb{R}^n$. The collection of open sets in X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are both open in X;

(ii) the union of any collection of open sets in X is itself open in X;

(iii) the intersection of any finite *collection of open sets in X is itself open in X.*

Proof of Proposition 1.6 The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole set X. This proves (i).

Let $\mathcal{A}$ be any collection of open sets in X, and let U denote the union of all the open sets belonging to $\mathcal{A}$. We must show that U is itself open in X. Let $\mathbf{x} \in U$. Then $\mathbf{x} \in V$ for some set V belonging to the collection $\mathcal{A}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset V$. But $V \subset U$, and thus $B_X(\mathbf{x}, \delta) \subset U$. This shows that U is open in X. This proves (ii).

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of subsets of X that are open in X, and let V denote the intersection $V_1 \cap V_2 \cap \cdots \cap V_k$ of these sets. Let $\mathbf{x} \in V$. Now $\mathbf{x} \in V_j$ for $j = 1, 2, \ldots, k$, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of sets.) Now $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(\mathbf{x}, \delta) \subset V$. Thus the intersection V of the sets $V_1, V_2, \ldots, V_k$ is itself open in X. This proves (iii). ∎

Proposition 1.7 *Let X be a subset of $\mathbb{R}^n$, and let U be a subset of X. Then U is open in X if and only if there exists some open set V in $\mathbb{R}^n$ for which $U = V \cap X$.*

Proof of Proposition 1.7 First suppose that $U = V \cap X$ for some open set V in $\mathbb{R}^n$. Let $\mathbf{u} \in U$. Then the definition of open sets in $\mathbb{R}^n$ ensures that there exists some positive real number δ such that

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta\} \subset V.$$

Then

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\} \subset U.$$

This shows that U is open in X.

Conversely suppose that the subset U of X is open in X. For each point $\mathbf{u}$ of U there exists some positive real number $\delta_{\mathbf{u}}$ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta_{\mathbf{u}}\} \subset U.$$

For each $\mathbf{u} \in U$, let $B(\mathbf{u}, \delta_{\mathbf{u}})$ denote the open ball in $\mathbb{R}^n$ of radius $\delta_{\mathbf{u}}$ about the point $\mathbf{u}$, so that

$$B(\mathbf{u}, \delta_{\mathbf{u}}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta_{\mathbf{u}}\}$$

for all $\mathbf{u} \in U$, and let V be the union of all the open balls $B(\mathbf{u}, \delta_{\mathbf{u}})$ as $\mathbf{u}$ ranges over all the points of U. Then V is an open set in $\mathbb{R}^n$. Indeed every open ball in $\mathbb{R}^n$ is an open set (Lemma 1.5), and any union of open sets in $\mathbb{R}^n$ is itself an open set (Proposition 1.6). The set V is a union of open balls. It is therefore a union of open sets, and so must itself be an open set.

Now $B(\mathbf{u}, \delta_{\mathbf{u}}) \cap X \subset U$. for all $\mathbf{u} \in U$. Also every point of V belongs to $B(\mathbf{u}, \delta_{\mathbf{u}})$ for at least one point $\mathbf{u}$ of U. It follows that $V \cap X \subset U$. But $\mathbf{u} \in B(\mathbf{u}, \delta_{\mathbf{u}})$ and $B(\mathbf{u}, \delta_{\mathbf{u}}) \subset V$ for all $\mathbf{u} \in U$, and therefore $U \subset V$, and thus $U \subset V \cap X$. It follows that $U = V \cap X$, as required. ∎

Lemma 1.8 *A sequence* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *of points in* $\mathbb{R}^n$ *converges to a point* $\mathbf{p}$ *if and only if, given any open set* U *which contains* $\mathbf{p}$*, there exists some positive integer* N *such that* $\mathbf{x}_j \in U$ *for all* j *satisfying* $j \geq N$*.*

Proof of Lemma 1.8 Suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ has the property that, given any open set U which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in U$ whenever $j \geq N$. Let $\varepsilon > 0$ be given. The open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is an open set by Lemma 1.5. Therefore there exists some positive integer N such that $\mathbf{x}_j \in B(\mathbf{p}, \varepsilon)$ whenever $j \geq N$. Thus $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$. This shows that the sequence converges to $\mathbf{p}$.

Conversely, suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Let U be an open set which contains $\mathbf{p}$. Then there exists some $\varepsilon > 0$ such that the open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is a subset of U. Thus there exists some $\varepsilon > 0$ such that U contains all points $\mathbf{x}$ of X that satisfy $|\mathbf{x} - \mathbf{p}| < \varepsilon$. But there exists some positive integer N with the property that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$, since the sequence converges to $\mathbf{p}$. Therefore $\mathbf{x}_j \in U$ whenever $j \geq N$, as required. ∎

Lemma 1.10 *Let* X *be a subset of* $\mathbb{R}^n$*, and let* F *be a subset of* X *which is closed in* X*. Let* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *be a sequence of points of* F *which converges to a point* $\mathbf{p}$ *of* X*. Then* $\mathbf{p} \in F$*.*

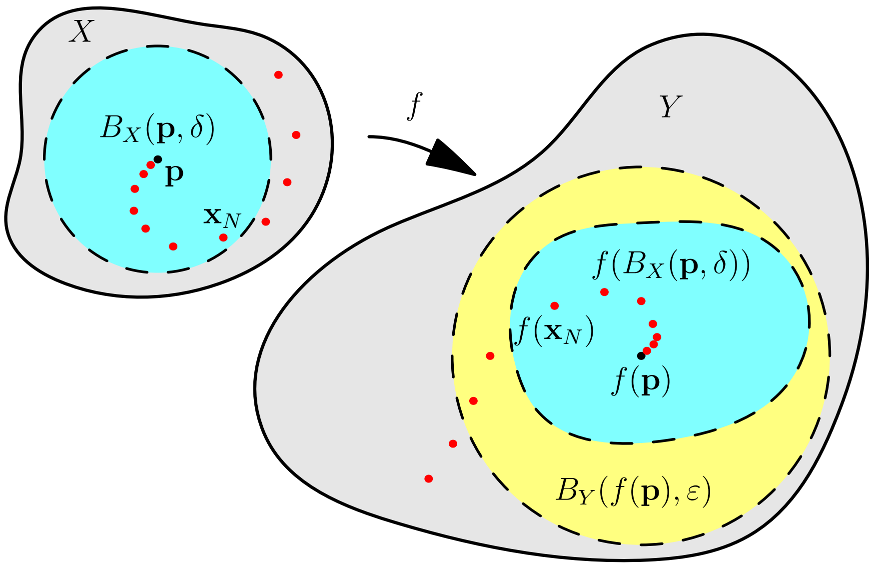
Proof of Lemma 1.10 The complement $X \setminus F$ of F in X is open, since F is closed. Suppose that $\mathbf{p}$ were a point belonging to $X \setminus F$. It would then follow from Lemma 1.8 that $\mathbf{x}_j \in X \setminus F$ for all values of j greater than some positive integer N, contradicting the fact that $\mathbf{x}_j \in F$ for all j. This contradiction shows that $\mathbf{p}$ must belong to F, as required. ∎

Lemma 1.11 *Let* X*,* Y *and* Z *be subsets of* $\mathbb{R}^m$*,* $\mathbb{R}^n$ *and* $\mathbb{R}^k$ *respectively, and let* $f\colon X \to Y$ *and* $g\colon Y \to Z$ *be functions satisfying* $f(X) \subset Y$*. Suppose that* f *is continuous at some point* $\mathbf{p}$ *of* X *and that* g *is continuous at* $f(\mathbf{p})$*. Then the composition function* $g \circ f\colon X \to Z$ *is continuous at* $\mathbf{p}$*.*

Proof of Lemma 1.11 Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(\mathbf{y}) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - f(\mathbf{p})| < \eta$. But then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \eta$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|g(f(\mathbf{x})) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus $g \circ f$ is continuous at $\mathbf{p}$, as required. ∎

Lemma 1.12 *Let* X *and* Y *be a subsets of* $\mathbb{R}^m$ *and* $\mathbb{R}^n$ *respectively, and let* $f\colon X \to Y$ *be a continuous function from* X *to* Y*. Let* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *be a sequence of points of* X *which converges to some point* $\mathbf{p}$ *of* X*. Then the sequence* $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ *converges to* $f(\mathbf{p})$*.*

Proof of Lemma 1.12 Let $\varepsilon > 0$ be given. Then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, since the function f is continuous at $\mathbf{p}$. Also there exists some positive integer N



such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$, since the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Thus if $j \geq N$ then $|f(\mathbf{x}_j) - f(\mathbf{p})| < \varepsilon$. Thus the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$, as required. ∎

Proposition 1.11 *Let X, Y and Z be subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions satisfying $f(X) \subset Y$. Suppose that f is continuous at some point $\mathbf{p}$ of X and that g is continuous at $f(\mathbf{p})$. Then the composition function $g \circ f\colon X \to Z$ is continuous at $\mathbf{p}$.*

Proof of Proposition 1.11 Note that the ith component f_i of f is given by $f_i = \pi_i \circ f$, where $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ is the continuous function which maps $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ onto its ith coordinate y_i. Now any composition of continuous functions is continuous, by Lemma 1.11. Thus if f is continuous at $\mathbf{p}$, then so are the components of f.

Conversely suppose that the components of f are continuous at $\mathbf{p} \in X$. Let $\varepsilon > 0$ be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $|f_i(\mathbf{x}) - f_i(\mathbf{p})| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - f(\mathbf{p})|^2 = \sum_{i=1}^{n} |f_i(\mathbf{x}) - f_i(\mathbf{p})|^2 < \varepsilon^2,$$

and hence $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$. Thus the function f is continuous at $\mathbf{p}$, as required. ∎

Proposition 1.14 *Let X be a subset of $\mathbb{R}^n$, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be continuous functions from X to $\mathbb{R}$. Then the functions $f+g$, $f-g$ and $f\cdot g$ are continuous. If in addition $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ then the quotient function f/g is continuous.*

Proof of Proposition 1.14 First we prove that $f+g$ is continuous. Let some strictly positive real number ε be given. Then there exist strictly positive real numbers δ_1 and δ_2 such that $|f(\mathbf{x}) - f(\mathbf{p})| < \frac{1}{2}\varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_1$ and $|g(\mathbf{x}) - g(\mathbf{p})| < \frac{1}{2}\varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then

$$|(f+g)(\mathbf{x}) - (f+g)(\mathbf{p})| \leq |f(\mathbf{x}) - f(\mathbf{p})| + |g(\mathbf{x}) - g(\mathbf{p})| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

Thus the function $f+g$ is continuous at $\mathbf{p}$.

The function $-g$ is also continuous at $\mathbf{p}$, and $f - g = f + (-g)$. It follows that the function $f - g$ is continuous at $\mathbf{p}$.

Next we prove that $f \cdot g$ is continuous. Let some strictly positive real number ε be given. There exists some strictly positive real number δ_0 such that $|f(\mathbf{x}) - f(\mathbf{p})| < 1$ and $|g(\mathbf{x}) - g(\mathbf{p})| < 1$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_0$. Let M be the maximum of $|f(\mathbf{p})| + 1$ and $|g(\mathbf{p})| + 1$. Then $|f(\mathbf{x})| < M$ and $|g(\mathbf{x})| < M$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_0$. Now

$$f(\mathbf{x})g(\mathbf{x}) - f(\mathbf{p})g(\mathbf{p}) \;=\; (f(\mathbf{x}) - f(\mathbf{p}))g(\mathbf{x}) + f(\mathbf{p})(g(\mathbf{x}) - g(\mathbf{p})),$$

and thus

$$|f(\mathbf{x})g(\mathbf{x}) - f(\mathbf{p})g(\mathbf{p})| \;\leq\; M\Big(|f(\mathbf{x}) - f(\mathbf{p})| + |g(\mathbf{x}) - g(\mathbf{p})|\Big)$$

whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_0$. There then exists some strictly positive real number δ, where $0 < \delta \leq \delta_0$, such that

$$|f(\mathbf{x}) - f(\mathbf{p})| < \frac{\varepsilon}{2M} \quad \text{and} \quad |g(\mathbf{x}) - g(\mathbf{p})| < \frac{\varepsilon}{2M}$$

whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$. But then

$$|f(\mathbf{x})g(\mathbf{x}) - f(\mathbf{p})g(\mathbf{p})| < \varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$. Thus the function $f \cdot g$ is continuous at $\mathbf{p}$.

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$. Note that $1/g = r \circ g$, where $r\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is the reciprocal function, defined by $r(t) = 1/t$. Now the reciprocal function r is continuous. Thus the function $1/g$ is a composition of continuous functions and is thus continuous. But then, using the fact that a product of continuous real-valued functions is continuous, we deduce that f/g is continuous. ∎

Lemma 1.15 *Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a continuous function mapping X into $\mathbb{R}^n$, and let $|f|\colon X \to \mathbb{R}$ be defined such that $|f|(\mathbf{x}) = |f(\mathbf{x})|$ for all $\mathbf{x} \in X$. Then the real-valued function $|f|$ is continuous on X.*

Proof of Lemma 1.15 Let $\mathbf{x}$ and $\mathbf{p}$ be elements of X. Then

$$|f(\mathbf{x})| = |(f(\mathbf{x}) - f(\mathbf{p})) + f(\mathbf{p})| \leq |f(\mathbf{x}) - f(\mathbf{p})| + |f(\mathbf{p})|$$

and

$$|f(\mathbf{p})| = |(f(\mathbf{p}) - f(\mathbf{x})) + f(\mathbf{x})| \leq |f(\mathbf{x}) - f(\mathbf{p})| + |f(\mathbf{x})|,$$

and therefore

$$\Bigl| |f(\mathbf{x})| - |f(\mathbf{p})| \Bigr| \leq |f(\mathbf{x}) - f(\mathbf{p})|.$$

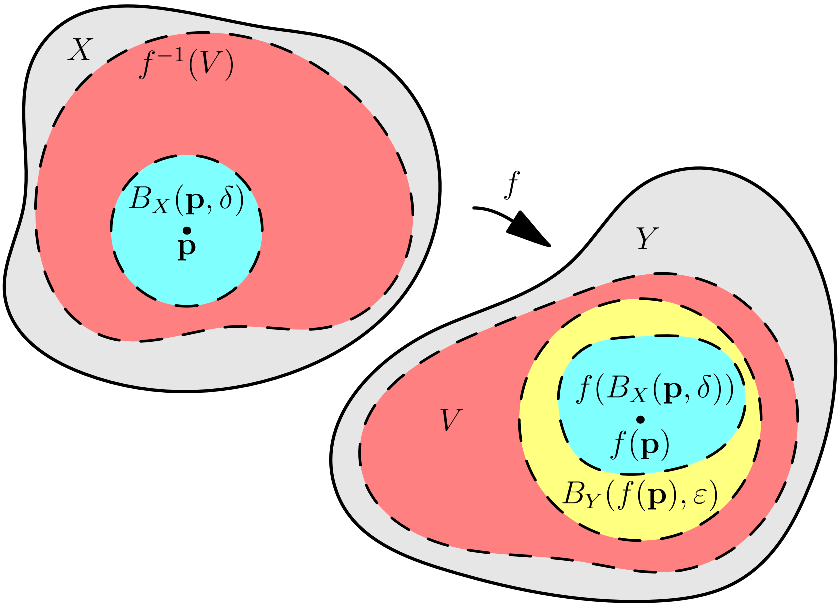
The result now follows from the definition of continuity, using the above inequality. Indeed let $\mathbf{p}$ be a point of X, and let some positive real number ε be given. Then there exists a positive real number δ small enough to ensure that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then

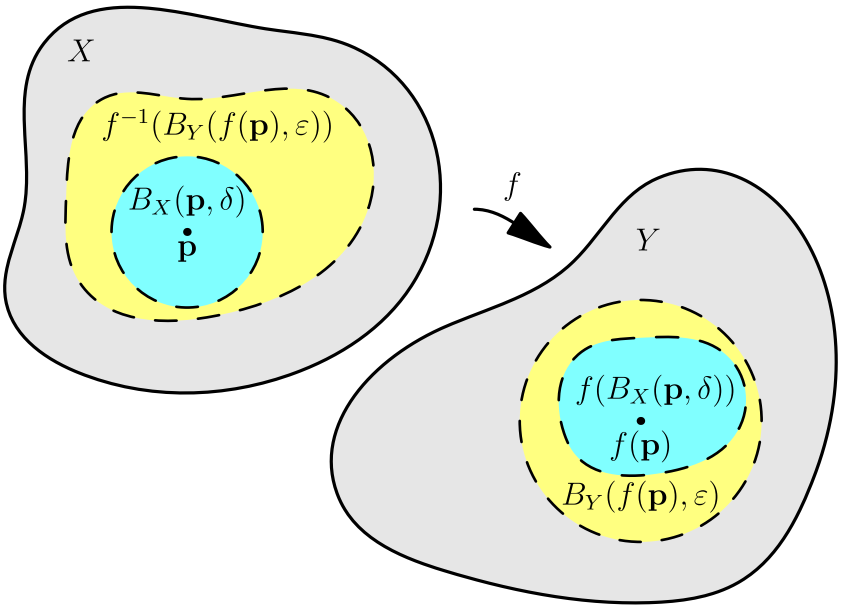
$$\Bigl| |f(\mathbf{x})| - |f(\mathbf{p})| \Bigr| \leq |f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus the function $|f|$ is continuous, as required. ∎

Proposition 1.16 *Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f\colon X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(V)$ is open in X for every open subset V of Y.*

Proof of Proposition 1.16 Suppose that $f\colon X \to Y$ is continuous. Let V be an open set in Y. We must show that $f^{-1}(V)$ is open in X. Let $\mathbf{p} \in f^{-1}(V)$. Then $f(\mathbf{p}) \in V$. But V is open, hence there exists some $\varepsilon > 0$ with the property that $B_Y(f(\mathbf{p}), \varepsilon) \subset V$. But f is continuous at $\mathbf{p}$. Therefore there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (see the remarks above). Thus $f(\mathbf{x}) \in V$ for all $\mathbf{x} \in B_X(\mathbf{p}, \delta)$, showing that


X
$f^{-1}(V)$
$B_X(\mathbf{p}, \delta)$
$\mathbf{p}$
f
Y
V
$f(B_X(\mathbf{p}, \delta))$
$f(\mathbf{p})$
$B_Y(f(\mathbf{p}), \varepsilon)$


X
$f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$
$B_X(\mathbf{p}, \delta)$
$\mathbf{p}$
f
Y
$f(B_X(\mathbf{p}, \delta))$
$f(\mathbf{p})$
$B_Y(f(\mathbf{p}), \varepsilon)$

$B_X(\mathbf{p}, \delta) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is open in X for every open set V in Y.

Conversely suppose that $f\colon X \to Y$ is a function with the property that $f^{-1}(V)$ is open in X for every open set V in Y. Let $\mathbf{p} \in X$. We must show that f is continuous at $\mathbf{p}$. Let $\varepsilon > 0$ be given. Then $B_Y(f(\mathbf{p}), \varepsilon)$ is an open set in Y, by Lemma 1.5, hence $f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$ is an open set in X which contains $\mathbf{p}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{p}, \delta) \subset f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$. Thus, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$. We conclude that f is continuous at $\mathbf{p}$, as required. ∎

B Alternative Proofs of Results concerning Correspondences

Proof of Proposition 2.9 using the Bolzano-Weierstrass Theorem Suppose that the proposition were false. Then there would exist infinite sequences $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ and $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \ldots$ such that $\mathbf{x}_j \in K$, $\mathbf{w}_j \in X \setminus V$ and $|\mathbf{w}_j - \mathbf{x}_j| < 1/j$ for all positive integers j. The set K is both closed and bounded in $\mathbb{R}^n$. The multidimensional Bolzano-Weierstrass Theorem (Theorem 1.4) would then ensure the existence of a subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converging to some point $\mathbf{q}$ of K. Moreover $\lim\limits_{j\to+\infty}(\mathbf{w}_j - \mathbf{x}_j) = \mathbf{0}$, and therefore

$$\lim_{j\to\infty} \mathbf{w}_{k_j} = \lim_{j\to\infty} \mathbf{x}_{k_j} = \mathbf{q}.$$

But $\mathbf{w}_j \in X \setminus V$. Moreover $X \setminus V$ is closed in X, and therefore any sequence of points in $X \setminus V$ that converges in X must converge to a point of $X \setminus V$ (see Lemma 1.10). It would therefore follow that $\mathbf{q} \in K \cap (X \setminus V)$. But this is impossible, because $K \subset V$ and therefore $K \cap (X \setminus V) = \emptyset$. Thus a contradiction would follow were the proposition false. The result follows. ∎

Proof of Proposition 2.9 using the Heine-Borel Theorem It follows from the multidimensional Heine-Borel Theorem (Theorem 1.23) that the set K is compact, and thus every open cover of K has a finite subcover. Given point $\mathbf{x}$ of K let $\varepsilon_{\mathbf{x}}$ be a positive real number with the property that

$$B_X(\mathbf{x}, 2\varepsilon_{\mathbf{x}}) \subset V,$$

where

$$B_X(\mathbf{x}, r) = \{\mathbf{x}' \in X : |\mathbf{x}' - \mathbf{x}| < r\}$$

for all positive integers r. The collection of open balls $B_X(\mathbf{x}, \varepsilon_{\mathbf{x}})$ determined by the points $\mathbf{x}$ of K covers K. By compactness this open cover of K has a finite subcover. Therefore there exist points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ of K such that

$$K \subset B(\mathbf{x}_1, \varepsilon_{\mathbf{x}_1}) \cup B(\mathbf{x}_2, \varepsilon_{\mathbf{x}_2}) \cup \cdots \cup B(\mathbf{x}_k, \varepsilon_{\mathbf{x}_k}).$$

Let ε be the minimum of $\varepsilon_{\mathbf{x}_1}, \varepsilon_{\mathbf{x}_2}, \ldots, \varepsilon_{\mathbf{x}_k}$. If $\mathbf{x}$ is a point of K then $\mathbf{x} \in B_X(\mathbf{x}_j, \varepsilon_{\mathbf{x}_j}$ for some integer j between 1 and k. But it then follows from the Triangle Inequality that

$$B(\mathbf{x}, \varepsilon) \subset B_X(\mathbf{x}_j, 2\varepsilon_{\mathbf{x}_j}) \subset V.$$

It follows from this that

$$B_X(K, \varepsilon) \subset V,$$

as required. ∎

Proof of Proposition 2.12 using the Bolzano-Weierstrass Theorem Let V be a subset of Y that is open in Y, and let $\mathbf{p}$ be a point of X for which $\Phi(\mathbf{p}) \subset V$. Let $F = Y \setminus V$. Then the set F is a subset of Y that is closed in Y, and $\Phi(\mathbf{p}) \cap F = \emptyset$. Now Y is a closed bounded subset of $\mathbb{R}^m$, because it is compact (Theorem 1.23). It follows that F is closed in $\mathbb{R}^m$ (Lemma 1.18).

Suppose that there did not exist any positive number δ such that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Then there would exist an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X converging to the point $\mathbf{p}$ with the property that $\Phi(\mathbf{x}_j) \cap F \neq \emptyset$ for all positive integers j. There would then exist an infinite sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ of elements of Y such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j) \cap F$ for all positive integers j. Then $(\mathbf{x}_j, \mathbf{y}_j) \in \mathrm{Graph}(\Phi)$ for all positive integers j. Moreover the infinite sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ would be bounded, because the set Y is bounded.

It would therefore follow from the multidimensional Bolzano-Weierstrass Theorem (Theorem 1.4) that there would exist a convergent subsequence

$$\mathbf{y}_{k_1}, \mathbf{y}_{k_2}, \mathbf{y}_{k_3}, \ldots$$

of the sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$. Let $\mathbf{q} = \lim_{j \to +\infty} \mathbf{y}_{k_j}$. Then $\mathbf{q} \in F$, because the set F is closed in Y and $\mathbf{y}_{k_j} \in F$ for all positive integers j (see Lemma 1.10). Similarly $(\mathbf{p}, \mathbf{q}) \in \mathrm{Graph}(\Phi)$, because the set $\mathrm{Graph}(\Phi)$ is closed in $X \times Y$, $(\mathbf{x}_{k_j}, \mathbf{y}_{k_j}) \in \mathrm{Graph}(\Phi)$ for all positive integers j, and

$$(\mathbf{p}, \mathbf{q}) = \lim_{j \to +\infty} (\mathbf{x}_{k_j}, \mathbf{y}_{k_j}).$$

But were there to exist $(\mathbf{p}, \mathbf{q}) \in X \times Y$ for which $\mathbf{q} \in F$ and $(\mathbf{p}, \mathbf{q}) \in \mathrm{Graph}(\Phi)$, it would follow that $\mathbf{q} \in \Phi(\mathbf{p}) \cap F$. But this is impossible, because $\Phi(\mathbf{p}) \cap F = \emptyset$. Thus a contradiction would arise were there to exist an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X for which $\Phi(\mathbf{x}_j) \cap F \neq \emptyset$ and $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$. Therefore no such infinite sequence can exist, and therefore there must exist some positive real number δ such that $\Phi(\mathbf{x}) \subset V$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$. We conclude that

$$\{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V\}$$

is open in X. The result follows. ∎

Proof of Proposition 2.18 using Proposition 2.10 Let

$$W = \{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\},$$

and let $\mathbf{p} \in W$. If $\Phi(\mathbf{p}) = \emptyset$ then it follows from Lemma 2.14 that there exists some positive real number δ such that $\Phi(\mathbf{x}) = \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Then $\mathbf{x} \in W$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$.

Suppose that $\Phi(\mathbf{p}) \neq 0$. Let $K = \Phi(\mathbf{p})$. Then K is a compact subset of Y, because the correspondence Φ is compact-valued. Also $(\mathbf{p}, \mathbf{y}) \in U$ for all $\mathbf{y} \in K$. It follows from Proposition 2.10 that there exists some positive real number δ_1 such that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_1$ and $d_Y(\mathbf{y}, K) < \delta_1$, where

$$d_Y(\mathbf{y}, K) = \inf\{|\mathbf{y} - \mathbf{z}| : \mathbf{z} \in K\}.$$

Let

$$V = \{\mathbf{y} \in Y : d_Y(\mathbf{y}, K) < \delta_1\}.$$

Then V is open in Y because the function sending $\mathbf{y} \in Y$ to $d(\mathbf{y}, K)$ is continuous on Y (see Lemma 2.8). Also $\Phi(\mathbf{p}) \subset V$. It follows from the upper hemicontinuity of the correspondence Φ that there exists some positive number δ_2 such that $\Phi(\mathbf{x}) \subset V$ whenever $|\mathbf{x} - \mathbf{p}| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then $\Phi(\mathbf{x}) \subset V$. But then $d(\mathbf{y}, K) < \delta_1$ for all $\mathbf{y} \in \Phi(\mathbf{x})$. Moreover $|\mathbf{x} - \mathbf{p}| < \delta_1$. It follows that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{y} \in \Phi(\mathbf{x})$, and therefore $\mathbf{x} \in W$. This shows that W is an open subset of X, as required. ∎

Proof of Proposition 2.18 using the Heine-Borel Theorem

Let $\Phi \colon X \to Y$ be a compact-valued upper hemicontinuous correspondence, and let U be a subset of $X \times Y$ that is open in $X \times Y$. Let

$$W = \{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}.$$

We must prove that W is open in X.

Let $K = \Phi(\mathbf{p})$. Then, given any point $\mathbf{y}$ of K, there exists an open set $M_{\mathbf{p},\mathbf{y}}$ in X and an open set $V_{\mathbf{p},\mathbf{y}}$ in Y such that $M_{\mathbf{p},\mathbf{y}} \times V_{\mathbf{p},\mathbf{y}} \subset U$ (see Lemma 2.5). Now every open cover of K has a finite subcover, because K is compact. Therefore there exist points $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k$ of K such that

$$K \subset V_{\mathbf{p},\mathbf{y}_1} \cup V_{\mathbf{p},\mathbf{y}_2} \cup \cdots \cup V_{\mathbf{p},\mathbf{y}_k}.$$

Let

$$M_{\mathbf{p}} = M_{\mathbf{p},\mathbf{y}_1} \cap M_{\mathbf{p},\mathbf{y}_2} \cap \cdots \cap M_{\mathbf{p},\mathbf{y}_k}$$

and

$$V_{\mathbf{p}} = V_{\mathbf{p},\mathbf{y}_1} \cup V_{\mathbf{p},\mathbf{y}_2} \cup \cdots \cup V_{\mathbf{p},\mathbf{y}_k}.$$

Then

$$M_{\mathbf{p}} \times V_{\mathbf{p}} \subset \bigcup_{j=1}^{k} (M_{\mathbf{p}} \times V_{\mathbf{p},\mathbf{y}_j}) \subset \bigcup_{j=1}^{k} (M_{\mathbf{p},\mathbf{y}_j} \times V_{\mathbf{p},\mathbf{y}_j}) \subset U.$$

Now $M_{\mathbf{p}}$ is open in X, because it is the intersection of a finite number of subsets of X that are open in X. Also it follows from the upper hemicontinuity of the correspondence Φ that $\Phi^+(V_{\mathbf{p}})$ is open in X, where

$$\Phi^+(V_{\mathbf{p}}) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V_{\mathbf{p}}\}$$

(see Lemma 2.1). Let $N_{\mathbf{p}} = M_{\mathbf{p}} \cap \Phi^+(V_{\mathbf{p}})$. Then $N_{\mathbf{p}}$ is open in X and $\mathbf{p} \in N_{\mathbf{p}}$. Now if $\mathbf{x} \in N_{\mathbf{p}}$ then $\mathbf{x} \in M_{\mathbf{p}}$ and $\Phi(\mathbf{x}) \subset V_{\mathbf{p}}$, and therefore $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{y} \in \Phi(\mathbf{x})$. We have thus shown that $N_{\mathbf{p}} \subset W$ for all $\mathbf{p} \in W$, where

$$W = \{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}.$$

Thus W is the union of the subsets $N_{\mathbf{p}}$ as $\mathbf{p}$ ranges over the points of W. Moreover the set $N_{\mathbf{p}}$ is open in X for each $\mathbf{p} \in W$. It follows that W must itself be open in X. Indeed, given any point $\mathbf{p}$ of W, there exists some positive real number δ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} \subset N_{\mathbf{p}} \subset W.$$

The result follows. ∎

Remark The various proofs of Proposition 2.18 were presented in the contexts of correspondences between subsets of Eucldean spaces. All these proofs generalize easily so as to apply to correspondence between subsets of metric spaces. The last of the proofs can be generalized without difficulty so as to apply to correspondences between topological spaces. Inded the notion of *correspondences* between topological spaces is defined so that a correspondence $\Phi\colon X \rightrightarrows Y$ between topological spaces X and Y associates to each point of X a subset $\Phi(\mathbf{x})$ of Y. Such a correspondence is said to be upper hemicontinuous at a point p of X if, given any open subset V of Y for which $\Phi(p) \subset V$, there exists an open set $N(p)$ in X such that $\Phi(x) \subset V$ for all $x \in N$.

The proof of Proposition 2.18 using the Heine-Borel Theorem presented above can be generalized to show that, given a compact-valued correspondence $\Phi\colon X \rightrightarrows Y$ between topological spaces X and Y, and given a subset U of Y, the set

$$\{x \in X : (x, y) \in U \text{ for all } y \in \Phi(x)\}$$

is open in X.

We describe another proof of the Berge Maximum Theorem using the characterization of compact-valued upper hemicontinuous correspondences using sequences established in Proposition 2.17 and the characterization of lower hemicontinuous correspondences using sequences established in Proposition 2.19. First we introduce some terminology.

Definition Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Let $(\mathbf{x}_j : j \in \mathbb{N})$ be a sequence of points of the domain X of the correspondence. We say that an infinite sequence $(\mathbf{y}_j : j \in \mathbb{N})$ in the codomain of the correspondence is a *companion sequence* for $(\mathbf{x}_j)$ with respect to the correspondence Φ if $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j.

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Then the continuity properties of $\Phi\colon X \rightrightarrows Y$ can be characterized in terms of companion sequences with respect to Φ as follows:—

- the correspondence $\Phi\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous at a point $\mathbf{p}$ of X if and only if, given any infinite sequence $(\mathbf{x}_j : j \in \mathbb{N})$ in X converging to the point $\mathbf{p}$, and given any companion sequence $(\mathbf{y}_j : j \in \mathbb{N})$ in Y, that companion sequence has a subsequence that converges to a point of $\Phi(\mathbf{p})$ (Proposition 2.17);

- the correspondence $\Phi\colon X \rightrightarrows Y$ is lower hemicontinuous at a point $\mathbf{p}$ of X if and only if, given any infinite sequence $(\mathbf{x}_j : j \in \mathbb{N})$ in X converging to the point $\mathbf{p}$, and given any point $\mathbf{q}$ of $\Phi(\mathbf{p})$, there exists a companion sequence $(\mathbf{y}_j : j \in \mathbb{N})$ in Y converging to the point $\mathbf{q}$. (Proposition 2.19).

Proof of Theorem 2.23 using Companion Sequences Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $f\colon X \times Y \to \mathbb{R}$ be a continuous real-valued function on $X \times Y$, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is both upper and lower hemicontinuous and that also has the property that $\Phi(\mathbf{x})$ is non-empty and compact for all $\mathbf{x} \in X$. Let

$$m(\mathbf{x}) = \sup\{f(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in \Phi(\mathbf{x})\}$$

for all $\mathbf{x} \in X$, and let the correspondence $M\colon X \rightrightarrows Y$ be defined such that

$$M(\mathbf{x}) = \{\mathbf{y} \in \Phi(\mathbf{x}) : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}$$

for all $\mathbf{x} \in X$. We must prove that $m\colon X \to \mathbb{R}$ is continuous, $M(\mathbf{x})$ is a non-empty compact subset of Y for all $\mathbf{x} \in X$, and the correspondence $M\colon X \rightrightarrows Y$ is upper hemicontinuous.

It follows from the continuity of $f\colon X \times Y \to \mathbb{R}$ that $M(\mathbf{x})$ is closed in $\Phi(\mathbf{x})$ for all $\mathbf{x} \in X$. It also follows from the Extreme Value Theorem (Theorem 1.20) that $M(\mathbf{x})$ is non-empty for all $\mathbf{x}$.

Let $(\mathbf{x}_j, j \in \mathbb{N})$ be a sequence in X which converges to a point $\mathbf{p}$ of X, and let $(\mathbf{y}_j^* : j \in \mathbb{N})$ be a companion sequence of $(\mathbf{x}_j)$ with respect to the correspondence M. Then, for each positive integer j, $\mathbf{y}_j^* \in \Phi(\mathbf{x}_j)$ and

$$f(\mathbf{x}_j, \mathbf{y}_j^*) \geq f(\mathbf{x}_j, \mathbf{y})$$

for all $\mathbf{y} \in \Phi(\mathbf{x}_j)$. Now the correspondence Φ is compact-valued and upper hemicontinuous. It follows from Proposition 2.17 that there exists a subsequence of $(\mathbf{y}_j^* : j \in \mathbb{N})$ that converges to an element $\mathbf{q}$ of $\Phi(\mathbf{q})$. Let that subsequence be the sequence $(\mathbf{y}_{k_j}^* : j \in \mathbb{N})$ whose members are

$$\mathbf{y}_{k_1}^*, \mathbf{y}_{k_2}^*, \mathbf{y}_{k_3}^*, \ldots,$$

where $k_1 < k_2 < k_3 < \cdots$. Then $\mathbf{q} = \lim\limits_{j\to+\infty} \mathbf{y}_{k_j}^*$.

We show that $\mathbf{q} \in M(\mathbf{p})$. Let $\mathbf{r} \in \Phi(\mathbf{p})$. The correspondence $\Phi\colon X \to Y$ is lower hemicontinuous. It follows that there exists a companion sequence $(\mathbf{z}_j\colon j \in N)$ to $(\mathbf{x}_j\colon j \in N)$ with respect to the correspondence Φ that converges to $\mathbf{r}$ (Proposition 2.19). Then

$$\lim_{j\to+\infty} \mathbf{y}_{k_j}^* = \mathbf{q} \quad \text{and} \quad \lim_{j\to+\infty} \mathbf{z}_{k_j} = \mathbf{r}.$$

It follows from the continuity of $f\colon X \times Y \to \mathbb{R}$ that

$$\lim_{j\to+\infty} f(\mathbf{x}_{k_j}, \mathbf{y}_{k_j}^*) = f(\mathbf{p}, \mathbf{q}) \quad \text{and} \quad \lim_{j\to+\infty} f(\mathbf{x}_{k_j}, \mathbf{z}_{k_j}) = f(\mathbf{p}, \mathbf{r}).$$

Now

$$f(\mathbf{x}_{k_j}, \mathbf{y}_{k_j}^*) \geq f(\mathbf{x}_{k_j}, \mathbf{z}_{k_j})$$

for all positive integers j, because $\mathbf{y}_{k_j}^* \in M(\mathbf{x}_{k_j})$. It follows that

$$f(\mathbf{p}, \mathbf{q}) = \lim_{j\to+\infty} f(\mathbf{x}_{k_j}, \mathbf{y}_{k_j}^*) \geq \lim_{j\to+\infty} f(\mathbf{x}_{k_j}, \mathbf{z}_{k_j}) = f(\mathbf{p}, \mathbf{r}).$$

Thus $f(\mathbf{p}, \mathbf{q}) \geq f(\mathbf{p}, \mathbf{r})$ for all $\mathbf{r} \in \Phi(\mathbf{p})$. It follows that $\mathbf{q} \in M(\mathbf{p})$.

We have now shown that, given any sequence $(\mathbf{x}_j : j \in \mathbb{R})$ in X converging to the point $\mathbf{p}$, and given any companion sequence $(\mathbf{y}_j^* : j \in \mathbb{R})$ with respect to the correspondence M, there exists a subsequence of $(\mathbf{y}_j^* : j \in \mathbb{R})$ that converges to a point of $M(\mathbf{x})$. It follows that the correspondence $M\colon X \to Y$ is compact-valued and upper hemicontinuous at the point $\mathbf{p}$ (Proposition 2.17).

It remains to show that the function $m\colon X \to \mathbb{R}$ is continuous at the point $\mathbf{p}$, where $m(\mathbf{x}) = f(\mathbf{x}, \mathbf{y}^*)$ for all $\mathbf{x} \in X$ and $\mathbf{y}^* \in M(\mathbf{x})$. Let $(\mathbf{x}_j : j \in \mathbb{R})$ be an infinite sequence converging to the point $\mathbf{p}$, and let $v_j = m(\mathbf{x}_j)$ for all positive integers j. Then there exists an infinite sequence Let $(\mathbf{y}_j^* : j \in \mathbb{R})$ in Y that is a companion sequence to $(\mathbf{x}_j)$ with respect to the correspondence M. Then $\mathbf{y}_j^* \in M(\mathbf{x}_j)$ and therefore $v_j = f(\mathbf{x}_j, \mathbf{y}_j^*)$ for all positive integers j. Now the correspondence $M\colon X \rightrightarrows Y$ has been shown to be compact-valued and upper hemicontinuous. There therefore exists a subsequence $(\mathbf{y}_{k_j}^* : j \in \mathbb{N})$ of $(\mathbf{y}_j)$ that converges to a point $\mathbf{q}$ of $M(\mathbf{p})$. It then follows from the continuity of the function $f\colon X \times Y \to \mathbb{R}$ that

$$\lim_{j\to+\infty} m(\mathbf{x}_{k_j}) = \lim_{j\to+\infty} v_{k_j} = \lim_{j\to+\infty} f(\mathbf{x}_{k_j}, \mathbf{y}_{k_j}^*) = f(\mathbf{p}, \mathbf{q}) = m(\mathbf{p}).$$

Now the result just proved can be applied with any subsequence of $(\mathbf{x}_j : j \in \mathbb{N})$ in place of the original sequence. It follows that *every subsequence* of of $(v_j : j \in \mathbb{R})$ itself has a subsequence that converges to $m(\mathbf{p})$.

Let some positive real number ε be given. Suppose that there did not exist any positive integer N with the property that $|v_j - m(\mathbf{p})| < \varepsilon$ whenever $j \geq N$. Then there would exist infinitely many positive integers j for which $|v_j - m(\mathbf{p})| \geq \varepsilon$. It follows that there would exist some subsequence

$$v_{l_1}, v_{l_2}, v_{l_3}, \ldots$$

of $v_1, v_2, v_3, \ldots$ with the property that $|v_{l_j} - m(\mathbf{p})| \geq \varepsilon$ for all positive integers j. This subsequence would not in turn contain any subsequences converging to the point $m(\mathbf{p})$.

But we have shown that every subsequence of $(v_j : j \in \mathbb{N})$ contains a subsequence converging to $m(\mathbf{p})$. It follows that there must exist some positive integer N with the property that $|v_j - m(\mathbf{p})| < \varepsilon$ whenever $j \geq N$. We conclude from this that $\lim_{j\to+\infty} m(\mathbf{x}_j) = m(\mathbf{p})$.

We have shown that if $(\mathbf{x}_j : j \in \mathbb{N})$ is an infinite sequence in X and if $\lim_{j\to+\infty} \mathbf{x}_j = \mathbf{p}$ then $\lim_{j\to+\infty} m(\mathbf{x}_j) = m(\mathbf{p})$. It follows that the function $m\colon X \to \mathbb{R}$ is continuous at $\mathbf{p}$. This completes the proof of Berge's Maximum Theorem. ▮

C Historical Note on Berge's Maximum Theorem

In 1959, the French mathematician Claude Berge published a book entitled *Espaces topologiques: fonctions multivoques* (Dunod, Paris, 1959). This book was subsequently translated into English by E.M. Patterson, and the translation was published with the title *Topological spaces, including a treatment of multi-valued functions, vector spaces and convexity* (Oliver and Boyd, Edinburgh and London, 1963).

Claude Berge had completed his Ph.D. at the University of Paris in 1953, supervised by the differential geometer and mathematical physicist *André Lichnerowicz*. His thesis was entitled *Sur une théorie ensembliste des jeux alternatifs*, and a paper of that name was published by him (*J. Math. Pures Appl.* **32** (1953), 129–184). He subsequently published *Théorie Générale des Jeux à N Personnes* (Gauthier Villars, Paris, 1957). The title translates as "General theory of n-person games".

Claude Berge was Professor at the Institute of Statistics at the University of Paris from 1957 to 1964, and subsequently directed the International Computing Center in Rome. Following his early work in game theory, his research developed in the fields of combinatorics and graph theory.

The preface of the 1959 book, *Espaces topologiques: fonctions multivoques*, includes a passage translated by E.M. Patterson as follows:—

> In Set Topology, with which we are concerned in this book, we study sets in topological spaces and topological vector spaces; whenever these sets are colletions of n-tuples or classes of functions, we recover well-known results of classical analysis.
>
> But the role of topology does not stop there; the majority of text-books seem to ignore certain problems posed by the calculus of probabilities, the decision functions of statistics, linear programming, cybernetics, economics; thus, in order to provide a topological tool which is of equal interest to the student of pure mathematics and the student of applied mathematics, we have felt it desirable to include a systematic devcelopment of the properties of *multi-valued functions*.

The following theorem is included in *Espaces topologiques* by Claude Berge (Chapter 6, Section 3, page 122):—

> **Théorème du maximum.** — Si $\varphi(y)$ est une fonction numérique continue dans Y, et si Γ est un application continue

de X dans Y telle que $\Gamma x \neq \emptyset$ pour tout x,

$$M(x) = \max\{\varphi(y)/y \in \Gamma x\}$$

est une fonction numérique continue dans X, et

$$\Phi x = \{y/y \in \Gamma x, \varphi(y) = M(x)\}$$

est une application u.s.c. de X dans Y.

This theorem is translated by E.M. Patterson as follows (*Topological Spaces*, Claude Berge, translated by E.M. Patterson, Oliver and Boyd, Edinburgh, 1963, in Chapter 6, Section 3, page 116):—

> **Maximum Theorem** — If φ is a continuous numerical function in Y and Γ is a continuous mapping of X into Y such that, for each x, $\Gamma x \neq \emptyset$, then the numerical function M defined by
>
> $$M(x) = \max\{\varphi(y)/y \in \Gamma x\}$$
>
> is continuous in X and the mapping Φ defined by
>
> $$\Phi x = \{y/y \in \Gamma x, \varphi(y) = M(x)\}$$
>
> is an u.s.c. mapping of X into Y.

In this context X and Y are Hausdorff topological spaces. Indeed in Chapter 4, Section 5 of *Espaces topologiques*, Berge introduces the concept of a *separated* (or *Hausdorff*) space and then, after some discussion of separation properties, makes that statement translated by E.M. Patterson as follows:—

> In what follows all the topological spaces which we consider will be assumed to be separated.

It seems that, in the original statement, the objective function φ was required to be a continuous function on Y, but the first sentence of the proof of the "Maximum Theorem" notes that φ is continuous on $X \times Y$. A "mapping" in Berge is a correspondence. A mapping (or correspondence) is said by Berge to be "upper semi-continuous" when it is both compact-valued and upper hemicontinuous; a mapping is said by Berge to be "lower semi-continuous" when it is lower hemicontinuous.

Berge's proof of the *Théorème du maximum* is just one short paragraph, but requires the work of earlier theorems. We discuss his proof using the terminology adopted in these lectures. In Theorem 1 of Chapter 6, Section 4, Berge shows that if the correpondence $\Gamma\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous then, given any point x_0 of X, and given any positive real number ε, the function $M(x)$ equal to the maximum value of the objective function ϕ on $\Gamma(x)$ satisfies $M(x) \leq M(x_0) + \varepsilon$ throughout some open neighbourhood of the point x_0. (This result can be compared with Lemma 2.21 and the first proof of Theorem 2.23 presented in these notes.) In Theorem 2 of Chapter 6, Section 4, Berge shows that if the correspondence Γ is lower hemicontinuous then, given any point x_0 of X, and given any positive real number ε, the function $M(x)$ equal to the maximum value of the objective function ϕ on $\Gamma(x)$ satisfies $M(x) \geq M(x_0) - \varepsilon$ throughout some open neighbourhood of the point x_0.

(This result can be compared with Lemma 2.22 and the first proof of Theorem 2.23 presented in these notes.) These two results ensure that if Γ is compact-valued, everywhere non-empty and both upper and lower hemicontinuous then the function function M is continuous on X. In Theorem 7 of Chapter 6, Section 1, Berge had proved that the intersection of a compact-valued upper hemicontinuous correspondence and a correspondence with closed graph is compact valued and upper hemicontinuous (see Proposition 2.20 of these notes). Berge completes the proof of the *Théorème du maximum* by putting these results together in a fashion to obtain a proof (in the contexts of correspondences between Hausdorff topological spaces) similar in structure to the first proof of Theorem 2.23 presented in these notes.

The definitions of "upper-semicontinuous" and "lower-semicontinuous" mappings (i.e., correspondences) Given by Claude Berge at the beginning of Chapter VI are accompanied by a footnote translated by E.M. Patterson as follows (C. Berge, translated E.M. Patterson, *Topological Spaces*, *loc. cit.*, p. 109):—

> The two kinds of semi-continuity of a multivalued function were introduced independently by Kuratowski (*Fund. Math.* **18**, 1932, p.148) and Bouligand (*Ens. Math.*, 1932, p. 14). In general, the definitions given by different authors do not coincide whenever we deal with non-compact spaces (at least for upper semi-continuity, which is the more important from the point of view of applications). The definitions adopted here, which we have developed elsewhere (C. Berge, *Mém. Sc. Math.* 138), enable us to include the case when the image of a point x can be empty.

In 1959, the year in which Claude Berge published *Espaces topologiques*, Gérard Debreu published his influential monograph *Theory of value: an axiomatic analysis of economic equilibrium* (Cowles Foundation Monographs 17, 1959). Section 1.8 of Debreu's monograph discusses "continuous correspondences", developing the theory of correspondences φ from S to T, where S is a subset of $\mathbb{R}^m$ and T is a compact subset of $\mathbb{R}^n$. Debreu also requires correspondences to be non-empty-valued. In consequence of these conventions, closed-valued correspondences from S to T must necessarily be compact-valued. Also a correspondence from S to T is upper hemicontinuous if and only if its graph is closed (see Propositions 2.11 2.12 of these notes).

In the notes to Chapter 1 of the *Theory of Value*, Debreu notes that "a study of the *continuity of correspondences* from a topological space to a topological space will be found in C. Berge [1], Chapter 6". The reference is to *Espace Topologiques*.

According to Debreu, the correspondence φ is *upper semicontinuous* at the point x^0 if the following condition is satisfied:

"$x^q \to x^0$, $y^q \in \varphi(x^q)$, $y^q \to y^0$" implies "$y^0 \in \varphi(x^0)$".

This condition is satisfied at each point of the domain of a correspondence if and only if that correspondence has closed graph. Thus Debreu's definition is in accordance with the definition of *upper hemicontinuity* for those correspondences, and only those correspondences, where the codomain of the correspondence is a compact subset of a Euclidean space. Indeed Debreu notes the following in Section 1.8 of the *Theory of Value*:—

> "(1) *The correspondence φ is upper semicontinuous on S if and only if its graph is closed in $S \times T$.*"

Again according to Debreu, the correspondence φ is *lower semicontinuous* at the point x^0 if the following condition is satisfied:

"$x^q \to x^0$, $y^0 \in \varphi(x^0)$" implies "there is (y^q) such that $y^q \in \varphi(x^q)$, $y^q \to y^0$".

This condition is satisfied at each point of the domain of a correspondence if and only if that correspondence is lower hemicontinuous (in accordance with the definitions adopted in those notes, see Proposition 2.19 of these notes).

A correspondence from S to T is said by Debreu to be continuous if it is both upper semicontinuous and lower semicontinuous according to his definitions.

Debreu discusses Berge's Maximum Theorem, in the context of a correspondence φ from a subset S of $\mathbb{R}^m$ to a compact subset T of $\mathbb{R}^n$, as follows (*Theory of Value*, Section 1.8, page 19):—

The interest of these concepts for economics lies, in particular, in the interpretations of an element x of S as the environment of a certain agent, of T as the set of actions *a priori* available to him, and of $\varphi(x)$ (assumed here to be closed for every x in S) as the subset of T to which his choice is actually restricted by the environment x. Let f be a continuous *real-valued function on* $S \times T$, and interpret $f(x, y)$ as the gain for that agent when his environment is x and his action y. *Given* x, one is interested in the *elements of* $\varphi(x)$ *which maximize* f (now *a function of* y *alone*) on $\varphi(x)$; they form a set $\mu(x)$. What can be said about the continuity of the correspondence μ from S to T?

One is also interested in $g(x)$, *the value of the maximum of* f *on* $\phi(x)$ *for a given* x. What can be said about the continuity of the real-valued function g on S? An answer to these two questions is given by the following result (the proof of the continuity of g should not be attempted).
(4) If f is continuous on $S \times T$, and if φ is continuous at $x \in S$, then μ is upper semicontinuous at x, and g is continuous on x.

The book *Infinite dimensional analysis: a hitchhiker's guide* by Charalambos D. Aliprantis and Kim C. Border (2nd edition, Springer-Verlag, 1999) discusses the theory of continuous correspondences between topological spaces (Chapter 16). Berge's Maximum Theorem is stated and proved, in the context of correspondences between topological spaces, as Theorem 16.31 (p. 539). The definitions of upper hemicontinuity and lower hemicontinuity for correspondences are consistent with the definitions adopted in these lecture notes. These definitions are accompanied by the following footnote:—

> J. C. Moore [...] identifies five slightly different definitions of upper semicontinuity in use by economists, and points out some of the differences for compositions, etc. T. Ichiishi [...] and E. Klein and A. C. Thompson [...] also give other notions of continuity.

The book *Mathematical Methods and Models for Economists* by Angel de la Fuente (Cambridge University Press, 2000) includes a section on continuity of correspondences between subsets of Euclidean spaces (Chapter 2, Section 11). The definitions of upper and lower hemicontinuity adopted there are consistent with those given in these lecture notes. The sequential characterization of compact-valued upper hemicontinuous correspondences in terms of companion sequences (Proposition 2.17 of these lecture notes) is stated and proved as Theorem 11.2 of Chapter 2 of Angel de la Fuente's textbook. Similarly the sequencial characterization of lower hemicontinuous correspondences in terms of companion sequences Proposition 2.19 is stated and proved as Theorem 11.3 of that textbook.

Theorem 11.6 in Chapter 2 of that textbook covers the result that a closed-valued upper hemicontinuous correspondence has a closed graph (see Proposition 2.11) and the result that a correspondence with closed graph whose codomain is compact is upper hemicontinuous (see Proposition 2.12). The result that the intersection of a compact-valued upper hemicontinuous correspondence and a correspondence with closed graph is compact-valued and upper hemicontinuous (see Proposition 2.20) is Theorem 11.7 in Chapter 2 of the textbook by Angel de la Fuente. Berge's Maximal Theorem is Theorem 2.1 in Chapter 7 of that textbook. The proof is based on the use of the sequential characterizations of upper and lower hemicontinuity in terms of existence and properties of companion sequences.

D Further Results Concerning Barycentric Subdivision

D.1 The Barycentric Subdivision of a Simplex

Proposition D.1 *Let σ be a simplex in $\mathbb{R}^N$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$, and let $m_0, m_1, \ldots, m_r$ be integers satisfying*

$$0 \leq m_0 < m_1 < \cdots < m_r \leq q.$$

Let ρ be the simplex in $\mathbb{R}^N$ with vertices $\hat{\tau}_0, \hat{\tau}_1, \ldots, \hat{\tau}_r$, where $\hat{\tau}_k$ denotes the barycentre of the simplex τ_k with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{m_k}$ for $k = 1, 2, \ldots, r$. Then the simplex ρ is the set consisting of all points of $\mathbb{R}^N$ that can be represented in the form $\sum_{j=0}^{q} t_j \mathbf{v}_j$, where $t_0, t_1, \ldots, t_q$ are real numbers satisfying the following conditions:

(i) $0 \leq t_j \leq 1$ *for* $j = 0, 1, \ldots, q$;

(ii) $\sum_{j=0}^{q} t_j = 1$;

(iii) $t_0 \geq t_1 \geq \cdots \geq t_q$;

(iv) $t_j = t_{m_0}$ *for all integers j satisfying* $j \leq m_0$;

(v) $t_j = t_{m_k}$ *for all integers j and k satisfying* $0 < k \leq r$ *and* $m_{k-1} < j \leq m_k$;

(vi) $t_j = 0$ *for all integers j satisfying* $j > m_r$.

Moreover the interior of the simplex ρ is the set consisting of all points of $\mathbb{R}^N$ that can be represented in the form $\sum_{j=0}^{q} t_j \mathbf{v}_j$, where $t_0, t_1, \ldots, t_q$ are real numbers satisfying conditions (i)–(iv) above together with the following extra condition:

(vii) $t_{m_{k-1}} > t_{m_k} > 0$ *for all integers k satisfying* $0 < k \leq r$.

Proof Let $\mathbf{w}_k = \hat{\tau}_k$ for $k = 0, 1, \ldots, r$. Then

$$\mathbf{w}_k = \frac{1}{m_k + 1} \sum_{j=0}^{m_k} \mathbf{v}_j.$$

Let $\mathbf{x} \in \rho$, and let the real numbers $u_0, u_1, \ldots, u_r$ be the barycentric coordinates of the point $\mathbf{x}$ with respect to the vertices $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_r$ of ρ, so that $0 \leq u_k \leq 1$ for $k = 0, 1, \ldots, r$, $\sum_{k=0}^{r} u_k \mathbf{w}_k = \mathbf{x}$, and $\sum_{k=0}^{r} u_k = 1$.

Also let

$$K(j) = \{k \in \mathbb{Z} : 0 \leq k \leq r \text{ and } m_k \geq j\}$$

for $j = 0, 1, \ldots, q$. Then $\mathbf{x} = \sum_{j=0}^{q} t_j \mathbf{v}_j$, where

$$t_j = \sum_{k \in K(j)} \frac{u_k}{m_k + 1}$$

when $0 \leq j \leq m_r$, and $t_j = 0$ when $m_r < j \leq q$. Moreover

$$\begin{aligned} \sum_{j=0}^{q} t_j &= \sum_{j=0}^{m_r} \sum_{k \in K(j)} \frac{u_k}{m_k + 1} = \sum_{(j,k) \in L} \frac{u_k}{m_k + 1} \\ &= \sum_{k=0}^{r} \sum_{j=0}^{m_k} \frac{u_k}{m_k + 1} = \sum_{k=0}^{r} u_k = 1, \end{aligned}$$

where

$$L = \{(j, k) \in \mathbb{Z}^2 : 0 \leq j \leq q, \ \ 0 \leq k \leq r \text{ and } j \leq m_k\}.$$

Now $t_j \geq 0$ for $j = 0, 1, \ldots, q$, because $u_k \geq 0$ for $k = 0, 1, \ldots, r$, and therefore

$$0 \leq t_j \leq \sum_{j=0}^{q} t_j = 1.$$

Also $t_{j'} \leq t_j$ for all integers j and j' satisfying $0 \leq j < j' \leq m_r$, because $K(j') \subset K(j)$. If $0 \leq j \leq m_0$ then $K(j) = K(m_0)$, and therefore $t_j = t_{m_0}$. Similarly if $0 < k \leq r$, and $m_{k-1} < j \leq m_k$ then $K(j) = K(m_k)$, and therefore $t_j = t_{m_k}$. Thus the real numbers $t_0, t_1, \ldots, t_k$ satisfy conditions (i)–(vi) above.

Now let $t_0, t_1, \ldots, t_q$ be real numbers satisfying conditions (i)-(vi), let

$$u_r = (m_r + 1) t_{m_r}$$

and

$$u_k = (m_k + 1)(t_{m_k} - t_{m_{k+1}})$$

for $k = 0, 1, \ldots, r - 1$. Then

$$t_{m_k} = \sum_{k'=k}^{r} \frac{u_{k'}}{m_{k'} + 1}$$

for $k = 0, 1, \ldots, r$. Also $u_k \geq 0$ for $k = 0, 1, \ldots, r$, and

$$\begin{aligned}
\sum_{k=0}^{r} u_k &= \sum_{k=0}^{r-1} (m_k + 1)(t_{m_k} - t_{m_{k+1}}) + (m_r + 1) t_{m_r} \\
&= (m_0 + 1) t_{m_0} + \sum_{k=1}^{r-1} (m_k + 1) t_{m_k} - \sum_{k=0}^{r-2} (m_k + 1) t_{m_{k+1}} \\
&\qquad - (m_{r-1} + 1) t_{m_r} + (m_r + 1) t_{m_r} \\
&= (m_0 + 1) t_{m_0} + \sum_{k=1}^{r-1} (m_k + 1) t_{m_k} - \sum_{k=1}^{r-1} (m_{k-1} + 1) t_{m_k} \\
&\qquad + (m_r - m_{r-1}) t_{m_r} \\
&= (m_0 + 1) t_{m_0} + \sum_{k=1}^{r} (m_k - m_{k-1}) t_{m_k},
\end{aligned}$$

But

$$\begin{aligned}
\sum_{j=0}^{q} t_q &= \sum_{j=0}^{m_0} t_j + \sum_{k=1}^{r} \sum_{j=m_{k-1}+1}^{m_k} t_j \\
&= (m_0 + 1) t_{m_0} + \sum_{k=1}^{r} (m_k - m_{k-1}) t_{m_k},
\end{aligned}$$

because conditions (i)-(vi) satisfied by the real numbers $t_0, t_1, \ldots, t_q$ ensure that $t_j = t_{m_0}$ when $0 \leq j \leq m_0$, $t_j = t_{m_k}$ when $1 \leq k \leq r$, and $m_{k-1} < j \leq m_k$ and $t_j = 0$ when $j > m_r$. Thus

$$\sum_{k=0}^{r} u_k = (m_0 + 1) t_{m_0} + \sum_{k=1}^{r} (m_k - m_{k-1}) t_{m_k} = \sum_{j=0}^{q} t_j = 1.$$

It follows that $u_0, u_1, \ldots, u_r$ are the barycentric coordinates of a point of the simplex with vertices $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_r$. Moreover

$$t_j = \sum_{k \in K(j)} \frac{u_k}{m_k + 1}$$

for $j = 0, 1, \ldots, q$, and therefore

$$\begin{aligned}
\sum_{k=0}^{r} u_k \mathbf{w_k} &= \sum_{k=0}^{r} \sum_{j=0}^{m_k} \frac{u_k}{m_k+1} \mathbf{v}_j \\
&= \sum_{(j,k)\in L} \frac{u_k}{m_k+1} \mathbf{v}_j \\
&= \sum_{j=0}^{q} \sum_{k\in K(j)} \frac{u_k}{m_k+1} \mathbf{v}_j \\
&= \sum_{j=0}^{q} t_j \mathbf{v}_j.
\end{aligned}$$

We conclude the the simplex ρ is the set of all points of $\mathbb{R}^N$ that are representable in the form $\sum_{j=0}^{q} t_j \mathbf{v}_j$, where the coefficients $t_0, t_1, \ldots, t_q$ are real numbers satisfying conditions (i)–(vi).

Now the point $\sum_{j=0}^{q} t_j \mathbf{v}_j$ belongs to the interior of the simplex ρ if and only if $u_k > 0$ for $k = 0, 1, \ldots, r$, where $u_r = (m_r + 1) t_{m_r}$ and $u_k = (m_k + 1)(t_{m_k} - t_{m_{k+1}})$ for $k = 0, 1, \ldots, r - 1$. This point therefore belongs to the interior of the simplex ρ if and only if $t_{m_r} > 0$ and $t_{m_k} > t_{m_{k+1}}$ for $k = 0, 1, \ldots, r - 1$. Thus the interior of the simplex ρ consists of those points $\sum_{j=0}^{q} t_j \mathbf{v}_j$ of σ whose barycentric coordinates $t_0, t_1, \ldots, t_q$ with respect to the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of σ satisfy conditions (i)–(vii), as required. ∎

Corollary D.2 *Let σ be a simplex in some Euclidean space $\mathbb{R}^N$, and let K_σ be the simplicial complex consisting of the simplex σ together with all of its faces. Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of σ, and let $t_0, t_1, \ldots, t_q$ be the barycentric coordinates of some point $\mathbf{x}$ of σ, so that $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$, $\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{x}$ and $\sum_{j=0}^{q} t_j = 1$. Then there exists a permutation π of the set $\{0, 1, \ldots, q\}$ and integers $m_0, m_1, \ldots, m_r$ satisfying*

$$0 \leq m_0 < m_1 < \cdots < m_r \leq q.$$

such the following conditions are satisfied:

(iii) $t_{\pi(0)} \geq t_{\pi(1)} \geq \cdots \geq t_{\pi(q)}$;

(iv) $t_{\pi(j)} = t_{\pi(m_0)}$ *for all integers j satisfying $j \leq m_0$;*

(v) $t_{\pi(j)} = t_{\pi(m_k)}$ *for all integers* j *and* k *satisfying* $0 < k \leq r$ *and* $m_{k-1} < j \leq m_k$*;*

(vi) $t_{\pi(j)} = 0$ *for all integers* j *satisfying* $j > m_r$*.*

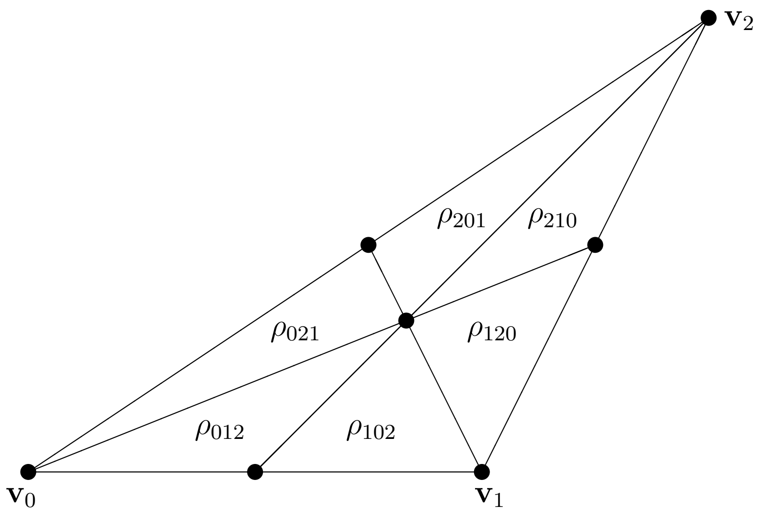
(vii) $t_{\pi(m_{k-1})} > t_{\pi(m_k)} > 0$ *for all integers* k *satisfying* $0 < k \leq r$*.*

Let ρ *be the simplex of the first barycentric subdivision* K'_σ *of the simplical complex* K_σ *with vertices* $\hat{\tau}_0, \hat{\tau}_1, \ldots, \hat{\tau}_r$*, where* $\hat{\tau}_k$ *is the barycentre of the simplex* τ_k *with vertices* $\mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(m_k)}$ *for* $k = 0, 1, \ldots, r$*. Then* ρ *is the unique simplex of* K'_σ *that contains the point* $\mathbf{x}$ *in its interior.*

Proof The required permutation π can be any permutation that rearranges the barycentric coordinates in descending order, so that $1 \geq t_{\pi(0)} \geq t_{\pi(1)} \geq \ldots \geq t_{\pi(q)} \geq 0$. The required result then follows immediately on applying Proposition D.1. ∎

Corollary D.2 may be applied to determine the simplices of the first barycentric subdivision K'_σ of the simplicial complex K_σ that consists of some simplex σ together with all of its faces.

Example Let K be the simplicial complex consisting of a triangle with vertices $\mathbf{v}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$, together with all its edges and vertices, and let K' be the first barycentric subdivision of the simplicial complex K. Then K' consists of six triangles ρ_{012}, ρ_{102}, ρ_{021}, ρ_{120}, ρ_{201} and ρ_{210}, together with all the edges and vertices of those triangles, where



$$
\begin{aligned}
\rho_{012} &= \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_0 \geq t_1 \geq t_2 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\}, \\
\rho_{102} &= \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_1 \geq t_0 \geq t_2 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\}, \\
\rho_{021} &= \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_0 \geq t_2 \geq t_1 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\}, \\
\rho_{120} &= \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_1 \geq t_2 \geq t_0 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\}, \\
\rho_{201} &= \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_2 \geq t_0 \geq t_1 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\}, \\
\rho_{210} &= \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_2 \geq t_1 \geq t_0 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\}.
\end{aligned}
$$

The intersection of any two of those triangles is a common edge or vertex of those triangles. For example, the intersection of the triangles ρ_{012} and ρ_{102} is the edge $\rho_{012} \cap \rho_{102}$, where

$$
\rho_{012} \cap \rho_{102} = \left\{ \sum_{j=0}^{2} t_j \mathbf{v}_j : 1 \geq t_0 = t_1 \geq t_2 \geq 0 \text{ and } \sum_{j=0}^{2} t_j = 1 \right\}.
$$

And the intersection of the triangle ρ_{012} and ρ_{120} is the barycentre of the triangle $\mathbf{v}_0\,\mathbf{v}_1\,\mathbf{v}_2$, and is thus the point $\sum\limits_{j=0}^{2} t_j \mathbf{v}_j$ whose barycentric coordinates t_0, t_1, t_2 satisfy $t_0 = t_1 = t_2 = \frac{1}{3}$.

Let σ be a q-simplex with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$, let K_σ be the simplicial complex consisting of the simplex σ, together with all its faces, and let K'_σ be the first barycentric subdivision of the simplicial complex K_σ. Then the q-simplices of K'_σ are the simplices of the form $\rho_{m_0\,m_1\,\ldots\,m_q}$, where the list $m_0, m_1, \ldots, m_q$ is a rearrangement of the list $0, 1, \ldots, q$ (so that each integer between 0 and q occurs exactly one in the list $m_0, m_1, \ldots, m_q$), and where

$$
\begin{aligned}
&\rho_{m_0\,m_1\,\ldots\,m_q} \\
&\quad = \left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j : 1 \geq t_{m_0} \geq t_{m_1} \geq \cdots \geq t_{m_q} \geq 0 \text{ and } \sum_{j=0}^{q} t_j = 1 \right\}.
\end{aligned}
$$

A point of σ belongs to the interior of one of the simplices of K'_σ if and only if its barycentric coordinates $t_0, t_1, \ldots, t_q$ are all distinct and strictly positive. Moreover if a point $\sum_{j=0}^{q} t_j \mathbf{v}_j$ of σ with barycentric coordinates $t_0, t_1, \ldots, t_q$ belongs to the interior of some r-simplex of K'_σ then there are exactly $r+1$ distinct values amongst the real numbers $t_0, t_1, \ldots, t_q$ (i.e., $\{t_0, t_1, \ldots, t_q\}$ is a set with exactly $r+1$ elements).