Worked Solution to Hemicontinuity Question

1. *Let* $\Phi\colon \mathbb{R} \rightrightarrows \mathbb{R}$ *be the correspondence defined such that*

$$\Phi(x) = \begin{cases} \left\{y \in \mathbb{R} : xy^2 + (1-x^2)y - x \geq 0\right\} & \textit{if } x < 0; \\ \{y \in \mathbb{R} : y \geq 0\} & \textit{if } x = 0; \\ \left\{y \in \mathbb{R} : y^2 - y - x - x^2 \leq 0\right\} & \textit{if } 0 < x < 1; \\ \{y \in \mathbb{R} : 0 \leq xy \leq 2\} & \textit{if } x \geq 1. \end{cases}$$

Answer the following, fully justifying your answers:—

(i) *Is* $\Phi\colon \mathbb{R} \to \mathbb{R}$ *upper hemicontinuous? at* $x = 0$*?*

We need to investigate the nature of the set $\Phi(x)$ when $x < 0$ and also when $0 < x < 1$. Now, if $x < 0$, Let x and y be real numbers, where $x < 0$. Then $y \in \Phi(x)$ if and only if $xy^2 + (1-x^2)y - x \geq 0$. Now $xy^2 + (1-x^2)y - x$ is a quadratic polynomial in y whose roots are

$$\frac{1}{2x}\left(-(1-x^2) \pm \sqrt{(1-x^2)^2 + 4x^2}\right).$$

But $(1-x^2)^2 + 4x^2 = (1+x^2)^2$. Thus the roots of the quadratic polynomial are

$$\frac{1}{2x}\left(x^2 - 1 \pm (1+x^2)\right)$$

and are thus x and $-1/x$. Moreover $0 \in \Phi(x)$ when $x < 0$. We deduce that

$$\Phi(x) = \left[x, -\frac{1}{x}\right] \quad \text{whenever } x < 0.$$

Next we consider the behaviour of $\Phi(x)$ when $0 < x < 1$. Now $y^2 - y - x - x^2$ is a quadratic polynomial in y whose roots are $\frac{1}{2}(1 \pm \sqrt{1 + 4x + 4x^2})$. Moreover $1 + 4x + 4x^2 = (2x+1)^2$. It follows that the roots of the quadratic polynomial are $x + 1$ and $-x$. Morever $0 \in \Phi(x)$ when $0 < x < 1$. It follows that

$$\Phi(x) = [-x, x+1] \quad \text{whenever } 0 < x < 1.$$

Note furthermore that $\Phi(0) = [0, +\infty)$.

We are now in a position to determine whether or not the correspondence is upper hemicontinuous at $x = 0$. Let V be an open set for which $\Phi(0) \subset V$. Then there exists $\delta > 0$ for which $[-\delta, +\infty) \subset V$. If $-\delta < x < 0$ then

$$\Phi(x) = [x, -1/x] \subset [x, +\infty) \subset [-\delta, +\infty) \subset V.$$

Also if $0 < x < \delta$ then

$$\Phi(x) = [-x, x+1] \subset [-x, +\infty) \subset [-\delta, +\infty) \subset V.$$

It follows that $\Phi(x) \subset V$ for all real numbers x satisfying $|x| < \delta$. Thus Φ is upper hemicontinuous at 0.

(ii) *Is $\Phi: \mathbb{R} \to \mathbb{R}$ lower hemicontinuous at $x = 0$?*

We make use of results obtained in investigating the previous part. For a counter-example, let $V = \{y \in \mathbb{R} : 2 < y < 3\}$. Then $\Phi(0) \cap V = V$ and thus $\Phi(0) \cap V \neq \emptyset$. But if $0 < x < 1$ then $y < 2$ for all $y \in \Phi(x)$, and therefore $\Phi(x) \cap V = \emptyset$. Thus the open set V provides the required counter-example, demonstrating that the correspondence Φ is not lower hemicontinuous at 0.

(iii) *Is $\Phi: \mathbb{R} \to \mathbb{R}$ upper hemicontinuous at $x = 1$?*

Note that

$$Phi(x) = \left[0, \frac{2}{x}\right] \quad \text{whenever } x \geq 1.$$

In particular $\Phi(1) = [0, 2]$.

The correspondence Φ is not upper hemicontinuous at 1. For a counter-example let $V = \{y \in \mathbb{R} : -\frac{1}{2} < x < 3\}$. Then $\Phi(1) \subset V$. But if $x > \frac{1}{2}$ then $-x \in \Phi(x)$ but $-x \notin V$. It follows that $\Phi(x)$ is not contained in V for any x satisfying $\frac{1}{2} < x < 1$. This counter-example demonstrates that Phi is not upper hemicontinuous at 1.

(iv) *Is $\Phi: \mathbb{R} \to \mathbb{R}$ lower hemicontinuous at $x = 1$?*

The correspondence Φ is lower hemicontinuous at 1. We have already noted that $\Phi(1) = [0, 2]$. If V is an open set in $\mathbb{R}$, and if either $0 \in V$ or $2 \in V$ then there exists $v \in V$ satisfying $0 < v < 2$. It follows that if V is an open set in $\mathbb{R}$ which has non-empty intersection with $[0, 2]$ then there exists some element v of V that satisfies $0 < v < 2$. If x satisfies $v - 1 < x < 1$ then $v < x + 1$ and therefore $v \in \Phi(x)$. Also if $1 \leq x < 2/v$ then $xv < 2$ and therefore $v \in \Phi(x)$. It follows that $\Phi(x) \cap V \neq \emptyset$ for all real numbers x satisfying $v - 1 < x < 2/v$. Thus Φ is lower hemicontinuous at 1.