Module MA3486: Annual Examination 2016
Worked solutions

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Module Website

The module website, with online lecture notes, problem sets. etc. are located at

`http://www.maths.tcd.ie/~dwilkins/Courses/MA3486/`

1. (a) [Seen similar.]

(i) Let V be an open set in $\mathbb{R}$ satisfying $\Phi(-1) \subset V$. Now $\Phi(-1) = \{y \in \mathbb{R} : y \geq 0\}$. Now $0 \in V$, and therefore there exists $\delta > 0$ such that $(-\delta, \delta) \subset V$. It follows that $\{y \in \mathbb{R} : y > -\delta\} \subset V$. But then $\Phi(x) \subset V$ whenever $-1 - \delta < x < -1 + \delta$. Thus $\Phi \colon \mathbb{R} \rightrightarrows \mathbb{R}$ is upper hemicontinuous at -1.

(ii) Let $V = \{y \in \mathbb{R} : 2 < y < 3\}$. Then $\Phi(-1) \cap V \neq \emptyset$ but $\Phi(x) \cap V = \emptyset$ for all real numbers x satisfying $x < -1$. It follows that the correspondence $\Phi \colon \mathbb{R} \rightrightarrows \mathbb{R}$ is not lower hemicontinuous at -1.

(iii) Let $V = \{y \in \mathbb{R} : 1 < y < 4\}$. Then V is open in $\mathbb{R}$ and $\Phi(1) \subset V$. Now if $\frac{9}{10} < x < 1$ then $0 < 1 - x^2 < \frac{19}{100} < \frac{1}{5}$. It follows that follows that $5 \in \Phi(x)$ for all real numbers x satisfying $\frac{9}{10} < x < 1$. But $5 \notin V$. Thus $\Phi(x) \not\subset V$ for all real numbers x satisfying $\frac{9}{10} < x < 1$. It follows that $\Phi \colon \mathbb{R} \rightrightarrows \mathbb{R}$ is not upper hemicontinuous at 1.

(iv) Let V be an open set in $\mathbb{R}$ satisfying $\Phi(1) \cap V \neq \emptyset$. Now $\Phi(1) = [2, 3]$. Therefore there exists $s \in V$ satisfying $2 \leq s \leq 3$. Moreover V is open in $\mathbb{R}$, and therefore there exists $\delta_1 > 0$ such that $y \in V$ for all real numbers y satisfying $s - \delta_1 < y < s + \delta_1$. If $1 \leq x < 1 + \delta_1$ and if $y = x + s - 1$ then $x + 1 \leq y \leq x + 2$, and therefore $y \in \Phi(x)$. Also $s \leq y < s + \delta_1$ and therefore $y \in V$. Thus $\Phi(x) \cap V \neq \emptyset$ whenever $1 \leq x < \delta_1$. Also we have already verified that $1 - x^2 \leq \frac{1}{5}$ whenever $\frac{9}{10} < x < 1$. It follows that $(1 - x^2)s \leq \frac{3}{5}$ and therefore $s \in \Phi(x)$ whenever $\frac{9}{10} < x < 1$. Let $\delta = \min(\delta_1, \frac{1}{10}$. Then $\Phi(x) \cap V \neq \emptyset$ for all $x \in \mathbb{R}$ satisfying $1 - \delta < x < 1 + \delta$. Thus $\Phi \colon \mathbb{R} \rightrightarrows \mathbb{R}$ is lower hemicontinuous at 1.

(b) [Bookwork.] Suppose that the proposition were false. Then there would exist infinite sequences $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ and $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \ldots$ such that $\mathbf{x}_j \in K$, $\mathbf{w}_j \in X \setminus V$ and $|\mathbf{w}_j - \mathbf{x}_j| < 1/j$ for all positive integers j. The set K is both closed and bounded in $\mathbb{R}^n$. The multidimensional Bolzano-Weierstrass Theorem would then ensure the existence of a subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converging to some point $\mathbf{q}$ of K. Moreover $\lim\limits_{j \to +\infty} (\mathbf{w}_j - \mathbf{x}_j) = \mathbf{0}$, and therefore

$$\lim_{j \to \infty} \mathbf{w}_{k_j} = \lim_{j \to \infty} \mathbf{x}_{k_j} = \mathbf{q}.$$

But $\mathbf{w}_j \in X \setminus V$. Moreover $X \setminus V$ is closed in X, and therefore any sequence of points in $X \setminus V$ that converges in X must converge to a point of $X \setminus V$. It would therefore follow that $\mathbf{q} \in K \cap (X \setminus V)$. But this is impossible, because $K \subset V$ and therefore $K \cap (X \setminus V) = \emptyset$. Thus a contradiction would follow were the proposition false. The result follows.

[N.B., the lecture notes contain three proofs of this result: a proof using the multidimensional Bolzano-Weierstrass Theorem, and proof using the multidimensional Heine-Borel Theorem, a proof using the Extreme Value Theorem. Any of these proofs, or any other correct proof that does not essentially beg the question, is acceptable.]

(c) [Bookwork.] Let V be an open set in Y that satisfies $\Phi(\mathbf{p}) \subset V$. Now $\Phi(\mathbf{p})$ is a compact subset of Y, because $\Phi: X \to Y$ is compact-valued. It follows from (ii) that there exists some positive real number ε such that $B_Y(\Phi(\mathbf{p}), \varepsilon) \subset V$. There then exists some positive number δ such that

$$\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon) \subset V$$

whenever $|\mathbf{x} - \mathbf{p}| < \delta$. Thus $\Phi: X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$.

2. (a) [Bookwork.] Suppose that the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent. Let $t_1, t_2, \ldots, t_q$ be real numbers which satisfy the equation

$$\sum_{j=1}^{q} t_j(\mathbf{v}_j - \mathbf{v}_0) = \mathbf{0}.$$

Then $\sum\limits_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{0}$ and $\sum\limits_{j=0}^{q} t_j = 0$, where $t_0 = -\sum\limits_{j=1}^{q} t_j$, and therefore

$$t_0 = t_1 = \cdots = t_q = 0.$$

It follows that the displacement vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent.

Conversely, suppose that these displacement vectors are linearly independent. Let $t_0, t_1, t_2, \ldots, t_q$ be real numbers which satisfy the equations $\sum\limits_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{0}$ and $\sum\limits_{j=0}^{q} t_j = 0$. Then $t_0 = -\sum\limits_{j=1}^{q} t_j$, and therefore

$$\mathbf{0} = \sum_{j=0}^{q} t_j \mathbf{v}_j = t_0 \mathbf{v}_0 + \sum_{j=1}^{q} t_j \mathbf{v}_j = \sum_{j=1}^{q} t_j(\mathbf{v}_j - \mathbf{v}_0).$$

It follows from the linear independence of the displacement vectors $\mathbf{v}_j - \mathbf{v}_0$ for $j = 1, 2, \ldots, q$ that

$$t_1 = t_2 = \cdots = t_q = 0.$$

But then $t_0 = 0$ also, because $t_0 = -\sum\limits_{j=1}^{q} t_j$. It follows that the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent, as required.

(b) [Definitions.] A *simplex* in $\mathbb{R}^k$ of dimension q with vertices

$$\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$$

is defined to be a set of the form

$$\left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j : 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^{q} t_j = 1 \right\},$$

where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent points of $\mathbb{R}^k$. Let $\mathbf{x}$ be a point of this simplex. Then $\mathbf{x} = \sum\limits_{j=0}^{q} t_j \mathbf{v}_j$ where $0 \leq t_j \leq 1$ for

$j = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$. The coefficients t_j of the vertices in this expression are the barycentric coordinate of the point $\mathbf{x}$. The barycentre of the simplex is the point whose barycentric coordinates are all equal to $1/(q+1)$, where q is the dimension of the simplex.

(c) [Seen similar.] The simplices of σ, ordered so that the barycentric coordinates of the point $\mathbf{x}$ occur in increasing order, are

$$\mathbf{v}_4, \mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_5, \mathbf{v}_2, \mathbf{v}_0.$$

The barycentric coordinate of vertex $\mathbf{v}_4$ is equal to zero. Let σ_0, σ_1, σ_2 and σ_3 be the simplices of K whose vertex sets are as follows:—

$$\operatorname{Vert}\sigma_0 = \{\mathbf{v}_0\}, \quad \operatorname{Vert}\sigma_1 = \{\mathbf{v}_0, \mathbf{v}_2\},$$

$$\operatorname{Vert}\sigma_2 = \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5\}, \quad \operatorname{Vert}\sigma_3 = \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5\}.$$

These sets are assigned as follows: the vertex set for σ_3 consists of all vertices where the associated barycentric coordinate of $\mathbf{x}$ is greater than zero; the vertex set for σ_2 consists of all vertices where the associated barycentric coordinate of $\mathbf{x}$ is greater than $\frac{1}{12}$; the vertex set for σ_1 consists of all vertices where the associated barycentric coordinate of $\mathbf{x}$ is greater than $\frac{1}{6}$; the vertex set for σ_0 consists of all vertices where the associated barycentric coordinate of $\mathbf{x}$ is greater than $\frac{1}{4}$.

The simplex τ then has vertices $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$, where

$$\mathbf{w}_0 = \hat{\sigma}_0 = \mathbf{v}_0, \quad \mathbf{w}_1 = \hat{\sigma}_1 = \tfrac{1}{2}(\mathbf{v}_0 + \mathbf{v}_2),$$

$$\mathbf{w}_2 = \hat{\sigma}_2 = \tfrac{1}{4}(\mathbf{v}_0+\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_5), \quad \mathbf{w}_3 = \hat{\sigma}_3 = \tfrac{1}{5}(\mathbf{v}_0+\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3+\mathbf{v}_5).$$

Then

$$\begin{aligned}
\mathbf{x} &= \tfrac{1}{12}\mathbf{v}_3 + \tfrac{1}{6}(\mathbf{v}_1 + \mathbf{v}_5) + \tfrac{1}{4}\mathbf{v}_2 + \tfrac{1}{3}\mathbf{v}_0 \\
&= \tfrac{1}{12}(\mathbf{v}_3 + \mathbf{v}_1 + \mathbf{v}_5 + \mathbf{v}_2 + \mathbf{v}_0) + \tfrac{1}{12}(\mathbf{v}_1 + \mathbf{v}_5 + \mathbf{v}_2 + \mathbf{v}_0) \\
&\qquad + \tfrac{1}{12}(\mathbf{v}_2 + \mathbf{v}_0) + \tfrac{1}{12}\mathbf{v}_0 \\
&= \tfrac{1}{12}\mathbf{w}_0 + \tfrac{1}{6}\mathbf{w}_1 + \tfrac{1}{3}\mathbf{w}_2 + \tfrac{5}{12}\mathbf{w}_3.
\end{aligned}$$

Thus the barycentric coordinates of the point $\mathbf{x}$ with respect to the vertices $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ of τ are $\frac{1}{12}$, $\frac{1}{6}$, $\frac{1}{3}$ and $\frac{5}{12}$ respectively.

3. (a) [Bookwork.] Every point of $|K|$ belongs to the interior of a unique simplex of K It follows that the complement $|K|\backslash \mathrm{st}_K(\mathbf{x})$ of $\mathrm{st}_K(\mathbf{x})$ in $|K|$ is the union of the interiors of those simplices of K that do not contain the point $\mathbf{x}$. But if a simplex of K does not contain the point $\mathbf{x}$, then the same is true of its faces. Moreover the union of the interiors of all the faces of some simplex is the simplex itself. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{x})$ is the union of all simplices of K that do not contain the point $\mathbf{x}$. But each simplex of K is closed in $|K|$. It follows that $|K| \setminus \mathrm{st}_K(\mathbf{x})$ is a finite union of closed sets, and is thus itself closed in $|K|$. We deduce that $\mathrm{st}_K(\mathbf{x})$ is open in $|K|$. Also $\mathbf{x} \in \mathrm{st}_K(\mathbf{x})$, since $\mathbf{x}$ belongs to the interior of at least one simplex of K.

(b) [Bookwork.] Let $s\colon K \to L$ be a simplicial approximation to $f\colon |K| \to |L|$, let $\mathbf{v}$ be a vertex of K, and let $\mathbf{x} \in \mathrm{st}_K(\mathbf{v})$. Then $\mathbf{x}$ and $f(\mathbf{x})$ belong to the interiors of unique simplices $\sigma \in K$ and $\tau \in L$. Moreover $\mathbf{v}$ must be a vertex of σ, by definition of $\mathrm{st}_K(\mathbf{v})$. Now $s(\mathbf{x})$ must belong to τ (since s is a simplicial approximation to the map f), and therefore $s(\mathbf{x})$ must belong to the interior of some face of τ. But $s(\mathbf{x})$ must belong to the interior of $s(\sigma)$, since $\mathbf{x}$ is in the interior of σ. It follows that $s(\sigma)$ must be a face of τ, and therefore $s(\mathbf{v})$ must be a vertex of τ. Thus $f(\mathbf{x}) \in \mathrm{st}_L(s(\mathbf{v}))$. We conclude that if $s\colon K \to L$ is a simplicial approximation to $f\colon |K| \to |L|$, then $f\left(\mathrm{st}_K(\mathbf{v})\right) \subset \mathrm{st}_L\left(s(\mathbf{v})\right)$.

Conversely let $s\colon \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ be a function with the property that $f\left(\mathrm{st}_K(\mathbf{v})\right) \subset \mathrm{st}_L\left(s(\mathbf{v})\right)$ for all vertices $\mathbf{v}$ of K. Let $\mathbf{x}$ be a point in the interior of some simplex of K with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. Then $\mathbf{x} \in \mathrm{st}_K(\mathbf{v}_j)$ and hence $f(\mathbf{x}) \in \mathrm{st}_L\left(s(\mathbf{v}_j)\right)$ for $j = 0, 1, \ldots, q$. It follows that each vertex $s(\mathbf{v}_j)$ must be a vertex of the unique simplex $\tau \in L$ that contains $f(\mathbf{x})$ in its interior. In particular, $s(\mathbf{v}_0), s(\mathbf{v}_1), \ldots, s(\mathbf{v}_q)$ span a face of τ, and $s(\mathbf{x}) \in \tau$. We conclude that the function $s\colon \mathrm{Vert}\, K \to \mathrm{Vert}\, L$ represents a simplicial map which is a simplicial approximation to $f\colon |K| \to |L|$, as required.

(c) [Bookwork.] The collection consisting of the stars $\mathrm{st}_L(\mathbf{w})$ of all vertices $\mathbf{w}$ of L is an open cover of $|L|$, since each star $\mathrm{st}_L(\mathbf{w})$ is open in $|L|$ and the interior of any simplex of L is contained in $\mathrm{st}_L(\mathbf{w})$ whenever $\mathbf{w}$ is a vertex of that simplex. It follows from the continuity of the map $f\colon |K| \to |L|$ that the collection consisting of the preimages $f^{-1}(\mathrm{st}_L(\mathbf{w}))$ of the stars of all vertices $\mathbf{w}$ of L is an open cover of $|K|$.

Now the set $|K|$ is a closed bounded subset of a Euclidean space. It follows that there exists a Lebesgue number δ_L for the open cover consisting of the preimages of the stars of all the vertices of L. This Lebesgue number δ_L is a positive real number with the following property: every subset of $|K|$ whose diameter is less than δ is contained in the preimage of the star of some vertex $\mathbf{w}$ of L. It follows that every subset of $|K|$ whose diameter is less than K is mapped by f into $\mathrm{st}_L(\mathbf{w})$ for some vertex $\mathbf{w}$ of L.

Now the mesh $\mu(K^{(j)})$ of the jth barycentric subdivision of K tends to zero as $j \to +\infty$. Thus we can choose j such that $\mu(K^{(j)}) < \frac{1}{2}\delta$. If $\mathbf{v}$ is a vertex of $K^{(j)}$ then each point of $\mathrm{st}_{K^{(j)}}(\mathbf{v})$ is within a distance $\frac{1}{2}\delta$ of $\mathbf{v}$, and hence the diameter of $\mathrm{st}_{K^{(j)}}(\mathbf{v})$ is at most δ. We can therefore choose, for each vertex $\mathbf{v}$ of $K^{(j)}$ a vertex $s(\mathbf{v})$ of L such that $f(\mathrm{st}_{K^{(j)}}(\mathbf{v})) \subset \mathrm{st}_L(s(\mathbf{v}))$. In this way we obtain a function $s\colon \mathrm{Vert}\, K^{(j)} \to \mathrm{Vert}\, L$ from the vertices of $K^{(j)}$ to the vertices of L. This is the desired simplicial approximation to f.

4. (a) [Bookwork.] The closed n-dimensional ball E^n is itself homeomorphic to an n-dimensional simplex Δ. It follows that there exists a homeomorphism $h: X \to \Delta$ mapping the set X onto the simplex Δ. Then the continuous map $f: X \to X$ determines a continuous map $g: \Delta \to \Delta$, where $g(h(\mathbf{x}) = h(f(\mathbf{x}))$ for all $\mathbf{x} \in X$. Suppose that it were the case that $f(\mathbf{x}) \neq \mathbf{x}$ for all $\mathbf{x} \in X$. Then $g(\mathbf{z}) \neq \mathbf{z}$ for all $\mathbf{z} \in \Delta$. There would then exist a well-defined continuous map $r: \Delta \to \partial\Delta$ mapping each point $\mathbf{z}$ of Δ to the unique point $r(\mathbf{z})$ of the boundary $\partial\Delta$ of Δ at which the half line starting at $g(\mathbf{z})$ and passing through $\mathbf{z}$ intersects $\partial\Delta$. Then $r: \Delta \to \partial\Delta$ would be continuous, and $r(\mathbf{z}) = \mathbf{z}$ for all $\mathbf{z} \in \partial\Delta$. However there does not exist any continuous map $r: \Delta \to \partial\Delta$ with these properties. Therefore the map f must have at least one fixed point, as required. ▮

(b) [Bookwork.] Let $\mathbf{v}: \Delta \to \mathbb{R}^n$ be the function with ith component v_i given by

$$v_i(\mathbf{p}) = \begin{cases} p_i + z_i(\mathbf{p}) & \text{if } z_i(\mathbf{p}) > 0; \\ p_i & \text{if } z_i(\mathbf{p}) \leq 0. \end{cases}$$

Note that $\mathbf{v}(\mathbf{p}) \neq \mathbf{0}$ and the components of $\mathbf{v}(\mathbf{p})$ are non-negative for all $\mathbf{p} \in \Delta$. It follows that there is a well-defined map $\varphi: \Delta \to \Delta$ given by

$$\varphi(\mathbf{p}) = \frac{1}{\sum_{i=1}^{n} v_i(\mathbf{p})} \mathbf{v}(\mathbf{p}),$$

The Brouwer Fixed Point Theorem ensures that there exists $\mathbf{p}^* \in \Delta$ satisfying $\varphi(\mathbf{p}^*) = \mathbf{p}^*$. Then $\mathbf{v}(\mathbf{p}^*) = \lambda \mathbf{p}^*$ for some $\lambda \geq 1$. We claim that $\lambda = 1$.

Suppose that it were the case that $\lambda > 1$. Then $v_i(\mathbf{p}^*) > p_i^*$, and thus $z_i(\mathbf{p}^*) > 0$ whenever $p_i^* > 0$. But $p_i^* \geq 0$ for all i, and $p_i^* > 0$ for at least one value of i, since $\mathbf{p}^* \in \Delta$. It would follow that $\mathbf{p}^*.\mathbf{z}(\mathbf{p}^*) > 0$, contradicting the requirement that $\mathbf{p}.\mathbf{z}(\mathbf{p}) \leq 0$ for all $\mathbf{p} \in \Delta$. We conclude that $\lambda = 1$, and thus $v_i = p_i^*$ and $z_i(\mathbf{p}^*) \leq 0$ for all i, as required.

(c) [Bookwork.] Let $f(\mathbf{p}, \mathbf{q}) = \mathbf{p}^T M \mathbf{q}$ for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$. Given $\mathbf{q} \in \Delta_Q$, let

$$\mu_P(\mathbf{q}) = \sup\{f(\mathbf{p}, \mathbf{q}) : \mathbf{p} \in \Delta_P\}$$

and let

$$P(\mathbf{q}) = \{\mathbf{p} \in \Delta_P : f(\mathbf{p}, \mathbf{q}) = \mu_P(\mathbf{q})\}.$$

Similarly given $\mathbf{q} \in \Delta_Q$, let

$$\mu_Q(\mathbf{p}) = \inf\{f(\mathbf{p}, \mathbf{q}) : \mathbf{q} \in \Delta_Q\}$$

and let

$$Q(\mathbf{p}) = \{\mathbf{q} \in \Delta_Q : f(\mathbf{p}, \mathbf{q}) = \mu_Q(\mathbf{q})\}.$$

An application of Berge's Maximum Theorem ensures that the functions $\mu_P \colon \Delta_P \to \mathbb{R}$ and $\mu_Q \colon \Delta_Q \to \mathbb{R}$ are continuous, and that the correspondences $P \colon \Delta_Q \rightrightarrows \Delta_P$ and $Q \colon \Delta_P \rightrightarrows \Delta_Q$ are non-empty, compact-valued and upper hemicontinuous. These correspondences therefore have closed graphs. Morever $P(\mathbf{q})$ is convex for all $\mathbf{q} \in \Delta_Q$ and $Q(\mathbf{p})$ is convex for all $\mathbf{p} \in \Delta_P$. Let $X = \Delta_P \times \Delta_Q$, and let $\Phi \colon X \rightrightarrows X$ be defined such that

$$\Phi(\mathbf{p}, \mathbf{q}) = P(\mathbf{q}) \times Q(\mathbf{p})$$

for all $(\mathbf{p}, \mathbf{q}) \in X$. Kakutani's Fixed Point Theorem then ensures that there exists $(\mathbf{p}^*, \mathbf{q}^*) \in X$ such that $(\mathbf{p}^*, \mathbf{q}^*) \in \Phi(\mathbf{p}^*, \mathbf{q}^*)$. Then $\mathbf{p}^* \in P(\mathbf{q}^*)$ and $\mathbf{q}^* \in Q(\mathbf{p}^*)$ and therefore

$$f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}{*}, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q})$$

for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$, as required.