MA3486, Sample Paper 2016
Worked Solutions to Selected Parts

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1(a) The correspondence Φ is both upper hemicontinuous and lower hemicontinuous at $x = 0$.

To show upper hemicontinuity, let V be an open set satisfying $\Phi(0) \subset V$. Then $[-1, 1] \subset V$. Now V is open, therefore there exists $\varepsilon > 0$ such that $y \in V$ whenever $|y - 1| < \varepsilon$ and $y \in V$ whenever $|y + 1| < \varepsilon$. It then follows that

$$\{y \in \mathbb{R} : -1 - \varepsilon < y < 1 + \varepsilon\} \subset V.$$

Let δ satisfy $0 < \delta \leq \min(\varepsilon, \sqrt{\varepsilon})$. If $-\delta < x < 0$ then $0 < x^2 < \varepsilon$ and therefore

$$\{y \in \mathbb{R} : -1 + x^2 < y < 1 + x^2\} \subset \{y \in \mathbb{R} : -1 - \varepsilon < y < 1 + \varepsilon\} \subset V.$$

Similarly if $0 \leq x < \delta$ then

$$\{y \in \mathbb{R} : x - 1 < y < 1 - x\} \subset \{y \in \mathbb{R} : -1 - \varepsilon < y < 1 + \varepsilon\} \subset V.$$

Thus $\Phi(x) \subset V$ whenever $-\delta < x < \delta$. This proves that the correspondence $\Phi(V)$ is upper hemicontinuous at $x = 0$.

The correspondence Φ is also lower hemicontinuous at $x = 0$. Indeed if V is an open set in $\mathbb{R}$, and if $\Phi(0) \cap V \neq \emptyset$ then there exists some real number u satisfying $-1 < x < 1$ for which $u \in V$. (Note that if $-1 \in V$, then the definition of open sets in $\mathbb{R}$ ensures the existence of some real number u satisfying $-1 < u < 1$ for which $u \in V$. Similarly if $1 \in V$.) Then there exists $\delta > 0$ such that $x^2 - 1 < u$ whenever $-\delta < x < 0$ and $x - 1 < u$ and $1 - x > u$ whenever $0 \leq x < \delta$. Then $u \in \Phi(x)$ and thus $\Phi(x) \cap V \neq \emptyset$ whenever $-\delta < x < \delta$. This proves that the correspondence $\Phi(V)$ is lower hemicontinuous at $x = 0$.

The correspondence Φ is upper hemicontinuous at $x = 1$. Indeed

$$\Phi(1) = [-1, 1] = \{y \in \mathbb{R} : -1 \leq y \leq 1\}.$$

Let V be an open set in $\mathbb{R}$ for which $\Phi(1) \subset V$. Then $\Phi(1) = [-1, 1]$, and thus $[-1, 1] \subset V$. But then $\Phi(x) \subset \Phi(1)$ whenever $0 \leq x < 1$, and therefore $\Phi(x) \subset V$ whenever $0 \leq x < 1$. If $x \geq 1$ then

$$\Phi(x) = \{y \in \mathbb{R} : y^2 x \leq 1\} \subset [-1, 1] \subset V.$$

Thus $\Phi(x) \subset V$ for all $x \geq 0$, and thus there exists $\delta > 0$ such that $\Phi(x) \subset V$ for all x satisfying $|x - 1| < \delta$. This proves that the correspondence $\Phi(V)$ is lower hemicontinuous at $x = 1$.

The correspondence Φ is not lower hemicontinuous at $x = 1$. Indeed let

$$V = \{y \in \mathbb{R} : -1 < y < -\tfrac{1}{2}\}.$$

Then $\Phi(1) \cap V \neq \emptyset$. But if $\frac{1}{2} < x < 1$ then $y > -\frac{1}{2}$ for all $y \in \Phi(x)$ and therefore $\Phi(x) \cap V = \emptyset$.

2(c) We need to find non-negative real numbers t_0, t_1 and t_2 such that

$$(3, 3) = t_0(-1, 0) + t_1(3, 2) + t_2(4, 4)$$

and $t_0 + t_1 + t_2 = 1$. Thus

$$-t_0 + 3t_1 + 4t_2 = 3, \quad 2t_1 + 4t_2 = 3, \quad t_0 + t_1 + t_2 = 1.$$

Adding the first and third of the above equations, we see that $4t_1 + 2t_2 = 2$, and therefore

$$1 = 2(4t_1 + 2t_2) - (2t_1 + 4t_2) = 6t_1,$$

and thus $t_1 = \frac{1}{6}$. It then follows that $2t_2 = 2 - \frac{2}{3}$, and therefore $t_2 = \frac{2}{3}$. Using $t_0 + t_1 + t_2$, we see that $t_0 = \frac{1}{6}$.

2(d) Looking at the sizes of the barycentric coordinates of $\mathbf{x}$, and ranking them in increasing order, it becomes clear that the vertices of τ will be $\mathbf{w}_0$, $\mathbf{w}_1$ and $\mathbf{w}_2$, where

$$\mathbf{w}_0 = \tfrac{1}{4}(\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_3 + \mathbf{v}_4), \quad \mathbf{w}_1 = \tfrac{1}{3}(\mathbf{v}_0 + \mathbf{v}_3 + \mathbf{v}_4), \quad \mathbf{w}_2 = \mathbf{v}_3.$$

(Only in this way can we achieve a representation of $\mathbf{x}$ as a linear combination of barycentres of faces of σ, with positive coefficients that sum up to one.) Then

$$\begin{aligned} \mathbf{x} &= \tfrac{1}{6}(\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_3 + \mathbf{v}_4) + \tfrac{1}{12}(\mathbf{v}_0 + \mathbf{v}_3 + \mathbf{v}_4) + \tfrac{1}{12}\mathbf{v}_3 \\ &= \tfrac{2}{3}\mathbf{w}_0 + \tfrac{1}{4}\mathbf{w}_1 + \tfrac{1}{12}\mathbf{w}_2 \end{aligned}$$

Thus $\frac{2}{3}$, $\frac{1}{4}$ and $\frac{1}{12}$ are the barycentric coordinates of $\mathbf{x}$ with respect to the vertices $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2$ of τ.