MA3428, Annual Examination 2013, Question 5

5. (a) Given the definitions of the following:

   (i) a *geometrically independent* list $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of points in some Euclidean space $\mathbb{R}^k$;

   (ii) a *simplex*;

   (iii) the *barycentre* of a simplex;

   (iv) the *barycentric coordinates* of a point of a simplex;

   (v) the *interior* of a simplex;

   (vi) a *simplicial complex*;

   (vii) the *polyhedron* $|K|$ of a simplicial complex K.

(10 marks)

(b) Let $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ be vertices of a tetrahedron σ in some Euclidean space, let K_σ be the simplicial complex consisting of that tetrahedron together with all of its triangular faces, edges and vertices. Let τ be the unique simplex of the first barycentric subdivision K'_σ of K_σ that contains the point $\mathbf{x}$ in its interior, where

$$\mathbf{x} = \frac{1}{12}\mathbf{v}_0 + \frac{1}{3}\mathbf{v}_1 + \frac{1}{4}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3.$$

Write down the vertices of the simplex τ (expressing those vertices as linear combinations of $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ with coefficients whose sum is equal to one), and find the barycentric coordinates of $\mathbf{x}$ with respect to the vertices of τ.

(6 marks)

(c) Let $\mathbf{x}$ be a point of the polyhedron $|K|$ of a simplicial complex K. Given the definition of the *star* $\mathrm{st}_K(\mathbf{x})$ of $\mathbf{x}$ in the polyhedron $|K|$ of K, and prove that the star of $\mathbf{x}$ is an open neighbourhood of $\mathbf{x}$ in $|K|$.

(4 marks)

MA421, Annual Examination 2007, Question 4

4. (a) Let K be a simplicial complex which is a subdivision of an n-dimensional simplex. What is a *Sperner labelling* of the vertices of K?

(4 marks)

(b) State and prove *Sperner's Lemma.*

(10 marks)

(c) Use Sperner's Lemma and the Simplicial Approximation Theorem to show that there is no continuous map $r\colon \Delta \to \partial\Delta$ from an n-simplex Δ to its boundary $\partial\Delta$ with the property that $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$.

(6 marks)

MA421, Annual Examination 2003, Question 4

4. (a) What is a *simplex*? What is a *simplicial complex*? What is the *polyhedron* $|K|$ of a simplicial complex K?

   (b) Let $\mathbf{x}$ be a point of the polyhedron $|K|$ of a simplicial complex K. What is the *star* $\mathrm{st}_K(\mathrm{x})$ of x in K?

   (c) Let K and L be simplicial complexes. What is meant by saying that a function $s\colon \mathrm{Vert}\,K \to \mathrm{Vert}\,L$ from the vertex set of K to that of L is a *simplicial approximation* to a continuous map $f\colon |K| \to |L|$?

   (d) Let K and L be simplicial complexes, and let $f\colon |K| \to |L|$ be a continuous map. Prove that a function $s\colon \mathrm{Vert}\,K \to \mathrm{Vert}\,L$ between the vertex sets is a simplicial map, and is a simplicial approximation to $f\colon |K| \to |L|$, if and only if $f(\mathrm{st}_K(\mathbf{v})) \subset \mathrm{st}_L(s(\mathbf{v}))$ for all vertices $\mathbf{v}$ of K.

   (e) State and prove the *Simplicial Approximation Theorem.* [You may use, without proof, the result that the mesh of the jth barycentric subdivision of a simplicial complex K converges to zero $j \to \infty$. You may also use, without proof, the result that $\mathrm{st}_K(\mathbf{x})$ is an open subset of $|K|$ for all $\mathbf{x} \in |K|$.]

MA421, Annual Examination 1993, Question 4

4. (a) What is meant by saying that point $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are *geometrically independent* (or *affine independent*)?

   (b) Define the concepts of *simplex* and *simplicial complex.*

(c) A subset K of $\mathbb{R}^k$ is said to be *convex* if $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in K$ for all points $\mathbf{x}$ and $\mathbf{y}$ of K and real numbers λ satisfying $0 \leq \lambda \leq 1$. Let σ be a simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. Show that σ is convex. Show also that $\sigma \subset K$ for any convex subset K of $\mathbb{R}^k$ that contains the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of σ.

MA421, Annual Examination 1993, Question 6

6. Give an account of the manner in which the Brouwer Fixed Point Theorem can be applied to prove the existence of a Walras equilibrium in an exchange economy (i.e., prove the existence of prices that ensure that the supply of each commodity matches its demand).