Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
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MA3486-1

Faculty of Engineering, Mathematics and Science

School of Mathematics

SAMPLE PAPER **Trinity Term 2016**

MA3486: Fixed Point Theorems and Economics Equilibria

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Instructions to Candidates:

Credit will be given for the best 3 questions answered.

Materials Permitted for this Examination:

Formulae and Tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. What is meant by saying that a collection of subsets of $\mathbb{R}^n$ *covers* X? What is an *open cover* of X.

(4 points)

(b) Let X be a closed bounded set in n-dimensional Euclidean space, and let $\mathcal{V}$ be an open cover of X. Prove that there exists a positive real number δ_L with the property that, given any point $\mathbf{u}$ of X, there exists a member V of the open cover $\mathcal{V}$ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta_L\} \subset V.$$

(16 points)

2. Let X and Y be subsets of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A *correspondence* $\Phi\colon X \rightrightarrows Y$ assigns to each point $\mathbf{x}$ of X a subset $\Phi(\mathbf{x})$ of Y. The correspondence $\Phi\colon X \rightrightarrows Y$ is said to be *upper hemicontinuous* at a point $\mathbf{p}$ of X if, given any set V in Y that is open in Y and satisfies $\Phi(\mathbf{p}) \subset V$, there exists some positive real number δ such that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Also the correspondence $\Phi\colon X \rightrightarrows Y$ is said to be *lower hemicontinuous* at a point $\mathbf{p}$ of X if, given any set V in Y that is open in Y and satisfies $\Phi(\mathbf{p}) \cap V \neq \emptyset$, there exists some positive real number δ such that $\Phi(\mathbf{x}) \cap V \neq \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$.

(a) Let $\Phi\colon \mathbb{R} \rightrightarrows \mathbb{R}$ be the correspondence defined such that

$$\Phi(x) = \begin{cases} \{y \in \mathbb{R} : x^2 - 1 \leq y \leq x^2 + 1\} & \text{if } x < 0; \\ \{y \in \mathbb{R} : x - 1 \leq y \leq 1 - x\} & \text{if } 0 \leq x < 1; \\ \{y \in \mathbb{R} : y^2 x \leq 1\} & \text{if } x \geq 1. \end{cases}$$

Answer the following, fully justifying your answers:—

(i) Is $\Phi\colon \mathbb{R} \to \mathbb{R}$ is *upper hemicontinuous* at $x = 0$?

(ii) Is $\Phi\colon \mathbb{R} \to \mathbb{R}$ is *lower hemicontinuous* at $x = 0$?

(iii) Is $\Phi\colon \mathbb{R} \to \mathbb{R}$ is *upper hemicontinuous* at $x = 1$?

(iv) Is $\Phi\colon \mathbb{R} \to \mathbb{R}$ is *lower hemicontinuous* at $x = 1$?

(10 points)

(b) Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is compact-valued and upper hemicontinuous. Let K be a compact subset of X. Prove that $\Phi(K)$ is a compact subset of Y.

(5 points)

(c) Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is lower hemicontinuous. Let $f\colon X \times Y \to \mathbb{R}$ be a continuous real-valued function on $X \times Y$, and let c be a real number. Prove that

$$\{\mathbf{x} \in X : \text{ there exists } \mathbf{y} \in \Phi(\mathbf{x}) \text{ for which } f(\mathbf{x}, \mathbf{y}) > c\}$$

is open in X.

(5 points)

(End of Question)

3. (a) What is a *simplex*? What is the *dimension* of a simplex? What are the *barycentric coordinates* of a point of simplex with respect to the vertices of that simplex? What is the *barycentre* of a simplex? What is meant by saying that a simplex τ is a *face* of a simplex σ? What is the *interior of a simplex*?

(6 points)

(Question continues on the next page)

(b) Prove that any point of a simplex belongs to the interior of a unique face of that simplex.

(3 points)

(c) The point $(3, 3)$ of $\mathbb{R}^2$ belongs to the triangle in $\mathbb{R}^2$ with vertices $(-1, 0)$, $(3, 2)$ and $(4, 4)$. Determine the barycentric coordinates of $(3, 3)$ with regard to the vertices of this triangle.

(4 points)

Let K be a simplicial complex. The *first barycentric subdivision* K' of K is the simplicial complex consisting of simplices whose vertex sets are of the form $\{\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_r\}$, where $\sigma_0, \sigma_1, \ldots, \sigma_r$ are simplices of K, σ_{i-1} is a proper face of σ_i for $i = 1, 2, \ldots, r$ and $\hat{\sigma}_i$ denote the barycentre of the simplex σ_i for $i = 0, 1, 2, \ldots, r$.

(d) Let σ be a 5-dimensional simplex with vertices $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, let K be the simplicial complex consisting of the simplex σ together with all its faces, let K' be the first barycentric subdivision of σ, let

$$\mathbf{x} = \tfrac{1}{4}\mathbf{v}_0 + \tfrac{1}{6}\mathbf{v}_1 + \tfrac{1}{3}\mathbf{v}_3 + \tfrac{1}{4}\mathbf{v}_4,$$

and let τ be the unique simplex of K' that contains the point $\mathbf{x}$ in its interior (so that $\mathbf{x}$ belongs to τ but does not belong to any proper face of τ). Determine the barycentric coordinates of the vertices of τ with respect to the vertices of σ, and determine the barycentric coordinates of the point $\mathbf{x}$ with respect to the vertices of τ.

(7 points)

(End of Question)

4. (a) State and prove *Sperner's Lemma*.

(13 points)

(Question continues on the next page)

(b) Let M be an $m \times n$ matrix, and let

$$\Delta_P = \left\{ (p_1, p_2, \ldots, p_m) \in \mathbb{R}^m : p_i \geq 0 \text{ for } i = 1, 2, \ldots, m, \text{ and } \sum_{i=1}^{m} p_i = 1 \right\},$$

$$\Delta_Q = \left\{ (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n : q_i \geq 0 \text{ for } i = 1, 2, \ldots, n, \text{ and } \sum_{j=1}^{n} q_j = 1 \right\},$$

and let

$$f(\mathbf{p}, \mathbf{q}) = \mathbf{p}^T M \mathbf{q} = \sum_{i=1}^{m} \sum_{j=1}^{n} M_{i,j} p_i q_j$$

for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$. Prove von Neumann's Minimax Theorem, which asserts that there exist $\mathbf{p}^* \in \Delta_P$ and $\mathbf{q}^* \in \Delta_Q$ such that

$$f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q})$$

for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$.

(7 points)

[You may use, without proof, Kakutani's Fixed Point Theorem which asserts that if $\Phi: X \rightrightarrows X$ is a correspondence with closed graph which maps points of a non-empty, compact and convex subset X of a Euclidean space to non-empty convex subsets of X then there exists at least one "fixed point" $\mathbf{x}^*$ for Φ characterized by the property $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$. You may also use, without proof, Berge's Maximum Theorem, which in this context ensures that if

$$\mu_P(\mathbf{q}) = \sup\{f(\mathbf{p}, \mathbf{q}) : \mathbf{p} \in \Delta_P\}$$

and

$$P(\mathbf{q}) = \{\mathbf{p} \in \Delta_P : f(\mathbf{p}, \mathbf{q}) = \mu_P(\mathbf{q})\}$$

for all $\mathbf{q} \in \Delta_Q$ then the function $\mu_P: \Delta_Q \to \mathbb{R}$ is continuous and the correspondence $P: \Delta_Q \rightrightarrows \Delta_P$ is non-empty-valued, compact-valued and upper hemicontinuous and thus has closed graph.]

(End of Question)