Module MA3486: Fixed Point Theorems and Economic Equilibria
Hilary Term 2016
Part I (Sections 1 to 4)

D. R. Wilkins

Contents

1 Ordered Fields and the Real Number System

1.1 Ordered Fields

The real numbers can be characterized as an ordered field with the Least Upper Bound Property. We give below the definition of ordered fields, and then described the Least Upper Bound Property, which requires that every non-empty subset of the set of real numbers that is bounded above has a least upper bound.

An *ordered field* $\mathbb{F}$ consists of a set $\mathbb{F}$ on which are defined binary operations $+$ of addition and $\times$ of multiplication, together with an ordering relation $<$, where these binary operations and ordering relation satisfy the following axioms:—

1. if u and v are elements of $\mathbb{F}$ then their sum $u+v$ is also a element of $\mathbb{F}$;
2. (*the Commutative Law for addition*) $u + v = v + u$ for all elements u and v of $\mathbb{F}$;
3. (*the Associative Law for addition*) $(u + v) + w = u + (v + w)$ for all elements u, v and w of $\mathbb{F}$;
4. there exists an element of $\mathbb{F}$, denoted by 0, with the property that $u + 0 = x = 0 + u$ for all elements u of $\mathbb{F}$;
5. for each element u of $\mathbb{F}$ there exists some element $-u$ of $\mathbb{F}$ with the property that $u + (-u) = 0 = (-u) + u$;
6. if u and v are elements of $\mathbb{F}$ then their product $u \times v$ is also a element of $\mathbb{F}$;
7. (*the Commutative Law for multiplication*) $u \times v = v \times u$ for all elements u and v of $\mathbb{F}$;
8. (*the Associative Law for multiplication*) $(u \times v) \times w = u \times (v \times w)$ for all elements u, v and w of $\mathbb{F}$,
9. there exists an element of $\mathbb{F}$, denoted by 1, with the property that $u \times 1 = u = 1 \times u$ for all elements u of $\mathbb{F}$, and moreover $1 \neq 0$,
10. for each element u of $\mathbb{F}$ satisfying $u \neq 0$ there exists some element u^{-1} of $\mathbb{F}$ with the property that $u \times u^{-1} = 1 = u^{-1} \times u$,

11. (*the Distributive Law*) $u \times (v + w) = (u \times v) + (u \times w)$ for all elements u, v and w of $\mathbb{F}$,

12. (*the Trichotomy Law*) if u and v are elements of $\mathbb{F}$ then one and only one of the three statements $u < v$, $u = v$ and $u < v$ is true,

13. (*transitivity of the ordering*) if u, v and w are elements of $\mathbb{F}$ and if $u < v$ and $v < w$ then $u < w$,

14. if u, v and w are elements of $\mathbb{F}$ and if $u < v$ then $u + w < v + w$,

15. if u and v are elements of $\mathbb{F}$ which satisfy $0 < u$ and $0 < v$ then $0 < u \times v$,

The operations of subtraction and division are defined on an ordered field $\mathbb{F}$ in terms of the operations of addition and multiplication on that field in the obvious fashion: $u - v = u + (-v)$ for all elements u and v of $\mathbb{F}$, and moreover $u/v = uv^{-1}$ provided that $v \neq 0$.

Example The rational numbers, with the standard ordering, and the standard operations of addition, subtraction, multiplication, and division constitute an ordered field.

The *absolute value* $|x|$ of an element number x of an ordered field $\mathbb{F}$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

Note that $|x| \geq 0$ for all x and that $|x| = 0$ if and only if $x = 0$. Also $|x + y| \leq |x| + |y|$ and $|xy| = |x||y|$ for all elements x and y of the ordered field $\mathbb{F}$.

Let D be a subset of an ordered field $\mathbb{F}$. An element u of $\mathbb{F}$ is said to be an *upper bound* of the set D if $x \leq u$ for all $x \in D$. The set D is said to be *bounded above* if such an upper bound exists.

Definition Let $\mathbb{F}$ be an ordered field, and let D be some subset of $\mathbb{F}$ which is bounded above. An element s of $\mathbb{F}$ is said to be the *least upper bound* (or *supremum*) of D (denoted by $\sup D$) if s is an upper bound of D and $s \leq u$ for all upper bounds u of D.

Example The rational number 2 is the least upper bound, in the ordered field of rational numbers, of the sets $\{x \in \mathbb{Q} : x \leq 2\}$ and $\{x \in \mathbb{Q} : x < 2\}$. Note that the first of these sets contains its least upper bound, whereas the second set does not.

The axioms (1)–(15) listed above that characterize ordered fields are not in themselves sufficient to fully characterize the real number system. (Indeed any property of real numbers that could be derived solely from these axioms would be equally valid in any ordered field whatsoever, and in particular would be valid were the system of real numbers replaced by the system of rational numbers.) We require as an additional axiom the following property.

> **The Least Upper Bound Property** given any non-empty set D of real numbers that is bounded above, there exists a real number $\sup D$ that is the least upper bound for the set D.

A *lower bound* of a set D of real numbers is a real number l with the property that $l \leq x$ for all $x \in D$. A set D of real numbers is said to be *bounded below* if such a lower bound exists. If D is bounded below, then there exists a greatest lower bound (or *infimum*) $\inf D$ of the set D. Indeed $\inf D = -\sup\{x \in \mathbb{R} : -x \in D\}$.

Remark We have simply listed above a complete set of axioms for the real number system. We have not however proved the existence of a system of real numbers satisfying these axioms. There are in fact several constructions of the real number system: one of the most popular of these is the representation of real numbers as *Dedekind sections* of the set of rational numbers. For an account of the this construction, and for a proof that these axioms are sufficient to characterize the real number system, see chapters 27–29 of *Calculus*, by M. Spivak. The construction of the real number system using Dedekind cuts is also described in detail in the Appendix to Chapter 1 of *Principles of Real Analysis* by W. Rudin.

1.2 The Real Number System

From the time of the ancient Greeks to the present day, mathematicians have recognized the necessity of establishing rigorous foundations for the discipline. This led mathematicians such as Bolzano, Cauchy and Weierstrass to establish in the nineteenth century the definitions of continuity, limits and convergence that are required in order to establish a secure foundation upon which to build theories of real and complex analysis that underpin the application of standard techiques of the differential calculus in one or more variables.

But mathematicians in the nineteenth century realised that, in order to obtain satisfactory proofs of basic theorems underlying the applications of calculus, they needed a deeper understanding of the nature of the real number system. Accordingly Dedekind developed a theory in which real numbers were represented by *Dedekind sections*, in which each real number was characterized by means of a partition of the set of rational numbers into two subsets, where every rational number belonging to the first subset is less than every rational number belonging to the second.

Dedekind published his construction of the real number system in 1872, in the work *Stetigkeit und irrationale Zahlen.* In the same year, Georg Cantor published a construction of the real number system in which real numbers are represented by sequences of rational numbers satisfying an appropriate convegence criterion.

It has since been shown that the system of real numbers is completely characterized by the statement that the real numbers constitute an ordered field which satisfies the Least Upper Bound Axiom.

1.3 Infinite Sequences of Real Numbers

An *infinite sequence* of real numbers is a sequence of the form $x_1, x_2, x_3, \ldots$, where x_j is a real number for each positive integer j. (More formally, one can view an infinite sequence of real numbers as a function from $\mathbb{N}$ to $\mathbb{R}$ which sends each positive integer j to some real number x_j.)

Definition An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to *converge* to some real number l if and only if the following criterion is satisfied:

> given any strictly positive real number ε, there exists some positive integer N such that $|x_j - l| < \varepsilon$ for all positive integers j satisfying $j \geq N$.

If the sequence $x_1, x_2, x_3, \ldots$ converges to the *limit* l then we denote this fact by writing '$x_j \to l$ as $j \to +\infty$', or by writing '$\lim\limits_{j\to+\infty} x_j = l$'.

Let x and l be real numbers, and let ε be a strictly positive real number. Then $|x - l| < \varepsilon$ if and only if both $x - l < \varepsilon$ and $l - x < \varepsilon$. It follows that $|x - l| < \varepsilon$ if and only if $l - \varepsilon < x < l + \varepsilon$. The condition $|x - l| < \varepsilon$ essentially requires that the value of the real number x should agree with l to within an error of at most ε. An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers converges to some real number l if and only if, given any positive real number ε, there exists some positive integer N such that $l - \varepsilon < x_j < l + \varepsilon$ for all positive integers j satisfying $j \geq N$.

Definition We say that an infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is *bounded above* if there exists some real number B such that $x_j \leq B$ for all positive integers j. Similarly we say that this sequence is *bounded below* if there exists some real number A such that $x_j \geq A$ for all positive integers j. A sequence is said to be *bounded* if it is bounded above and bounded below. Thus a sequence is bounded if and only if there exist real numbers A and B such that $A \leq x_j \leq B$ for all positive integers j.

Lemma 1.1 *Every convergent sequence of real numbers is bounded.*

Proof Let $x_1, x_2, x_3, \ldots$ be a sequence of real numbers converging to some real number l. On applying the formal definition of convergence (with $\varepsilon = 1$), we deduce the existence of some positive integer N such that $|x_j - l| < 1$ for all $j \geq N$. But then $A \leq x_j \leq B$ for all positive integers j, where A is the minimum of $x_1, x_2, \ldots, x_{N-1}$ and $l - 1$, and B is the maximum of $x_1, x_2, \ldots, x_{N-1}$ and $l + 1$. ∎

Proposition 1.2 *Let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3,$ be convergent infinite sequences of real numbers. Then the sum, difference and product of these sequences are convergent, and*

$$\begin{aligned} \lim_{j\to+\infty}(x_j + y_j) &= \lim_{j\to+\infty} x_j + \lim_{j\to+\infty} y_j, \\ \lim_{j\to+\infty}(x_j - y_j) &= \lim_{j\to+\infty} x_j - \lim_{j\to+\infty} y_j, \\ \lim_{j\to+\infty}(x_j y_j) &= \left(\lim_{j\to+\infty} x_j\right)\left(\lim_{j\to+\infty} y_j\right). \end{aligned}$$

If in addition $y_j \neq 0$ for all positive integers j and $\lim_{j\to+\infty} y_j \neq 0$, then the quotient of the sequences (x_j) and (y_j) is convergent, and

$$\lim_{j\to+\infty} \frac{x_j}{y_j} = \frac{\lim_{j\to+\infty} x_j}{\lim_{j\to+\infty} y_j}.$$

Proof Throughout this proof let $l = \lim_{j\to+\infty} x_j$ and $m = \lim_{j\to+\infty} y_j$.

First we prove that $x_j + y_j \to l + m$ as $j \to +\infty$. Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j + y_j - (l + m)| < \varepsilon$ whenever $j \geq N$.

Now $x_j \to l$ as $j \to +\infty$, and therefore, given any strictly positive real number ε_1, there exists some positive integer N_1 with the property that $|x_j - l| < \varepsilon_1$ whenever $j \geq N_1$. In particular, there exists a positive integer N_1 with the property that $|x_j - l| < \frac{1}{2}\varepsilon$ whenever $j \geq N_1$. (To see this, let $\varepsilon_1 = \frac{1}{2}\varepsilon$.) Similarly there exists some positive integer N_2 such that $|y_j - m| < \frac{1}{2}\varepsilon$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then

$$\begin{aligned} |x_j + y_j - (l + m)| &= |(x_j - l) + (y_j - m)| \leq |x_j - l| + |y_j - m| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Thus $x_j + y_j \to l + m$ as $j \to +\infty$.

Let c be some real number. We show that $cy_j \to cm$ as $j \to +\infty$. The case when $c = 0$ is trivial. Suppose that $c \neq 0$. Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|y_j - m| < \varepsilon/|c|$ whenever $j \geq N$. But then $|cy_j - cm| = |c||y_j - m| < \varepsilon$ whenever $j \geq N$. Thus $cy_j \to cm$ as $j \to +\infty$.

If we combine this result, for $c = -1$, with the previous result, we see that $-y_j \to -m$ as $j \to +\infty$, and therefore $x_j - y_j \to l - m$ as $j \to +\infty$.

Next we show that if $u_1, u_2, u_3, \ldots$ and $v_1, v_2, v_3, \ldots$ are infinite sequences, and if $u_j \to 0$ and $v_j \to 0$ as $j \to +\infty$, then $u_j v_j \to 0$ as $j \to +\infty$. Let some strictly positive real number ε be given. Then there exist positive integers N_1 and N_2 such that $|u_j| < \sqrt{\varepsilon}$ whenever $j \geq N_1$ and $|v_j| < \sqrt{\varepsilon}$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then $|u_j v_j| < \varepsilon$. We deduce that $u_j v_j \to 0$ as $j \to +\infty$.

We can apply this result with $u_j = x_j - l$ and $v_j = y_j - m$ for all positive integers j. Using the results we have already obtained, we see that

$$\begin{aligned} 0 &= \lim_{j \to +\infty} (u_j v_j) = \lim_{j \to +\infty} (x_j y_j - x_j m - l y_j + lm) \\ &= \lim_{j \to +\infty} (x_j y_j) - m \lim_{j \to +\infty} x_j - l \lim_{j \to +\infty} y_j + lm \\ &= \lim_{j \to +\infty} (x_j y_j) - lm. \end{aligned}$$

Thus $x_j y_j \to lm$ as $j \to +\infty$.

Next we show that if $w_1, w_2, w_3, \ldots$ is an infinite sequence of non-zero real numbers, and if $w_j \to 1$ as $j \to +\infty$ then $1/w_j \to 1$ as $j \to +\infty$. Let some strictly positive real number ε be given. Let ε_0 be the minimum of $\frac{1}{2}\varepsilon$ and $\frac{1}{2}$. Then there exists some positive integer N such that $|w_j - 1| < \varepsilon_0$ whenever $j \geq N$. Thus if $j \geq N$ then $|w_j - 1| < \frac{1}{2}\varepsilon$ and $\frac{1}{2} < w_j < \frac{3}{2}$. But then

$$\left| \frac{1}{w_j} - 1 \right| = \left| \frac{1 - w_j}{w_j} \right| = \frac{|w_j - 1|}{|w_j|} < 2|w_j - 1| < \varepsilon.$$

We deduce that $1/w_j \to 1$ as $j \to +\infty$.

Finally suppose that $\lim\limits_{j \to +\infty} x_j = l$ and $\lim\limits_{j \to +\infty} y_j = m$, where $m \neq 0$. Let $w_j = y_j/m$. Then $w_j \to 1$ as $j \to +\infty$, and hence $1/w_j \to 1$ as $j \to +\infty$. We see therefore that $m/y_j \to 1$, and thus $1/y_j \to 1/m$, as $j \to +\infty$. The result we have already obtained for products of sequences then enables us to deduce that $x_j/y_j \to l/m$ as $j \to +\infty$. ∎

1.4 Monotonic Sequences

An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to be *strictly increasing* if $x_{j+1} > x_j$ for all positive integers j, *strictly decreasing* if $x_{j+1} < x_j$ for all positive integers j, *non-decreasing* if $x_{j+1} \geq x_j$ for all positive integers j, *non-increasing* if $x_{j+1} \leq x_j$ for all positive integers j. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

Theorem 1.3 *Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.*

Proof Let $x_1, x_2, x_3, \ldots$ be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Axiom that there exists a least upper bound l for the set $\{x_j : j \in \mathbb{N}\}$. We claim that the sequence converges to l.

Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j - l| < \varepsilon$ whenever $j \geq N$. Now $l - \varepsilon$ is not an upper bound for the set $\{x_j : j \in \mathbb{N}\}$ (since l is the least upper bound), and therefore there must exist some positive integer N such that $x_N > l - \varepsilon$. But then $l - \varepsilon < x_j \leq l$ whenever $j \geq N$, since the sequence is non-decreasing and bounded above by l. Thus $|x_j - l| < \varepsilon$ whenever $j \geq N$. Therefore $x_j \to l$ as $j \to +\infty$, as required.

If the sequence $x_1, x_2, x_3, \ldots$ is non-increasing and bounded below then the sequence $-x_1, -x_2, -x_3, \ldots$ is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence $x_1, x_2, x_3, \ldots$ is also convergent. ∎

Definition Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. A *subsequence* of this infinite sequence is a sequence of the form $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$ where $j_1, j_2, j_3, \ldots$ is an infinite sequence of positive integers with

$$j_1 < j_2 < j_3 < \cdots.$$

Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. The following sequences are examples of subsequences of the above sequence:—

$$x_1, x_3, x_5, x_7, \ldots$$
$$x_1, x_4, x_9, x_{16}, \ldots$$

1.5 The Bolzano-Weierstrass Theorem

Proposition 1.4 *Let $x_1, x_2, x_3, \ldots$ be a bounded infinite sequence of real numbers. Then there exists a real number c with the property that, given any strictly positive real number ε, there are infinitely many positive integers j for which $c - \varepsilon < x_j < c + \varepsilon$.*

Proof The infinite sequence $(x_j : j \in \mathbb{N})$ is bounded, and therefore there exist real numbers A and B such that $A \leq x_j \leq B$ for all positive integers j.

Let S denote the set of all real numbers s with the property that

$$\{j \in \mathbb{N} : x_j > s\}$$

is an infinite set. Let $c = \sup S$ (so that c is the least upper bound of the set S).

Let u and v be real numbers satisfying $u < c < v$. Choose v' satisfying $c < v' < v$. Then $v' \notin S$, and therefore

$$\{j \in \mathbb{N} : x_j > v'\}$$

is a finite set. It follows that

$$\{j \in \mathbb{N} : x_j \geq v\}$$

is also a finite set.

Also u is not an upper bound for the set S (because c is the least upper bound, and therefore there exists $u' \in S$ satisfying $u' > u$. It then follows that

$$\{j \in \mathbb{N} : x_j > u'\}$$

is an infinite set, and therefore

$$\{j \in \mathbb{N} : x_j > u\}$$

is an infinite set. But then

$$\{j \in \mathbb{N} : u < x_j < v\}$$

must be an infinite set, since it is obtained by removing from $\{j \in \mathbb{N} : x_j > u\}$ a finite number of values of j for which $x_j \geq v$. The result therefore follows on taking $u = c - \varepsilon$ and $v = c + \varepsilon$. ∎

Theorem 1.5 (Bolzano-Weierstrass) *Every bounded sequence of real numbers has a convergent subsequence.*

First Proof Let $x_1, x_2, x_3, \ldots$ be an bounded infinite sequence of real numbers. It follows from Proposition 1.4 that there exists a real number c with the property that, given any strictly positive real number ε, there are infinitely many positive integers j for which $c - \varepsilon < x_j < c + \varepsilon$. There then exists some positive integer k_1 such that $c - 1 < x_{k_1} < c + 1$.

Now suppose that positive integers $k_1, k_2, \ldots, k_m$ have been determined such that $k_1 < k_2 < \cdots < k_m$ and

$$c - \frac{1}{j} < x_{k_j} < c + \frac{1}{j}$$

for $j = 1, 2, \ldots, m$. The interval

$$\left\{ x \in \mathbb{R} : c - \frac{1}{m+1} < x < c + \frac{1}{m+1} \right\}$$

must then contain infinitely many members of the original sequence, and therefore there exists some positive integer k_{m+1} for which $k_m < k_{m+1}$ and

$$c - \frac{1}{m+1} < x_{k_{m+1}} < c + \frac{1}{m+1}.$$

Thus we can construct in this fashion a subsequence $x_{k_1}, x_{k_2}, x_{k_3}, \ldots$ of the original sequence with the property that

$$c - \frac{1}{j} < x_{k_j} < c + \frac{1}{j}$$

for all positive integers j. This subsequence then converges to c. The given sequence therefore has a convergent subsequence, as required. ∎

Second Proof Let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers, and let

$$S = \{j \in \mathbb{N} : a_j \geq a_k \text{ for all } k \geq j\}$$

(i.e., S is the set of all positive integers j with the property that a_j is greater than or equal to all the succeeding members of the sequence).

First let us suppose that the set S is infinite. Arrange the elements of S in increasing order so that $S = \{j_1, j_2, j_3, j_4, \ldots\}$, where $j_1 < j_2 < j_3 < j_4 < \cdots$. It follows from the manner in which the set S was defined that $a_{j_1} \geq a_{j_2} \geq a_{j_3} \geq a_{j_4} \geq \cdots$. Thus $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a non-increasing subsequence of the original sequence $a_1, a_2, a_3, \ldots$. This subsequence is bounded below (since the original sequence is bounded). It follows from Theorem 1.3 that $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a convergent subsequence of the original sequence.

Now suppose that the set S is finite. Choose a positive integer j_1 which is greater than every positive integer belonging to S. Then j_1 does not belong to S. Therefore there must exist some positive integer j_2 satisfying $j_2 > j_1$ such that $a_{j_2} > a_{j_1}$. Moreover j_2 does not belong to S (since j_2 is greater than j_1 and j_1 is greater than every positive integer belonging to S). Therefore there must exist some positive integer j_3 satisfying $j_3 > j_2$ such that $a_{j_3} > a_{j_2}$. We can continue in this way to construct (by induction on j) a strictly increasing subsequence $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ of our original sequence. This increasing subsequence is bounded above (since the original sequence is bounded) and thus is convergent, by Theorem 1.3. This completes the proof of the Bolzano-Weierstrass Theorem. ∎

1.6 Cauchy's Criterion for Convergence

Definition A sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to be a *Cauchy sequence* if the following condition is satisfied:

> given any strictly positive real number ε, there exists some positive integer N such that $|x_j - x_k| < \varepsilon$ for all positive integers j and k satisfying $j \geq N$ and $k \geq N$.

Lemma 1.6 *Every Cauchy sequence of real numbers is bounded.*

Proof Let $x_1, x_2, x_3, \ldots$ be a Cauchy sequence. Then there exists some positive integer N such that $|x_j - x_k| < 1$ whenever $j \geq N$ and $k \geq N$. In particular, $|x_j| \leq |x_N| + 1$ whenever $j \geq N$. Therefore $|x_j| \leq R$ for all positive integers j, where R is the maximum of the real numbers $|x_1|, |x_2|, \ldots, |x_{N-1}|$ and $|x_N| + 1$. Thus the sequence is bounded, as required. ∎

The following important result is known as *Cauchy's Criterion for convergence*, or as the *General Principle of Convergence*.

Theorem 1.7 (Cauchy's Criterion for Convergence) *An infinite sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Proof First we show that convergent sequences are Cauchy sequences. Let $x_1, x_2, x_3, \ldots$ be a convergent sequence of real numbers, and let $l = \lim\limits_{j \to +\infty} x_j$. Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|x_j - l| < \frac{1}{2}\varepsilon$ for all $j \geq N$. Thus if $j \geq N$ and $k \geq N$ then $|x_j - l| < \frac{1}{2}\varepsilon$ and $|x_k - l| < \frac{1}{2}\varepsilon$, and hence

$$|x_j - x_k| = |(x_j - l) - (x_k - l)| \leq |x_j - l| + |x_k - l| < \varepsilon.$$

Thus the sequence $x_1, x_2, x_3, \ldots$ is a Cauchy sequence.

Conversely we must show that any Cauchy sequence $x_1, x_2, x_3, \ldots$ is convergent. Now Cauchy sequences are bounded, by Lemma 1.6. The sequence $x_1, x_2, x_3, \ldots$ therefore has a convergent subsequence $x_{k_1}, x_{k_2}, x_{k_3}, \ldots$, by the Bolzano-Weierstrass Theorem (Theorem 1.5). Let $l = \lim_{j \to +\infty} x_{k_j}$. We claim that the sequence $x_1, x_2, x_3, \ldots$ itself converges to l.

Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|x_j - x_k| < \frac{1}{2}\varepsilon$ whenever $j \geq N$ and $k \geq N$ (since the sequence is a Cauchy sequence). Let m be chosen large enough to ensure that $k_m \geq N$ and $|x_{k_m} - l| < \frac{1}{2}\varepsilon$. Then

$$|x_j - l| \leq |x_j - x_{k_m}| + |x_{k_m} - l| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

whenever $j \geq N$. It follows that $x_j \to l$ as $j \to +\infty$, as required. ∎

2 Real Analysis in Euclidean Spaces

2.1 Basic Properties of Vectors and Norms

We denote by $\mathbb{R}^n$ the set consisting of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. The set $\mathbb{R}^n$ represents n-dimensional *Euclidean space* (with respect to the standard Cartesian coordinate system). Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad \mathbf{y} = (y_1, y_2, \ldots, y_n),$$

and let λ be a real number. We define

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\ \mathbf{x} - \mathbf{y} &= (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n), \\ \lambda \mathbf{x} &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n), \\ \mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \\ |\mathbf{x}| &= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \end{aligned}$$

The quantity $\mathbf{x} \cdot \mathbf{y}$ is the *scalar product* (or *inner product*) of $\mathbf{x}$ and $\mathbf{y}$, and the quantity $|\mathbf{x}|$ is the *Euclidean norm* of $\mathbf{x}$. Note that $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$. The *Euclidean distance* between two points $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^n$ is defined to be the Euclidean norm $|\mathbf{y} - \mathbf{x}|$ of the vector $\mathbf{y} - \mathbf{x}$.

Proposition 2.1 (Schwarz's Inequality) *The inequality* $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$ *is satisfied by all elements* $\mathbf{x}$ *and* $\mathbf{y}$ *of* $\mathbb{R}^n$*.*

Proof We note that $|\lambda \mathbf{x} + \mu \mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. But

$$|\lambda \mathbf{x} + \mu \mathbf{y}|^2 = (\lambda \mathbf{x} + \mu \mathbf{y}).(\lambda \mathbf{x} + \mu \mathbf{y}) = \lambda^2 |\mathbf{x}|^2 + 2\lambda\mu \mathbf{x} \cdot \mathbf{y} + \mu^2 |\mathbf{y}|^2.$$

Therefore $\lambda^2 |\mathbf{x}|^2 + 2\lambda\mu \mathbf{x} \cdot \mathbf{y} + \mu^2 |\mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. In particular, suppose that $\lambda = |\mathbf{y}|^2$ and $\mu = -\mathbf{x} \cdot \mathbf{y}$. We conclude that

$$|\mathbf{y}|^4 |\mathbf{x}|^2 - 2|\mathbf{y}|^2 (\mathbf{x} \cdot \mathbf{y})^2 + (\mathbf{x} \cdot \mathbf{y})^2 |\mathbf{y}|^2 \geq 0,$$

so that $\left(|\mathbf{x}|^2 |\mathbf{y}|^2 - (\mathbf{x} \cdot \mathbf{y})^2\right) |\mathbf{y}|^2 \geq 0$. Thus if $\mathbf{y} \neq \mathbf{0}$ then $|\mathbf{y}| > 0$, and hence

$$|\mathbf{x}|^2 |\mathbf{y}|^2 - (\mathbf{x} \cdot \mathbf{y})^2 \geq 0.$$

But this inequality is trivially satisfied when $\mathbf{y} = \mathbf{0}$. Thus $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$, as required. ∎

Corollary 2.2 (Triangle Inequality) *The inequality* $|\mathbf{x}+\mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$ *is satisfied for all elements* $\mathbf{x}$ *and* $\mathbf{y}$ *of* $\mathbb{R}^n$.

Proof Using Schwarz's Inequality, we see that

$$\begin{aligned} |\mathbf{x}+\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y}).(\mathbf{x}+\mathbf{y}) = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x}\cdot\mathbf{y} \\ &\leq |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2|\mathbf{x}||\mathbf{y}| = (|\mathbf{x}| + |\mathbf{y}|)^2. \end{aligned}$$

The result follows directly. ∎

It follows immediately from the Triangle Inequality (Corollary 2.2) that

$$|\mathbf{z}-\mathbf{x}| \leq |\mathbf{z}-\mathbf{y}| + |\mathbf{y}-\mathbf{x}|$$

for all points $\mathbf{x}$, $\mathbf{y}$ and $|\mathbf{z}|$ of $\mathbb{R}^n$. This important inequality expresses the geometric fact the the length of any triangle in a Euclidean space is less than or equal to the sum of the lengths of the other two sides.

2.2 Convergence of Sequences in Euclidean Spaces

Definition A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to *converge* to a point $\mathbf{p}$ if and only if the following criterion is satisfied:—

> given any real number ε satisfying $\varepsilon > 0$ there exists some positive integer N such that $|\mathbf{p} - \mathbf{x}_j| < \varepsilon$ whenever $j \geq N$.

We refer to $\mathbf{p}$ as the *limit* $\lim\limits_{j\to+\infty} \mathbf{x}_j$ of the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$.

Lemma 2.3 *Let* $\mathbf{p}$ *be a point of* $\mathbb{R}^n$, *where* $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. *Then a sequence* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *of points in* $\mathbb{R}^n$ *converges to* $\mathbf{p}$ *if and only if the* i*th components of the elements of this sequence converge to* p_i *for* $i = 1, 2, \ldots, n$.

Proof Let x_{ji} and p_i denote the ith components of $\mathbf{x}_j$ and $\mathbf{p}$, where $\mathbf{p} = \lim\limits_{j\to+\infty} \mathbf{x}_j$. Then $|x_{ji} - p_i| \leq |\mathbf{x}_j - \mathbf{p}|$ for all j. It follows directly from the definition of convergence that if $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ then $x_{ji} \to p_i$ as $j \to +\infty$.

Conversely suppose that, for each i, $x_{ji} \to p_i$ as $j \to +\infty$. Let $\varepsilon > 0$ be given. Then there exist positive integers $N_1, N_2, \ldots, N_n$ such that $|x_{ji} - p_i| < \varepsilon/\sqrt{n}$ whenever $j \geq N_i$. Let N be the maximum of $N_1, N_2, \ldots, N_n$. If $j \geq N$ then

$$|\mathbf{x}_j - \mathbf{p}|^2 = \sum_{i=1}^{n} (x_{ji} - p_i)^2 < n(\varepsilon/\sqrt{n})^2 = \varepsilon^2,$$

so that $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$. ∎

Definition A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to be a *Cauchy sequence* if and only if the following criterion is satisfied:—

> given any real number ε satisfying $\varepsilon > 0$ there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < \varepsilon$ whenever $j \geq N$ and $k \geq N$.

Lemma 2.4 *A sequence of points in $\mathbb{R}^n$ is convergent if and only if it is a Cauchy sequence.*

Proof Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of $\mathbb{R}^n$ converging to some point $\mathbf{p}$. Let $\varepsilon > 0$ be given. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$ whenever $j \geq N$. If $j \geq N$ and $k \geq N$ then

$$|\mathbf{x}_j - \mathbf{x}_k| \leq |\mathbf{x}_j - \mathbf{p}| + |\mathbf{p} - \mathbf{x}_k| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

by the Triangle Inequality. Thus every convergent sequence in $\mathbb{R}^n$ is a Cauchy sequence.

Now let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a Cauchy sequence in $\mathbb{R}^n$. Then the ith components of the elements of this sequence constitute a Cauchy sequence of real numbers. This Cauchy sequence must converge to some real number p_i, by Cauchy's Criterion for Convergence (Theorem 1.7). It follows from Lemma 2.3 that the Cauchy sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to the point $\mathbf{p}$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. ∎

2.3 The Multidimensional Bolzano-Weierstrass Theorem

A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to be *bounded* if there exists some constant K such that $|\mathbf{x}_j| \leq K$ for all j.

Example Let

$$(x_j, y_j, z_j) = \left(\sin(\pi\sqrt{j}),\ (-1)^j, \cos\left(\frac{2\pi \log j}{\log 2}\right)\right)$$

for $j = 1, 2, 3, \ldots$. This sequence of points in $\mathbb{R}^3$ is bounded, because the components of its members all take values between -1 and 1. Moreover $x_j = 0$ whenever j is the square of a positive integer, $y_j = 1$ whenever j is even and $z_j = 1$ whenever j is a power of two.

The infinite sequence $x_1, x_2, x_3, \ldots$ has a convergent subsequence

$$x_1, x_4, x_9, x_{16}, x_{25}, \ldots$$

which includes those x_j for which j is the square of a positive integer. The corresponding subsequence $y_1, y_4, y_9, \ldots$ of $y_1, y_2, y_3, \ldots$ is not convergent, because its values alternate between 1 and -1. However this subsequence is bounded, and we can extract from this sequence a convergent subsequence

$$y_4, y_{16}, y_{36}, y_{64}, y_{100}, \ldots$$

which includes those x_j for which j is the square of an even positive integer.

The subsequence

$$x_4, x_{16}, x_{36}, y_{64}, y_{100}, \ldots$$

is also convergent, because it is a subsequence of a convergent subsequence. However the corresponding subsequence

$$z_4, z_{16}, z_{36}, z_{64}, z_{100}, \ldots$$

does not converge. (Indeed $z_j = 1$ when j is an even power of 2, but $z_j = \cos(2\pi \log(9)/\log(2))$ when $j = 9 \times 2^{2p}$ for some positive integer p.) However this subsequence is bounded, and we can extract from it a convergent subsequence

$$z_4, z_{16}, z_{64}, z_{256}, z_{1024}, \ldots$$

which includes those x_j for which j is equal to two raised to the power of an even positive integer.

Then the first, second and third components of the following subsequence

$$(x_4, y_4, z_4), \ (x_{16}, y_{16}, z_{16}), \ (x_{64}, y_{64}, z_{64}), \ (x_{256}, y_{256}, z_{256}), \ldots$$

of the original sequence of points in $\mathbb{R}^3$ converge, and it therefore follows from Lemma 2.3 that this sequence is a convergent subsequence of the given sequence of points in $\mathbb{R}^3$.

Example Let

$$x_j = \begin{cases} 1 & \text{if } j = 4k \text{ for some integer } k \\ 0 & \text{if } j = 4k+1 \text{ for some integer } k \\ -1 & \text{if } j = 4k+2 \text{ for some integer } k \\ 0 & \text{if } j = 4k+3 \text{ for some integer } k \end{cases}$$

and

$$y_j = \begin{cases} 0 & \text{if } j = 4k \text{ for some integer } k, \\ 1 & \text{if } j = 4k+1 \text{ for some integer } k, \\ 0 & \text{if } j = 4k+2 \text{ for some integer } k, \\ -1 & \text{if } j = 4k+3 \text{ for some integer } k, \end{cases},$$

and let $\mathbf{u}_j = (x_j, y_j)$ for $j = 1, 2, 3, 4, \ldots$.

Then the first components x_j for which the index j is odd constitute a convergent sequence $x_1, x_3, x_5, x_7, \ldots$ of real numbers, and the second components y_j for which the index j is even also constitute a convergent sequence $y_2, y_4, y_6, y_8, \ldots$ of real numbers.

However one would not obtain a convergent subsequence of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ simply by selecting those indices j for which x_j is in the convergent subsequence $x_1, x_3, x_5, \ldots$ and y_j is in the convergent subsequence $y_2, y_4, y_6, \ldots$, because there no values of the index j for which x_j and y_j both belong to the respective subsequences. However the one-dimensional Bolzano-Weierstrass Theorem (Theorem 1.5) guarantees that there is a convergent subsequence of $y_1, y_3, y_5, y_7, \ldots$, and indeed $y_1, y_5, y_9, y_{13}, \ldots$ is such a convergent subsequence. This yields a convergent subsequence $\mathbf{u}_1, \mathbf{u}_5, \mathbf{u}_9, \mathbf{u}_{13}, \ldots$ of the given bounded sequence of points in $\mathbb{R}^2$.

Theorem 2.5 (The Multidimensional Bolzano-Weierstrass Theorem) *Every bounded sequence of points in $\mathbb{R}^n$ has a convergent subsequence.*

Proof We prove the result by induction on the dimension n of the Euclidean space $\mathbb{R}^n$ that contains the infinite sequence in question. It follows from the one-dimensional Bolzano-Weierstrass Theorem (Theorem 1.5) that the theorem is true when $n = 1$. Suppose that $n > 1$, and that every bounded sequence in $\mathbb{R}^{n-1}$ has a convergent subsequence. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a bounded infinite sequence of elements of $\mathbb{R}^n$, and let $x_{j,i}$ denote the ith component of $\mathbf{x}_j$ for $i = 1, 2, \ldots, n$ and for all positive integers j.

The induction hypothesis requires that all bounded sequences in $\mathbb{R}^{n-1}$ contain convergent subsequences. It follows that there exist real numbers $p_1, p_2, \ldots, p_{n-1}$ and an increasing sequence $m_1, m_2, m_3, \ldots$ of positive integers such that $\lim\limits_{k \to +\infty} x_{m_k,i} = p_i$ for $i = 1, 2, \ldots, n-1$. The nth components $x_{m_1,n}, x_{m_2,n}, x_{m_3,n}, \ldots$ of the members of the subsequence $\mathbf{x}_{m_1}, \mathbf{x}_{m_2}, \mathbf{x}_{m_3}, \ldots$ then constitute a bounded sequence of real numbers. It follows from the one-dimensional Bolzano-Weierstrass Theorem (Theorem 1.5) that there exists an increasing sequence $k_1, k_2, k_3, \ldots$ of positive integers for which the sequence $x_{m_{k_1},n}, x_{m_{k_2},n}, x_{m_{k_3},n}, \ldots$ converges.

Let $s_j = m_{k_j}$ for all positive integers j, and let

$$p_n = \lim_{j \to +\infty} x_{m_{k_j},n} = \lim_{j \to +\infty} x_{s_j,n}.$$

Then the sequence $x_{s_1,i}, x_{s_2,i}, x_{s_3,i}, \ldots$ converges for values of i between 1 and $n-1$, because it is a subquence of the convergent sequence

$$x_{m_1,i}, x_{m_2,i}, x_{m_3,i}, \ldots.$$

Moreover

$$x_{s_1,n}, x_{s_2,n}, x_{s_3,n}, \ldots$$

also converges. Thus the ith components of the infinite sequence

$$\mathbf{x}_{m_1}, \mathbf{x}_{m_2}, \mathbf{x}_{m_3}, \ldots$$

converge for $i = 1, 2, \ldots, n$. It then follows from Lemma 2.3 that

$$\lim_{j \to +\infty} \mathbf{x}_{s_k} = \mathbf{p},$$

where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. The result follows. ∎

2.4 Continuity of Functions of Several Real Variables

Definition Let X and Y be a subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A function $\varphi: X \to Y$ from X to Y is said to be *continuous* at a point $\mathbf{p}$ of X if and only if the following criterion is satisfied:—

> given any strictly positive real number ε, there exists some strictly positive real number δ such that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$.

The function $\varphi: X \to Y$ is said to be continuous on X if and only if it is continuous at every point $\mathbf{p}$ of X.

Lemma 2.6 *Let X, Y and Z be subsets of $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^k$ respectively, and let $\varphi: X \to Y$ and $\psi: Y \to Z$ be functions satisfying $\varphi(X) \subset Y$. Suppose that φ is continuous at some point $\mathbf{p}$ of X and that ψ is continuous at $\varphi(\mathbf{p})$. Then the composition function $\psi \circ \varphi: X \to Z$ is continuous at $\mathbf{p}$.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|\psi(\mathbf{y}) - \psi(\varphi(\mathbf{p}))| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - \varphi(\mathbf{p})| < \eta$. But then there exists some $\delta > 0$ such that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \eta$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|\psi(\varphi(\mathbf{x})) - \psi(\varphi(\mathbf{p}))| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus $\psi \circ \varphi$ is continuous at $\mathbf{p}$, as required. ∎

Lemma 2.7 *Let X and Y be a subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\mathbf{p}$ be a point of X. A function $\varphi: X \to Y$ from X to Y is continuous at $\mathbf{p}$ if and only if, given any infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X converging to the point $\mathbf{p}$, the infinite sequence $\varphi(\mathbf{x}_1), \varphi(\mathbf{x}_2), \varphi(\mathbf{x}_3), \ldots$ converges to $\varphi(\mathbf{p})$.*

Proof First suppose that $\varphi\colon X \to Y$ is continuous at the point p of X. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be an infinite sequence of points of X converging to the point $\mathbf{p}$. Let $\varepsilon > 0$ be given. Then there exists some $\delta > 0$ such that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, because the function φ is continuous at $\mathbf{p}$. Also there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$, because the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Thus if $j \geq N$ then $|\varphi(\mathbf{x}_j) - \varphi(\mathbf{p})| < \varepsilon$. Thus the sequence $\varphi(\mathbf{x}_1), \varphi(\mathbf{x}_2), \varphi(\mathbf{x}_3), \ldots$ converges to $\varphi(\mathbf{p})$.

Now suppose that $\varphi\colon X \to Y$ is not continuous at the the point $\mathbf{p}$. Then there exists a positive real number ε with the property that, given any positive real number δ there exists some point $\mathbf{x}$ of X for which $|\mathbf{x} - \mathbf{p}| < \delta$ and $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| \geq \varepsilon$. There then exists an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X such that $|\mathbf{x}_j - \mathbf{p}| < 1/j$ and $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| \geq \varepsilon$ for all integers j. The sequence $\varphi(\mathbf{x}_1), \varphi(\mathbf{x}_2), \varphi(\mathbf{x}_3), \ldots$ converges to the point $\mathbf{p}$, but the sequence $\varphi(\mathbf{x}_1), \varphi(\mathbf{x}_2), \varphi(\mathbf{x}_3), \ldots$ does not converge to $\varphi(\mathbf{p})$. The result follows. ∎

Let X and Y be a subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a function from X to Y. Then

$$\varphi(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))$$

for all $\mathbf{x} \in X$, where $f_1, f_2, \ldots, f_m$ are functions from X to $\mathbb{R}$, referred to as the *components* of the function φ.

Proposition 2.8 *Let X and Y be a subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\mathbf{p} \in X$. A function $\varphi\colon X \to Y$ is continuous at the point $\mathbf{p}$ if and only if its components are all continuous at $\mathbf{p}$.*

Proof Note that the ith component f_i of φ is given by $f_i = \pi_i \circ \varphi$, where $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ is the continuous function which maps $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ onto its ith coordinate y_i. Now any composition of continuous functions is continuous, by Lemma 2.6. Thus if φ is continuous at $\mathbf{p}$, then so are the components of φ.

Conversely suppose that the components of φ are continuous at $\mathbf{p} \in X$. Let $\varepsilon > 0$ be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_m$ such that $|f_i(\mathbf{x}) - f_i(\mathbf{p})| < \varepsilon/\sqrt{m}$ for $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_m$. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{p})|^2 = \sum_{i=1}^{m} |f_i(\mathbf{x}) - f_i(\mathbf{p})|^2 < \varepsilon^2,$$

and hence $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$. Thus the function φ is continuous at $\mathbf{p}$, as required. ∎

Lemma 2.9 *The functions $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $m\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $s(x,y) = x+y$ and $m(x,y) = xy$ are continuous.*

Proof Let $(u,v) \in \mathbb{R}^2$. We first show that $s\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u,v). Let $\varepsilon > 0$ be given. Let $\delta = \frac{1}{2}\varepsilon$. If (x,y) is any point of $\mathbb{R}^2$ whose distance from (u,v) is less than δ then $|x-u| < \delta$ and $|y-v| < \delta$, and hence

$$|s(x,y) - s(u,v)| = |x+y-u-v| \leq |x-u| + |y-v| < 2\delta = \varepsilon.$$

This shows that $s\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u,v).

Next we show that $m\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u,v). Now

$$m(x,y) - m(u,v) = xy - uv = (x-u)(y-v) + u(y-v) + (x-u)v.$$

for all points (x,y) of $\mathbb{R}^2$. Thus if the distance from (x,y) to (u,v) is less than δ then $|x-u| < \delta$ and $|y-v| < \delta$, and hence

$$|m(x,y) - m(u,v)| < \delta^2 + (|u| + |v|)\delta.$$

Let $\varepsilon > 0$ is given. If $\delta > 0$ is chosen to be the minimum of 1 and $\varepsilon/(1 + |u| + |v|)$ then $\delta^2 + (|u| + |v|)\delta < (1 + |u| + |v|)\delta < \varepsilon$, and thus $|m(x,y) - m(u,v)| < \varepsilon$ for all points (x,y) of $\mathbb{R}^2$ whose distance from (u,v) is less than δ. This shows that $p\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u,v). ∎

Proposition 2.10 *Let X be a subset of $\mathbb{R}^n$, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be continuous functions from X to $\mathbb{R}$. Then the functions $f+g$, $f-g$ and $f \cdot g$ are continuous. If in addition $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ then the quotient function f/g is continuous.*

Proof Note that $f + g = s \circ h$ and $f \cdot g = m \circ h$, where $h\colon X \to \mathbb{R}^2$, $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $m\colon \mathbb{R}^2 \to \mathbb{R}$ are given by $h(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$, $s(u,v) = u+v$ and $m(u,v) = uv$ for all $\mathbf{x} \in X$ and $u,v \in \mathbb{R}$. It follows from Proposition 2.8, Lemma 2.9 and Lemma 2.6 that $f+g$ and $f \cdot g$ are continuous, being compositions of continuous functions. Now $f - g = f + (-g)$, and both f and $-g$ are continuous. Therefore $f - g$ is continuous.

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$. Note that $1/g = r \circ g$, where $r\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is the reciprocal function, defined by $r(t) = 1/t$. Now the reciprocal function r is continuous. Thus the function $1/g$ is a composition of continuous functions and is thus continuous. But then, using the fact that a product of continuous real-valued functions is continuous, we deduce that f/g is continuous. ∎

Example Consider the function $f\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ defined by

$$f(x,y) = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right).$$

The continuity of the components of the function f follows from straightforward applications of Proposition 2.10. It then follows from Proposition 2.8 that the function f is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$.

2.5 Open Sets in Euclidean Spaces

Let X be a subset of $\mathbb{R}^n$. Given a point $\mathbf{p}$ of X and a non-negative real number r, the *open ball* $B_X(\mathbf{p}, r)$ in X of *radius* r about $\mathbf{p}$ is defined to be the subset of X given by

$$B_X(\mathbf{p}, r) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < r\}.$$

(Thus $B_X(\mathbf{p}, r)$ is the set consisting of all points of X that lie within a sphere of radius r centred on the point $\mathbf{p}$.)

Definition Let X be a subset of $\mathbb{R}^n$. A subset V of X is said to be *open* in X if and only if, given any point $\mathbf{p}$ of V, there exists some $\delta > 0$ such that $B_X(\mathbf{p}, \delta) \subset V$.

By convention, we regard the empty set $\emptyset$ as being an open subset of X. (The criterion given above is satisfied vacuously in the case when V is the empty set.)

In particular, a subset V of $\mathbb{R}^n$ is said to be an *open set* (in $\mathbb{R}^n$) if and only if, given any point $\mathbf{p}$ of V, there exists some $\delta > 0$ such that $B(\mathbf{p}, \delta) \subset V$, where $B(\mathbf{p}, r) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < r\}$.

Example Let $H = \{(x, y, z) \in \mathbb{R}^3 : z > c\}$, where c is some real number. Then H is an open set in $\mathbb{R}^3$. Indeed let $\mathbf{p}$ be a point of H. Then $\mathbf{p} = (u, v, w)$, where $w > c$. Let $\delta = w - c$. If the distance from a point (x, y, z) to the point (u, v, w) is less than δ then $|z - w| < \delta$, and hence $z > c$, so that $(x, y, z) \in H$. Thus $B(\mathbf{p}, \delta) \subset H$, and therefore H is an open set.

The previous example can be generalized. Given any integer i between 1 and n, and given any real number c_i, the sets

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i > c_i\}$$

and

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i < c_i\}$$

are open sets in $\mathbb{R}^n$.

Example Let U be an open set in $\mathbb{R}^n$. Then for any subset X of $\mathbb{R}^n$, the intersection $U \cap X$ is open in X. (This follows directly from the definitions.) Thus for example, let S^2 be the unit sphere in $\mathbb{R}^3$, given by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

and let N be the subset of S^2 given by

$$N = \{(x, y, z) \in \mathbb{R}^n : x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}.$$

Then N is open in S^2, since $N = H \cap S^2$, where H is the open set in $\mathbb{R}^3$ given by

$$H = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}.$$

Note that N is not itself an open set in $\mathbb{R}^3$. Indeed the point $(0, 0, 1)$ belongs to N, but, for any $\delta > 0$, the open ball (in $\mathbb{R}^3$ of radius δ about $(0, 0, 1)$ contains points (x, y, z) for which $x^2 + y^2 + z^2 \neq 1$. Thus the open ball of radius δ about the point $(0, 0, 1)$ is not a subset of N.

Lemma 2.11 *Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any positive real number r, the open ball $B_X(\mathbf{p}, r)$ in X of radius r about $\mathbf{p}$ is open in X.*

Proof Let $\mathbf{x}$ be an element of $B_X(\mathbf{p}, r)$. We must show that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. Let $\delta = r - |\mathbf{x} - \mathbf{p}|$. Then $\delta > 0$, since $|\mathbf{x} - \mathbf{p}| < r$. Moreover if $\mathbf{y} \in B_X(\mathbf{x}, \delta)$ then

$$|\mathbf{y} - \mathbf{p}| \leq |\mathbf{y} - \mathbf{x}| + |\mathbf{x} - \mathbf{p}| < \delta + |\mathbf{x} - \mathbf{p}| = r,$$

by the Triangle Inequality, and hence $\mathbf{y} \in B_X(\mathbf{p}, r)$. Thus $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. This shows that $B_X(\mathbf{p}, r)$ is an open set, as required. ∎

Lemma 2.12 *Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any non-negative real number r, the set $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| > r\}$ is an open set in X.*

Proof Let $\mathbf{x}$ be a point of X satisfying $|\mathbf{x} - \mathbf{p}| > r$, and let $\mathbf{y}$ be any point of X satisfying $|\mathbf{y} - \mathbf{x}| < \delta$, where $\delta = |\mathbf{x} - \mathbf{p}| - r$. Then

$$|\mathbf{x} - \mathbf{p}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{p}|,$$

by the Triangle Inequality, and therefore

$$|\mathbf{y} - \mathbf{p}| \geq |\mathbf{x} - \mathbf{p}| - |\mathbf{y} - \mathbf{x}| > |\mathbf{x} - \mathbf{p}| - \delta = r.$$

Thus $B_X(\mathbf{x}, \delta)$ is contained in the given set. The result follows. ∎

Proposition 2.13 *Let X be a subset of $\mathbb{R}^n$. The collection of open sets in X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are both open in X;

(ii) the union of any collection of open sets in X is itself open in X;

(iii) the intersection of any finite *collection of open sets in X is itself open in X.*

Proof The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole set X. This proves (i).

Let $\mathcal{A}$ be any collection of open sets in X, and let U denote the union of all the open sets belonging to $\mathcal{A}$. We must show that U is itself open in X. Let $\mathbf{x} \in U$. Then $\mathbf{x} \in V$ for some set V belonging to the collection $\mathcal{A}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset V$. But $V \subset U$, and thus $B_X(\mathbf{x}, \delta) \subset U$. This shows that U is open in X. This proves (ii).

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of subsets of X that are open in X, and let V denote the intersection $V_1 \cap V_2 \cap \cdots \cap V_k$ of these sets. Let $\mathbf{x} \in V$. Now $\mathbf{x} \in V_j$ for $j = 1, 2, \ldots, k$, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of sets.) Now $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(\mathbf{x}, \delta) \subset V$. Thus the intersection V of the sets $V_1, V_2, \ldots, V_k$ is itself open in X. This proves (iii). ∎

Example The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ and } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the intersection of the open ball of radius 2 about the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

Example The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ or } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the union of the open ball of radius 2 about the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

Example The set

$$\{(x, y, z) \in \mathbb{R}^3 : (x - n)^2 + y^2 + z^2 < \tfrac{1}{4} \text{ for some } n \in \mathbb{Z}\}$$

is an open set in $\mathbb{R}^3$, since it is the union of the open balls of radius $\frac{1}{2}$ about the points $(n, 0, 0)$ for all integers n.

Example For each positive integer k, let

$$V_k = \{(x, y, z) \in \mathbb{R}^3 : k^2(x^2 + y^2 + z^2) < 1\}.$$

Now each set V_k is an open ball of radius $1/k$ about the origin, and is therefore an open set in $\mathbb{R}^3$. However the intersection of the sets V_k for all positive integers k is the set $\{(0, 0, 0)\}$, and thus the intersection of the sets V_k for all positive integers k is not itself an open set in $\mathbb{R}^3$. This example demonstrates that infinite intersections of open sets need not be open.

Lemma 2.14 *A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to a point $\mathbf{p}$ if and only if, given any open set U which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in U$ for all j satisfying $j \geq N$.*

Proof Suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ has the property that, given any open set U which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in U$ whenever $j \geq N$. Let $\varepsilon > 0$ be given. The open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is an open set by Lemma 2.11. Therefore there exists some positive integer N such that $\mathbf{x}_j \in B(\mathbf{p}, \varepsilon)$ whenever $j \geq N$. Thus $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$. This shows that the sequence converges to $\mathbf{p}$.

Conversely, suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Let U be an open set which contains $\mathbf{p}$. Then there exists some $\varepsilon > 0$ such that the open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is a subset of U. Thus there exists some $\varepsilon > 0$ such that U contains all points $\mathbf{x}$ of X that satisfy $|\mathbf{x} - \mathbf{p}| < \varepsilon$. But there exists some positive integer N with the property that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$, since the sequence converges to $\mathbf{p}$. Therefore $\mathbf{x}_j \in U$ whenever $j \geq N$, as required. ∎

2.6 Closed Sets in Euclidean Spaces

Let X be a subset of $\mathbb{R}^n$. A subset F of X is said to be *closed* in X if and only if its complement $X \setminus F$ in X is open in X. (Recall that $X \setminus F = \{\mathbf{x} \in X : \mathbf{x} \notin F\}$.)

Example The sets $\{(x, y, z) \in \mathbb{R}^3 : z \geq c\}$, $\{(x, y, z) \in \mathbb{R}^3 : z \leq c\}$, and $\{(x, y, z) \in \mathbb{R}^3 : z = c\}$ are closed sets in $\mathbb{R}^3$ for each real number c, since the complements of these sets are open in $\mathbb{R}^3$.

Example Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{x}_0$ be a point of X. Then the sets $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{x}_0| \leq r\}$ and $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{x}_0| \geq r\}$ are closed for each non-negative real number r. In particular, the set $\{\mathbf{x}_0\}$ consisting of the single point $\mathbf{x}_0$ is a closed set in X. (These results follow immediately using Lemma 2.11 and Lemma 2.12 and the definition of closed sets.)

Let $\mathcal{A}$ be some collection of subsets of a set X. Then

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S), \qquad X \setminus \bigcap_{S \in \mathcal{A}} S = \bigcup_{S \in \mathcal{A}} (X \setminus S)$$

(i.e., the complement of the union of some collection of subsets of X is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of X is the union of the complements of those sets). The following result therefore follows directly from Proposition 2.13.

Proposition 2.15 *Let X be a subset of $\mathbb{R}^n$. The collection of closed sets in X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are both closed in X;

(ii) the intersection of any collection of closed sets in X is itself closed in X;

(iii) the union of any finite *collection of closed sets in X is itself closed in X.*

Lemma 2.16 *Let X be a subset of $\mathbb{R}^n$, and let F be a subset of X which is closed in X. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of F which converges to a point $\mathbf{p}$ of X. Then $\mathbf{p} \in F$.*

Proof The complement $X \setminus F$ of F in X is open, since F is closed. Suppose that $\mathbf{p}$ were a point belonging to $X \setminus F$. It would then follow from Lemma 2.14 that $\mathbf{x}_j \in X \setminus F$ for all values of j greater than some positive integer N, contradicting the fact that $\mathbf{x}_j \in F$ for all j. This contradiction shows that $\mathbf{p}$ must belong to F, as required. ∎

2.7 Continuous Functions and Open Sets

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi: X \to Y$ be a function from X to Y. We recall that the function φ is continuous at a point $\mathbf{p}$ of X if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|\varphi(\mathbf{u}) - \varphi(\mathbf{p})| < \varepsilon$ for all points $\mathbf{u}$ of X satisfying $|\mathbf{u} - \mathbf{p}| < \delta$. Thus the function $\varphi: X \to Y$ is continuous at $\mathbf{p}$ if and only if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that the function φ maps $B_X(\mathbf{p}, \delta)$ into $B_Y(\varphi(\mathbf{p}), \varepsilon)$ (where $B_X(\mathbf{p}, \delta)$ and $B_Y(\varphi(\mathbf{p}), \varepsilon)$ denote the open balls in X and Y of radius δ and ε about $\mathbf{p}$ and $\varphi(\mathbf{p})$ respectively).

Given any function $\varphi: X \to Y$, we denote by $\varphi^{-1}(V)$ the *preimage* of a subset V of Y under the map φ, defined by $\varphi^{-1}(V) = \{\mathbf{x} \in X : \varphi(\mathbf{x}) \in V\}$.

Proposition 2.17 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi: X \to Y$ be a function from X to Y. The function φ is continuous if and only if $\varphi^{-1}(V)$ is open in X for every subset V of Y that is open in Y.*

Proof Suppose that $\varphi: X \to Y$ is continuous. Let V be an open set in Y. We must show that $\varphi^{-1}(V)$ is open in X. Let $\mathbf{p} \in \varphi^{-1}(V)$. Then $\varphi(\mathbf{p}) \in V$. But V is open, hence there exists some $\varepsilon > 0$ with the property that $B_Y(\varphi(\mathbf{p}), \varepsilon) \subset V$. But φ is continuous at $\mathbf{p}$. Therefore there exists some $\delta > 0$ such that φ maps $B_X(\mathbf{p}, \delta)$ into $B_Y(\varphi(\mathbf{p}), \varepsilon)$ (see the remarks above). Thus $\varphi(\mathbf{x}) \in V$ for all $\mathbf{x} \in B_X(\mathbf{p}, \delta)$, showing that $B_X(\mathbf{p}, \delta) \subset \varphi^{-1}(V)$. This shows that $\varphi^{-1}(V)$ is open in X for every open set V in Y.

Conversely suppose that $\varphi: X \to Y$ is a function with the property that $\varphi^{-1}(V)$ is open in X for every open set V in Y. Let $\mathbf{p} \in X$. We must show that φ is continuous at $\mathbf{p}$. Let $\varepsilon > 0$ be given. Then $B_X(\varphi(\mathbf{p}), \varepsilon)$ is an open set in Y, by Lemma 2.11, hence $\varphi^{-1}\left(B_Y(\varphi(\mathbf{p}), \varepsilon)\right)$ is an open set in X which contains $\mathbf{p}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{p}, \delta) \subset \varphi^{-1}\left(B_Y(\varphi(\mathbf{p}), \varepsilon)\right)$. Thus, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that φ maps $B_X(\mathbf{p}, \delta)$ into $B_Y(\varphi(\mathbf{p}), \varepsilon)$. We conclude that φ is continuous at $\mathbf{p}$, as required. ∎

Let X be a subset of $\mathbb{R}^n$, let $f: X \to \mathbb{R}$ be continuous, and let c be some real number. Then the sets $\{\mathbf{x} \in X : f(\mathbf{x}) > c\}$ and $\{\mathbf{x} \in X : f(\mathbf{x}) < c\}$ are open in X, and, given real numbers a and b satisfying $a < b$, the set $\{\mathbf{x} \in X : a < f(\mathbf{x}) < b\}$ is open in X.

Corollary 2.18 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi: X \to Y$ be a continuous function from X to Y. Then $\varphi^{-1}(F)$ is closed in X for every subset F of Y that is closed in Y.*

Proof Let F be a subset of Y that is closed in Y, and let let $V = Y \setminus F$. Then V is open in Y. It follows from Proposition 2.17 that $\varphi^{-1}(V)$ is open in X. But

$$\varphi^{-1}(V) = \varphi^{-1}(Y \setminus F) = X \setminus \varphi^{-1}(F).$$

Indeed let $\mathbf{x} \in X$. Then

$$\begin{aligned} & \mathbf{x} \in \varphi^{-1}(V) \\ \iff & \mathbf{x} \in \varphi^{-1}(Y \setminus F) \\ \iff & \varphi(\mathbf{x}) \in Y \setminus F \\ \iff & \varphi(\mathbf{x}) \notin F \\ \iff & \mathbf{x} \notin \varphi^{-1}(F) \\ \iff & \mathbf{x} \in X \setminus \varphi^{-1}(F). \end{aligned}$$

It follows that the complement $X \setminus \varphi^{-1}(F)$ of $\varphi^{-1}(F)$ in X is open in X, and therefore $\varphi^{-1}(F)$ itself is closed in X, as required. ∎

Proposition 2.19 *Let $\varphi: X \to \mathbb{R}^m$ be a function mapping a subset X of $\mathbb{R}^n$ into $\mathbb{R}^m$. Let $F_1, F_2, \ldots, F_k$ be a finite collection of subsets of X such that F_i is closed in X for $i = 1, 2, \ldots, k$ and*

$$F_1 \cup F_2 \cup \cdots \cup F_k = X.$$

Then the function φ is continuous on X if and only if the restriction of φ to F_i is continuous on F_i for $i = 1, 2, \ldots, k$.

Proof Suppose that $\varphi: X \to \mathbb{R}^m$ is continuous. Then it follows directly from the definition of continuity that the restriction of φ to each subset of X is continuous on that subset. Therefore the restriction of φ to F_i is continuous on F_i for $i = 1, 2, \ldots, k$.

Conversely we must prove that if the restriction of the function φ to F_i is continuous on F_i for $i = 1, 2, \ldots, k$ then the function $\varphi: X \to \mathbb{R}^m$ is continuous. Let $\mathbf{p}$ be a point of X, and let some positive real number ε be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots \delta_k$ satisfying the following conditions:—

(i) if $\mathbf{p} \in F_i$, where $1 \leq i \leq k$, and if $\mathbf{x} \in F_i$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_i$ then $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$;

(ii) if $\mathbf{p} \notin F_i$, where $1 \leq i \leq k$, and if $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_i$ then $\mathbf{x} \notin F_i$.

Indeed the continuity of the function φ on each set F_i ensures that δ_i may be chosen to satisfy (i) for each integer i between 1 and k for which $\mathbf{p} \in F_i$. Also the requirement that F_i be closed in X ensures that $X \setminus F_i$ is open in X and therefore δ_i may be chosen to to satisfy (ii) for each integer i between 1 and k for which $\mathbf{p} \notin F_i$.

Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. Let $\mathbf{x} \in X$ satisfy $|\mathbf{x} - \mathbf{p}| < \delta$. If $\mathbf{p} \notin F_i$ then the choice of δ_i ensures that if $\mathbf{x} \notin F_i$. But X is the union of the sets $F_1, F_2, \ldots, F_k$, and therefore there must exist some integer i between 1 and k for which $\mathbf{x} \in F_i$. Then $\mathbf{p} \in F_i$, and the choice of δ_i ensures that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$. We have thus shown that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$ for all points $\mathbf{x}$ of X that satisfy $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $\varphi: X \to \mathbb{R}^m$ is continuous, as required. ∎

2.8 The Multidimensional Extreme Value Theorem

Theorem 2.20 (The Multidimensional Extreme Value Theorem) *Let X be a closed bounded set in n-dimensional Euclidean space, and let $f\colon X \to \mathbb{R}$ be a continuous real-valued function defined on X. Then there exist points $\mathbf{u}$ and $\mathbf{v}$ of X such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.*

Proof We prove the result for an arbitrary continuous real-valued function $f\colon X \to \mathbb{R}$ by showing that the result holds for a related continuous function $g\colon X \to \mathbb{R}$ that is known to be bounded above and below on X. Let $h\colon \mathbb{R} \to \mathbb{R}$ be the continuous function defined such that

$$h(t) = \frac{t}{1 + |t|}$$

for all $t \in \mathbb{R}$. Then the continuous function $h\colon \mathbb{R} \to \mathbb{R}$ is increasing. Moreover $-1 < h(t) < 1$ for all $t \in \mathbb{R}$.

Let $f\colon X \to \mathbb{R}$ be a continuous real-valued function on the closed bounded set X, and let $g\colon X \to \mathbb{R}$ be the continuous real-valued function defined on X such that

$$g(\mathbf{x}) = h(f(\mathbf{x})) = \frac{f(\mathbf{x})}{1 + |f(\mathbf{x})|}$$

for all $\mathbf{x} \in X$. Then $-1 < g(\mathbf{x}) < 1$ for all $\mathbf{x} \in X$. The set of values of the function g is then non-empty and bounded above, and therefore has a least upper bound. Let

$$M = \sup\{g(\mathbf{x}) : \mathbf{x} \in X\}.$$

Then, for each positive integer j, the real number $M - j^{-1}$ is not an upper bound for the set of values of the function g, and therefore there exists some point $\mathbf{x}_j$ in the set X for which $M - j^{-1} < g(\mathbf{x}_j) \leq M$. The sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is then a bounded sequence of points in $\mathbb{R}^m$, because the set X is bounded. It follows from the multidimensional Bolzano-Weierstrass Theorem (Theorem 2.5) that this sequence has a subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ which converges to some point $\mathbf{v}$ of $\mathbb{R}^n$. Moreover this point $\mathbf{v}$ belongs to the set X because X is closed (see Lemma 2.16).

Now

$$M - \frac{1}{k_j} < g(\mathbf{x}_{k_j}) \leq M$$

for all positive integers j, and therefore $g(\mathbf{x}_{k_j}) \to M$ as $j \to +\infty$. It then follows from Lemma 2.7 that

$$g(\mathbf{v}) = g\left(\lim_{j \to +\infty} \mathbf{x}_{k_j}\right) = \lim_{j \to +\infty} g(\mathbf{x}_{k_j}) = M.$$

But $g(\mathbf{x}) \leq M$ for all $\mathbf{x} \in X$. It follows that $h(f(\mathbf{x})) = g(\mathbf{x}) \leq g(\mathbf{v}) = h(f(\mathbf{v}))$ for all $\mathbf{x} \in X$. Moreover $h\colon \mathbb{R} \to \mathbb{R}$ is an increasing function. It follows therefore that $f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.

On applying this result with the continuous function f replaced by the function $-f$, we conclude also that there exists some point $\mathbf{u}$ of X such that $f(\mathbf{u}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in X$. The result follows. ∎

Proposition 2.21 *Let X be a closed bounded set in $\mathbb{R}^n$ and let $\varphi\colon X \to \mathbb{R}^m$ be a continuous function. Then $\varphi(X)$ is a closed bounded set in $\mathbb{R}^m$.*

Proof Let $g\colon X \to \mathbb{R}$ be the real-valued function on X defined such that $g(\mathbf{x}) = |\varphi(\mathbf{x})|$ for all $\mathbf{x} \in X$. Then the function $g\colon X \to \mathbb{R}$ is continuous (because it is a composition of continuous functions). It follows from the Extreme Value Theorem (Theorem 2.20) that there exists some point $\mathbf{v}$ of X such that $g(\mathbf{x}) \leq g(\mathbf{v})$ for all $\mathbf{x} \in X$. Let $M = g(\mathbf{v})$. Then $|\varphi(\mathbf{x})| \leq M$ for all $\mathbf{x} \in X$. Thus the set $\varphi(X)$ is bounded.

Let $\mathbf{q}$ be a point of $\mathbb{R}^m \setminus \varphi(X)$ and let $h\colon X \to \mathbb{R}^m$ be the real-valued function on X defined such that $h(\mathbf{x}) = |\varphi(\mathbf{x}) - \mathbf{q}|$ for all $\mathbf{x} \in X$. Then the function $h\colon X \to \mathbb{R}$ is continuous. It follows from the Extreme Value Theorem (Theorem 2.20) that there exists some point $\mathbf{u}$ of X such that $h(\mathbf{x}) \geq h(\mathbf{u})$ for all $\mathbf{x} \in X$. Let $\delta = h(\mathbf{u})$. Then $|\varphi(\mathbf{x}) - \mathbf{q}| \geq \delta$ for all $\mathbf{x} \in X$, and thus the open ball in $\mathbb{R}^m$ of radius δ about the point $\mathbf{q}$ is contained in the complement $\mathbb{R}^m \setminus \varphi(X)$ of $\varphi(X)$. It follows that $\mathbb{R}^m \setminus \varphi(X)$ is open in $\mathbb{R}^m$, and thus the set $\varphi(X)$ is closed in $\mathbb{R}^m$. Thus $\varphi(X)$ is both closed and bounded, as required. ∎

2.9 Uniform Continuity for Functions of Several Real Variables

Definition Let X be a subset of $\mathbb{R}^n$. A function $\varphi\colon X \to \mathbb{R}^m$ from X to $\mathbb{R}^n$ is said to be *uniformly continuous* if, given any $\varepsilon > 0$, there exists some $\delta > 0$ (which does not depend on either $\mathbf{x}'$ or $\mathbf{x}$) such that $|\varphi(\mathbf{x}') - \varphi(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}'$ and $\mathbf{x}$ of X satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$.

Theorem 2.22 *Let X be a subset of $\mathbb{R}^n$ that is both closed and bounded. Then any continuous function $\varphi\colon X \to \mathbb{R}^m$ is uniformly continuous.*

Proof Let $\varepsilon > 0$ be given. Suppose that there did not exist any $\delta > 0$ such that $|\varphi(\mathbf{x}') - \varphi(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}', \mathbf{x} \in X$ satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$. Then, for each positive integer j, there would exist points $\mathbf{u}_j$ and $\mathbf{v}_j$ in X such that $|\mathbf{u}_j - \mathbf{v}_j| < 1/j$ and $|\varphi(\mathbf{u}_j) - \varphi(\mathbf{v}_j)| \geq \varepsilon$. But the sequence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$

would be bounded, since X is bounded, and thus would possess a subsequence $\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \mathbf{u}_{j_3}, \ldots$ converging to some point $\mathbf{p}$ (Theorem 2.5). Moreover $\mathbf{p} \in X$, since X is closed.

The sequence $\mathbf{v}_{j_1}, \mathbf{v}_{j_2}, \mathbf{v}_{j_3}, \ldots$ would also converge to $\mathbf{p}$, since

$$\lim_{k\to+\infty} |\mathbf{v}_{j_k} - \mathbf{u}_{j_k}| = 0.$$

But then the sequences

$$\varphi(\mathbf{u}_{j_1}), \varphi(\mathbf{u}_{j_2}), \varphi(\mathbf{u}_{j_3}), \ldots \quad \text{and} \quad \varphi(\mathbf{v}_{j_1}), \varphi(\mathbf{v}_{j_2}), \varphi(\mathbf{v}_{j_3}), \ldots$$

would converge to $\varphi(\mathbf{p})$, since f is continuous (Lemma 2.7), and thus

$$\lim_{k\to+\infty} |\varphi(\mathbf{u}_{j_k}) - \varphi(\mathbf{v}_{j_k})| = 0.$$

But this is impossible, since $\mathbf{u}_j$ and $\mathbf{v}_j$ have been chosen so that $|\varphi(\mathbf{u}_j) - \varphi(\mathbf{v}_j)| \geq \varepsilon$ for all j. We conclude therefore that there must exist some $\delta > 0$ such that $|\varphi(\mathbf{x}') - \varphi(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}', \mathbf{x} \in X$ satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$, as required. ∎

2.10 Homeomorphisms between Subsets of Euclidean Spaces

Lemma 2.23 *Let X be a closed subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then a subset of X is closed in X if and only if it is closed in $\mathbb{R}^n$.*

Proof Let F be a subset of X. Then F is closed in X if and only if, given any point $\mathbf{p}$ of X for which $\mathbf{p} \notin F$, there exists some strictly positive real number δ such that there is no point of F whose distance from the point $\mathbf{p}$ is less than δ. It follows easily from this that is F is closed in $\mathbb{R}^n$ then F is closed in X.

Conversely suppose that F is closed in X, where X itself is closed in $\mathbb{R}^n$. Let $\mathbf{p}$ be a point of $\mathbb{R}^n$ that satisfies $\mathbf{p} \notin F$. Then either $\mathbf{p} \in X$ or $\mathbf{p} \notin X$.

Suppose that $\mathbf{p} \in X$. Then there exists some strictly positive real number δ such that there is no point of F whose distance from the point $\mathbf{p}$ is less than δ.

Otherwise $\mathbf{p} \notin X$. Then there exists some strictly positive real number δ such that there is no point of X whose distance from the point $\mathbf{p}$ is less than δ, because X is closed in $\mathbb{R}^n$. But $F \subset X$. It follows that there is no point of F whose distance from the point $\mathbf{p}$ is less than δ. We conclude that the set F is closed in $\mathbb{R}^n$, as required. ∎

Let X and Y be subsets of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi: X \to Y$ be a function from X to Y. A function $\psi: Y \to X$ is the inverse of $\varphi: X \to Y$ if and only if $\psi(\varphi(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in X$ and $\varphi(\psi(\mathbf{y})) = \mathbf{y}$ for all $\mathbf{y} \in Y$. The function X is *bijective* if and only if it has a well-defined inverse $\psi: Y \to X$.

Definition Let X and Y be subsets of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A function $\varphi: X \to Y$ is said to be a *homeomorphism* if and only if it is bijective and both $\varphi: X \to Y$ itself and its inverse are continuous functions.

Proposition 2.24 *Let X and Y be subsets of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi: X \to Y$ be a continuous bijective function from X to Y. Suppose that X is closed and bounded. Then $\varphi: X \to Y$ is a homeomorphism.*

Proof Let $\varphi: X \to Y$ be a continuous bijective function from X to Y, and let $\psi: Y \to X$ be the inverse of φ. Then $\varphi: X \to Y$ establishes a one-to-one correspondence between points of X and points of Y: given any point $\mathbf{x}$ of X, the point $\varphi(\mathbf{x})$ is the unique point of Y that corresponds to $\mathbf{x}$; given any point $\mathbf{y}$ of Y, the point $\psi(\mathbf{y})$ is the unique point of X that corresponds to $\mathbf{y}$.

In order to prove that the continuous bijective function $\varphi: X \to Y$ is a homeomorphism, we need to prove that its inverse $\psi: Y \to X$ is continuous. Let W be an open set in X. We must prove that its preimage $\psi^{-1}(W)$ is open in W. Let $F = X \setminus W$. Then F is closed in X, and X itself is closed in $\mathbb{R}^n$. It follows that F is closed in $\mathbb{R}^n$ (see Lemma 2.23). Also F is bounded, because X is bounded.

Now continuous functions between subsets of Euclidean spaces map closed bounded sets to closed bounded sets (see Proposition 2.21). It follows that $\varphi(F)$ is a closed subset of $\mathbb{R}^m$ and is thus closed in Y, and therefore its complement $Y \setminus \varphi(F)$ is open in Y.

But $Y \setminus \varphi(F) = \psi^{-1}(V)$. Indeed let $\mathbf{y} \in Y$. Then

$$\begin{aligned} & \mathbf{y} \in Y \setminus \varphi(F) \\ \iff & \mathbf{y} \notin \varphi(F) \\ \iff & \psi(\mathbf{y}) \notin F \\ \iff & \psi(\mathbf{y}) \in V \\ \iff & \mathbf{y} \in \psi^{-1}(V). \end{aligned}$$

It follows that $\psi^{-1}(V)$ is open in Y. We have shown that the preimage under ψ of every subset of X open in X is open in Y. It follows that

$\psi: Y \to X$ is continuous (see Proposition 2.17). We conclude that $\varphi: X \to Y$ is a homeomorphism, as required. ∎

Example A regular dodecahedron is a regular convex polyhedron in 3-dimensional Euclidean space with twelve faces that are regular pentagons. Let the closed bounded subset X of $\mathbb{R}^3$ be the surface of a regular dodecahedron centred on the origin. Then every straight ray with an endpoint at the origin will cut X in exactly one point. Let S^2 be the unit sphere in $\mathbb{R}^3$, so that

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\},$$

and let $\varphi: X \to S^2$ be defined so that

$$\varphi(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|}$$

for all $\mathbf{x} \in X$. Then $\varphi: X \to S^2$ is a continuous bijective function from X to S^2. Let $\psi S^2 \to X$ be the inverse of φ. It follows from Proposition 2.24 That $\varphi: X \to S^2$ is a homeomorphism, and therefore the map $\psi: S^2 \to X$ is continuous.

It might be instructive to ponder how one might set about constructing a proof that $\psi: S^2 \to X$ is continuous, using directly the "ε–δ" definition of continuity. Presumably one would first have to come up with algebraic expressions that specify what the map is, and presumably there would need to be twelve such algebraic expressions, each specifying map ψ over some portion of the unit sphere that gets mapped onto a pentagonal face of the dodecahedron. And moreover the regions over which these algebraic expressions apply would probably need to be specified using appropriate inequalities satisfied by appropriate angles that arise from some curvilinear coordinate system on the sphere.

3 Open Covers, Lebesgue Numbers and Compactness

3.1 Lebesgue Numbers

Definition Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. A collection of subsets of $\mathbb{R}^n$ is said to *cover* X if and only if every point of X belongs to at least one of these subsets.

Definition Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. An *open cover* of X is a collection of subsets of X that are open in X and cover the set X.

Proposition 3.1 *Let X be a closed bounded set in n-dimensional Euclidean space, and let $\mathcal{V}$ be an open cover of X. Then there exists a positive real number δ_L with the property that, given any point $\mathbf{u}$ of X, there exists a member V of the open cover $\mathcal{V}$ for which*

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta_L\} \subset V.$$

Proof Let

$$B_X(\mathbf{u}, \delta) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\}$$

for all $\mathbf{u} \in X$ and for all positive real numbers δ. Suppose that there did not exist any positive real number δ_L with the stated property.

Then, given any positive number δ, there would exist a point $\mathbf{u}$ of X for which the ball $B_X(\mathbf{u}, \delta)$ would not be wholly contained within any open set V belonging to the open cover $\mathcal{V}$. Then

$$B_X(\mathbf{u}, \delta) \cap (X \setminus V) \neq \emptyset$$

for all members V of the open cover $\mathcal{V}$. There would therefore exist an infinite sequence

$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$$

of points of X with the property that, for all positive integers j, the open ball

$$B_X(\mathbf{u}_j, 1/j) \cap (X \setminus V) \neq \emptyset$$

for all members V of the open cover $\mathcal{V}$.

The sequence

$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$$

would be bounded, because the set X is bounded. It would then follow from the multidimensional Bolzano-Weierstrass Theorem (Theorem 2.5) that there would exist a convergent subsequence

$$\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \mathbf{u}_{j_3}, \ldots$$

of

$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots.$$

Let $\mathbf{p}$ be the limit of this convergent subsequence. Then the point $\mathbf{p}$ would then belong to X, because X is closed (see Lemma 2.16). But then the point $\mathbf{p}$ would belong to an open set V belonging to the open cover $\mathcal{V}$. It would then follow from the definition of open sets that there would exist a positive real number δ for which $B_X(\mathbf{p}, 2\delta) \subset V$. Let $j = j_k$ for a positive integer k large enough to ensure that both $1/j < \delta$ and $\mathbf{u}_j \in B_X(\mathbf{p}, \delta)$. The Triangle Inequality would then ensure that every point of X within a distance $1/j$ of the point $\mathbf{u}_j$ would lie within a distance 2δ of the point $\mathbf{p}$, and therefore

$$B_X(\mathbf{u}_j, 1/j) \subset B_X(\mathbf{p}, 2\delta) \subset V.$$

But $B(\mathbf{u}_j, 1/j) \cap (X \setminus V) \neq \emptyset$ for all members V of the open cover $\mathcal{V}$, and therefore it would not be possible for this open set to be contained in the open set V. Thus the assumption that there is no positive number δ_L with the required property has led to a contradiction. Therefore there must exist some positive number δ_L with the property that, for all $\mathbf{u} \in X$ the open ball $B_X(\mathbf{u}, \delta_L)$ in X is contained wholly within at least one open set belonging to the open cover $\mathcal{V}$, as required. ∎

Definition Let X be a subset of n-dimensional Euclidean space, and let $\mathcal{V}$ be an open cover of X. A positive real number δ_L is said to be a *Lebesgue number* for the open cover $\mathcal{V}$ if, given any point $\mathbf{p}$ of X, there exists some member V of the open cover $\mathcal{V}$ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta_L\} \subset V.$$

Proposition 3.1 ensures that, given any open cover of a closed bounded subset of n-dimensional Euclidean space, there exists a positive real number that is a Lebesgue number for that open cover.

Definition The *diameter* $\operatorname{diam}(A)$ of a bounded subset A of n-dimensional Euclidean space is defined so that

$$\operatorname{diam}(A) = \sup\{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in A\}.$$

It follows from this definition that $\text{diam}(A)$ is the smallest real number K with the property that $|\mathbf{x} - \mathbf{y}| \leq K$ for all $\mathbf{x}, \mathbf{y} \in A$.

A *hypercube* in n-dimensional Euclidean space $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ of the form

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : u_i \leq x_i \leq u_i + l\},$$

where l is a positive constant that is the length of the edges of the hypercube and $(u_1, u_2, \ldots, u_n)$ is the point in $\mathbb{R}^n$ at which the Cartesian coordinates of points in the hypercube attain their minimum values. The diameter of a hypercube with edges of length l is $l\sqrt{n}$.

Lemma 3.2 *Let X be a bounded subset of n-dimensional Euclidean space, and let δ be a positive real number. Then there exists a finite collection $A_1, A_2, \ldots, A_s$ of subsets of X such that the* $\text{diam}(A_i) < \delta$ *for $i = 1, 2, \ldots, s$ and*

$$X = A_1 \cup A_2 \cup \cdots \cup A_k.$$

Proof The set X is bounded, and therefore there exists some positive real number M such that that if $(x_1, x_2, \ldots, x_n) \in X$ then $-M \leq x_j \leq M$ for $j = 1, 2, \ldots, n$. Choose k large enough to ensure that $2M/k < \delta_L/\sqrt{n}$. Then the large hypercube

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -M \leq x_j \leq M \text{ for } j = 1, 2, \ldots, n\}$$

can be subdivided into k^n hypercubes with edges of length l, where $l = 2M/k$.

Each of the smaller hypercubes is a set of the form

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : u_j \leq x_j \leq u_j + l \text{ for } j = 1, 2, \ldots, n\},$$

where $(u_1, u_2, \ldots, u_n)$ is the corner of the hypercube at which the Cartesian coordinates have their minimum values. If $\mathbf{p}$ is a point belonging to such a small hypercube, then all points of the hypercube lie within a distance $l\sqrt{n}$ of the point $\mathbf{p}$. It follows that the small hypercube is wholly contained within the open ball $B_{\mathbb{R}^n}(\mathbf{p}, \delta_L)$ of radius δ about the point $\mathbf{p}$.

Now the number of small hypercubes resulting from the subdivision is finite. Let $H_1, H_2, \ldots, H_s$ be a listing of the small hypercubes that intersect the set X, and let $A_i = H_i \cap X$. Then $\text{diam}(H_i) \leq \sqrt{n}l < \delta_L$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_k,$$

as required. ∎

Definition Let $\mathcal{V}$ and $\mathcal{W}$ be open covers of some subset X of a Euclidean space. Then $\mathcal{W}$ is said to be a *subcover* of $\mathcal{V}$ if and only if every open set belonging to $\mathcal{W}$ also belongs to $\mathcal{V}$.

Definition A subset X of a Euclidean space is said to be *compact* if and only if every open cover of X possesses a finite subcover.

Theorem 3.3 (The Multidimensional Heine-Borel Theorem) *A subset of n-dimensional Euclidean space $\mathbb{R}^n$ is compact if and only if it is both closed and bounded.*

Proof Let X be a compact subset of $\mathbb{R}^n$ and let

$$V_j = \{\mathbf{x} \in X : |\mathbf{x}| < j\}$$

for all positive integers j. Then the sets $V_1, V_2, V_3, \ldots$ constitute an open cover of X. This open cover has a finite subcover, and therefore there exist positive integers $j_1, j_2, \ldots, j_k$ such that

$$X \subset V_{j_1} \cup V_{j_2} \cup \cdots \cup V_{j_k}.$$

Let M be the largest of the positive integers $j_1, j_2, \ldots, j_k$. Then $|\mathbf{x}| \leq M$ for all $\mathbf{x} \in X$. Thus the set X is bounded.

Let $\mathbf{q}$ be a point of $\mathbb{R}^n$ that does not belong to X, and let

$$W_j = \left\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{q}| > \frac{1}{j}\right\}$$

for all positive integers j. Then the sets $W_1, W_2, W_3, \ldots$ constitute an open cover of X. This open cover has a finite subcover, and therefore there exist positive integers $j_1, j_2, \ldots, j_k$ such that

$$X \subset W_{j_1} \cup W_{j_2} \cup \cdots \cup W_{j_k}.$$

Let $\delta = 1/M$, where M is the largest of the positive integers $j_1, j_2, \ldots, j_k$. Then $|\mathbf{x} - \mathbf{q}| \geq \delta$ for all $\mathbf{x} \in X$ and thus the open ball of radius δ about the point $\mathbf{q}$ does not intersect the set X. We conclude that the set X is closed. We have now shown that every compact subset of $\mathbb{R}^n$ is both closed and bounded.

We now prove the converse. Let X be a closed bounded subset of $\mathbb{R}^n$, and let $\mathcal{V}$ be an open cover of X. It follows from Proposition 3.1 that there exists a Lebesgue number δ_L for the open cover $\mathcal{V}$. It then follows from Lemma 3.2 that there exist subsets $A_1, A_2, \ldots, A_s$ of X such that $\operatorname{diam}(A_i) < \delta_L$ for $i = 1, 2, \ldots, s$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_s.$$

We may suppose that A_i is non-empty for $i = 1, 2, \ldots, s$ (because if $A_i = \emptyset$ then A_i could be deleted from the list). Choose $\mathbf{p}_i \in A_i$ for $i = 1, 2, \ldots, s$.

Then $A_i \subset B_X(\mathbf{p}_i, \delta_L)$ for $i = 1, 2, \ldots, s$. The definition of the Lebesgue number δ_L then ensures that there exist members $V_1, V_2, \ldots, V_s$ of the open cover $\mathcal{V}$ such that $B_X(\mathbf{p}_i, \delta_L) \subset V_i$ for $i = 1, 2, \ldots, s$. Then $A_i \subset V_i$ for $i = 1, 2, \ldots, s$, and therefore

$$X \subset V_1 \cup V_2 \cup \cdots \cup V_s.$$

Thus $V_1, V_2, \ldots, V_s$ constitute a finite subcover of the open cover $\mathcal{U}$. We have therefore proved that every closed bounded subset of n-dimensional Euclidean space is compact, as required. ∎

4 Correspondences and Hemicontinuity

4.1 Correspondences

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A *correspondence* $\Phi\colon X \rightrightarrows Y$ assigns to each point $\mathbf{x}$ of X a subset $\Phi(\mathbf{x})$ of Y.

The *power set* $\mathcal{P}(Y)$ of Y is the set whose elements are the subsets of Y. A correspondence $\Phi\colon X \rightrightarrows Y$ may be regarded as a function from X to $\mathcal{P}(Y)$.

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Then the following definitions apply:—

- the correspondence $\Phi\colon X \to Y$ is said to be *non-empty-valued* if $\Phi(\mathbf{x})$ is a non-empty subset of Y for all $\mathbf{x} \in X$;
- the correspondence $\Phi\colon X \to Y$ is said to be *closed-valued* if $\Phi(\mathbf{x})$ is a closed subset of Y for all $\mathbf{x} \in X$;
- the correspondence $\Phi\colon X \to Y$ is said to be *compact-valued* if $\Phi(\mathbf{x})$ is a compact subset of Y for all $\mathbf{x} \in X$.

It follows from the multidimensional Heine-Borel Theorem (Theorem 3.3) that the correspondence $\Phi\colon X \to Y$ is compact-valued if and only if $\Phi(\mathbf{x})$ is a closed bounded subset of Y for all $\mathbf{x} \in X$.

Definition Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is said to be *upper hemicontinuous* at a point $\mathbf{p}$ of X if, given any set V in Y that is open in Y and satisfies $\Phi(\mathbf{p}) \subset V$, there exists some positive real number δ such that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. The correspondence Φ is upper hemicontinuous on X if it is upper hemicontinuous at each point of X.

Example Let $F\colon \mathbb{R} \rightrightarrows \mathbb{R}$ and $G\colon \mathbb{R} \rightrightarrows \mathbb{R}$ be the correspondences from $\mathbb{R}$ to $\mathbb{R}$ defined such that

$$F(x) = \begin{cases} [1,2] & \text{if } x < 0, \\ [0,3] & \text{if } x \geq 0, \end{cases}$$

and

$$G(x) = \begin{cases} [1,2] & \text{if } x \leq 0, \\ [0,3] & \text{if } x > 0, \end{cases}$$

The correspondences F and G are upper hemicontinuous at x for all non-zero real numbers x. The correspondence F is also upper hemicontinuous at 0, for if V is an open set in $\mathbb{R}$ and if $F(0) \subset V$ then $[0,3] \subset V$ and therefore $F(x) \in V$ for all real numbers x.

However the correspondence G is not upper hemicontinuous at 0. Indeed let

$$V = \{y \in \mathbb{R} : \tfrac{1}{2} < y < \tfrac{5}{2}\}.$$

Then $G(0) \subset V$, but $G(x)$ is not contained in V for any positive real number x. Therefore there cannot exist any positive real number δ such that $G(x) \subset V$ whenever $|x| < \delta$.

Let

$$\operatorname{Graph}(F) = \{(x,y) \in \mathbb{R}^2 : y \in F(x)\}$$

and

$$\operatorname{Graph}(G) = \{(x,y) \in \mathbb{R}^2 : y \in G(x)\}.$$

Then $\operatorname{Graph}(F)$ is a closed subset of $\mathbb{R}^2$ but $\operatorname{Graph}(G)$ is not a closed subset of $\mathbb{R}^2$.

Example Let S^1 be the unit circle in $\mathbb{R}^2$, defined such that

$$S^1 = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 = 1\},$$

let Z be the closed square with corners at $(1,1)$, $(-1,1)$, $(-1,-1)$ and $(1,-1)$, so that

$$Z = \{(x,y) \in \mathbb{R}^2 : -1 \le x \le 1 \text{ and } -1 \le y \le 1\}.$$

Let $g_{(u,v)}\colon \mathbb{R}^2 \to \mathbb{R}$ be defined for all $(u,v) \in S^1$ such that

$$g_{(u,v)}(x,y) = ux + vy,$$

and let $\Phi\colon S^1 \rightrightarrows \mathbb{R}^2$ be defined such that, for all $(u,v) \in S^1$, $\Phi(u,v)$ is the subset of $\mathbb{R}^2$ consisting of the point of points of Z at which the linear functional $g_{(u,v)}$ attains its maximum value on Z.

Thus a point (x,y) of Z belongs to $\Phi(u,v)$ if and only if $g_{(u,v)}(x,y) \ge g_{(u,v)}(x',y')$ for all $(x',y') \in Z$. Then

$$\Phi(u,v) = \begin{cases} \{(1,1)\} & \text{if } u > 0 \text{ and } v > 0; \\ \{(x,1) : -1 \le x \le 1\} & \text{if } u = 0 \text{ and } v > 0; \\ \{(-1,1)\} & \text{if } u < 0 \text{ and } v > 0; \\ \{(-1,y) : -1 \le y \le 1\} & \text{if } u < 0 \text{ and } v = 0; \\ \{(-1,-1)\} & \text{if } u < 0 \text{ and } v < 0; \\ \{(x,-1) : -1 \le x \le 1\} & \text{if } u = 0 \text{ and } v < 0; \\ \{(1,-1)\} & \text{if } u > 0 \text{ and } v < 0; \\ \{(1,y) : -1 \le y \le 1\} & \text{if } u > 0 \text{ and } v = 0. \end{cases}$$

It is a straightforward exercise to verify that the correspondence $\Phi\colon S^1 \rightrightarrows \mathbb{R}^2$ is upper hemicontinuous.

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi: X \rightrightarrows Y$ be a correspondence between X and Y. Given any subset V of Y, we denote by $\Phi^+(V)$ the subset of X defined such that

$$\Phi^+(V) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V\}.$$

Lemma 4.1 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi: X \rightrightarrows Y$ is upper hemicontinuous on X if and only if, given any set V in Y that is open in Y, the set $\Phi^+(V)$ is open in X.*

Proof First suppose that $\Phi: X \rightrightarrows Y$ is upper hemicontinuous at each point of X. Let V be an open set in Y and let $\mathbf{p} \in \Phi^+(V)$. Then $\Phi(\mathbf{p}) \subset V$. It then follows from the definition of upper hemicontinuity that there exists some positive real number δ such that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then $\mathbf{x} \in \Phi^+(V)$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $\Phi^+(V)$ is open in X.

Conversely suppose that $\Phi: X \rightrightarrows Y$ is a correspondence with the property that, for all subsets V of Y that are open in Y, $\Phi^+(V)$ is open in X. Let $\mathbf{p} \in X$, and let V be an open set in Y satisfying $\Phi(\mathbf{p}) \subset V$. Then $\Phi^+(V)$ is open in X and $\mathbf{p} \in \Phi^+(V)$, and therefore there exists some positive number δ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} \subset \Phi^+(V).$$

Then $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus $\Phi: X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$. The result follows. ∎

Definition Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi: X \rightrightarrows Y$ is said to be *lower hemicontinuous* at a point $\mathbf{p}$ of X if, given any set V in Y that is open in Y and satisfies $\Phi(\mathbf{p}) \cap V \neq \emptyset$, there exists some positive real number δ such that $\Phi(\mathbf{x}) \cap V \neq \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. The correspondence Φ is lower hemicontinuous on X if it is lower hemicontinuous at each point of X.

Example Let $F: \mathbb{R} \rightrightarrows \mathbb{R}$ and $G: \mathbb{R} \rightrightarrows \mathbb{R}$ be the correspondences from $\mathbb{R}$ to $\mathbb{R}$ defined such that

$$F(x) = \begin{cases} [1,2] & \text{if } x < 0, \\ [0,3] & \text{if } x \geq 0, \end{cases}$$

and

$$G(x) = \begin{cases} [1,2] & \text{if } x \leq 0, \\ [0,3] & \text{if } x > 0, \end{cases}$$

The correspondences F and G are lower hemicontinuous at x for all non-zero real numbers x. The correspondence G is also lower hemicontinuous at 0, for if V is an open set in $\mathbb{R}$ and if $G(0) \cap V \neq \emptyset$ then $[0, 1] \cap V \neq \emptyset$ and therefore $G(x) \cap V \neq \emptyset$ for all real numbers x.

However the correspondence F is not lower hemicontinuous at 0. Indeed let

$$V = \{y \in \mathbb{R} : 0 < y < \tfrac{1}{2}\}.$$

Then $F(0) \cap V \neq \emptyset$, but $F(x) \cap V = \emptyset$ for all negative real numbers x. Therefore there cannot exist any positive real number δ such that $F(x) \cap V = \emptyset$ whenever $|x| < \delta$.

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi: X \rightrightarrows Y$ be a correspondence between X and Y. Given any subset V of Y, we denote by $\Phi^-(V)$ the subset of X defined such that

$$\Phi^-(V) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \cap V \neq \emptyset\}.$$

Lemma 4.2 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi: X \rightrightarrows Y$ is lower hemicontinuous on X if and only if, given any set V in Y that is open in Y, the set $\Phi^-(V)$ is open in X.*

Proof First suppose that $\Phi: X \rightrightarrows Y$ is lower hemicontinuous at each point of X. Let V be an open set in Y and let $\mathbf{p} \in \Phi^-(V)$. Then $\Phi(\mathbf{p}) \cap V$ is non-empty. It then follows from the definition of lower hemicontinuity that there exists some positive real number δ such that $\Phi(\mathbf{x}) \cap V$ is non-empty for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then $\mathbf{x} \in \Phi^-(V)$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $\Phi^-(V)$ is open in X.

Conversely suppose that $\Phi: X \rightrightarrows Y$ is a correspondence with the property that, for all subsets V of Y that are open in Y, $\Phi^-(V)$ is open in X. Let $\mathbf{p} \in X$, and let V be an open set in Y satisfying $\Phi(\mathbf{p}) \cap V \neq \emptyset$. Then $\Phi^-(V)$ is open in X and $\mathbf{p} \in \Phi^-(V)$, and therefore there exists some positive number δ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} \subset \Phi^-(V).$$

Then $\Phi(\mathbf{x}) \cap V \neq \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus $\Phi: X \rightrightarrows Y$ is lower hemicontinuous at $\mathbf{p}$. The result follows. ∎

Definition Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi: X \rightrightarrows Y$ is said to be *continuous* at a point $\mathbf{p}$ of X if it is both upper hemicontinuous and lower hemicontinuous at $\mathbf{p}$. The correspondence Φ is continuous on X if it is continuous at each point of X.

Lemma 4.3 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $\varphi: X \to Y$ be a function from X to Y, and let $\Phi: X \rightrightarrows Y$ be the correspondence defined such that $\Phi(\mathbf{x}) = \{\varphi(\mathbf{x})\}$ for all $\mathbf{x} \in X$. Then $\Phi: X \rightrightarrows Y$ is upper hemicontinuous if and only if $\varphi: X \to Y$ is continuous. Similarly $\Phi: X \rightrightarrows Y$ is lower hemicontinuous if and only if $\varphi: X \to Y$ is continuous.*

Proof The function $\varphi: X \to Y$ is continuous if and only if

$$\{\mathbf{x} \in X : \varphi(\mathbf{x}) \in V\}$$

is open in X for all subsets V of Y that are open in Y (see Proposition 2.17). Let V be a subset of Y that is open in Y. Then $\Phi(\mathbf{x}) \subset V$ if and only if $\varphi(\mathbf{x}) \in V$. Also $\Phi(\mathbf{x}) \cap V \neq \emptyset$ if and only if $\varphi(\mathbf{x}) \in V$. The result therefore follows from the definitions of upper and lower hemicontinuity. ∎

4.2 The Graph of a Correspondence

Let m and n be integers. Then the Cartesian product $\mathbb{R}^n \times \mathbb{R}^m$ of the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ of dimensions n and m is itself a Euclidean space of dimension $n + m$ whose Euclidean norm is characterized by the property that

$$|(\mathbf{x}, \mathbf{y})|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2$$

for all $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$.

Lemma 4.4 *Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ and $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ be infinite sequences of points in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\mathbf{p} \in \mathbb{R}^n$ and $\mathbf{q} \in \mathbb{R}^m$. Then the infinite sequence*

$$(\mathbf{x}_1, \mathbf{y}_1),\ (\mathbf{x}_2, \mathbf{y}_2),\ (\mathbf{x}_3, \mathbf{y}_3), \ldots$$

converges in $\mathbb{R}^n \times \mathbb{R}^m$ to the point $(\mathbf{p}, \mathbf{q})$ if and only if the infinite sequence Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to the point $\mathbf{p}$ and the infinite sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ converges to the point $\mathbf{q}$.

Proof Suppose that the infinite sequence

$$(\mathbf{x}_1, \mathbf{y}_1),\ (\mathbf{x}_2, \mathbf{y}_2),\ (\mathbf{x}_3, \mathbf{y}_3), \ldots$$

converges in $\mathbb{R}^n \times \mathbb{R}^m$ to the point $(\mathbf{p}, \mathbf{q})$. Let some strictly positive real number ε be given. Then there exists some positive integer N such that

$$|\mathbf{x}_j - \mathbf{p}|^2 + |\mathbf{y}_j - \mathbf{q}|^2 < \varepsilon^2$$

whenever $j \geq N$. But then

$$|\mathbf{x}_j - \mathbf{p}| < \varepsilon \quad \text{and} \quad |\mathbf{y}_j - \mathbf{q}| < \varepsilon$$

whenever $j \geq N$. It follows that $\mathbf{x}_j \to \mathbf{p}$ and $\mathbf{y}_j \to \mathbf{q}$ as $j \to +\infty$.

Conversely suppose that $\mathbf{x}_j \to \mathbf{p}$ and $\mathbf{y}_j \to \mathbf{q}$ as $j \to +\infty$. Let some positive real number ε be given. Then there exist positive integers N_1 and N_2 such that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon/\sqrt{2}$ whenever $j \geq N_1$ and $|\mathbf{y}_j - \mathbf{q}| < \varepsilon/\sqrt{2}$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. Then

$$|\mathbf{x}_j - \mathbf{p}|^2 + |\mathbf{y}_j - \mathbf{q}|^2 < \varepsilon^2$$

whenever $j \geq N$. It follows that $(\mathbf{x}_j, \mathbf{y}_j) \to (\mathbf{p}, \mathbf{q})$ as $j \to +\infty$, as required. ∎

Lemma 4.5 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let V be a subset of $X \times Y$. Then V is open in $X \times Y$ if and only if, given any point $(\mathbf{p}, \mathbf{q})$ of V, where $\mathbf{p} \in X$ and $\mathbf{q} \in Y$, there exist subsets W_X and W_Y of X and Y respectively such that $\mathbf{p} \in W_X$, $\mathbf{q} \in W_Y$, W_X is open in X, W_Y is open in Y and $W_X \times W_Y \subset V$.*

Proof Let V be a subset of $X \times Y$ and let $(\mathbf{p}, \mathbf{q}) \in V$, where $\mathbf{p} \in X$ and $\mathbf{q} \in Y$.

Suppose that V is open in $X \times Y$. Then there exists a positive real number δ such that $(\mathbf{x}, \mathbf{y}) \in V$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying

$$|\mathbf{x} - \mathbf{p}|^2 + |\mathbf{y} - \mathbf{q}|^2 < \delta^2.$$

Let

$$W_X = \left\{ \mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \frac{\delta}{\sqrt{2}} \right\}$$

and

$$W_Y = \left\{ \mathbf{y} \in Y : |\mathbf{y} - \mathbf{q}| < \frac{\delta}{\sqrt{2}} \right\}$$

If $\mathbf{x} \in W_X$ and $\mathbf{y} \in W_Y$ then

$$|\mathbf{x} - \mathbf{p}|^2 + |\mathbf{y} - \mathbf{q}|^2 < 2\left(\frac{\delta}{\sqrt{2}}\right)^2 = \delta^2$$

and therefore $(\mathbf{x}, \mathbf{y}) \in V$. It follows that $W_X \times W_Y \subset V$.

Conversely suppose that there exist open sets W_X and W_Y in X and Y respectively such that $\mathbf{p} \in W_X$, $\mathbf{q} \in W_Y$ and $W_X \times W_Y \subset V$. Then there exists some positive real number δ such that $\mathbf{x} \in W_X$ for all $\mathbf{x} \in X$ satisfying

$|\mathbf{x}-\mathbf{p}| < \delta$ and also $\mathbf{y} \in W_Y$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y}-\mathbf{q}| < \delta$. If $(\mathbf{x}, \mathbf{y})$ is a point of $X \times Y$ that lies within a distance δ of $(\mathbf{p}, \mathbf{q})$ then $|\mathbf{x}-\mathbf{p}| < \delta$ and $|\mathbf{y}-\mathbf{q}| < \delta$, and therefore $(\mathbf{x}, \mathbf{y}) \in W_X \times W_Y$. But $W_X \times W_Y \subset V$. It follows that the open ball of radius δ about the point $(\mathbf{p}, \mathbf{q})$ is wholly contained within the subset V of $X \times Y$. The result follows. ∎

Proposition 4.6 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let G be a subset of $X \times Y$. Then G is closed in $X \times Y$ if and only if*

$$(\lim_{j\to\infty} \mathbf{x}_j,\ \lim_{j\to\infty} \mathbf{y}_j) \in G$$

for all convergent infinite sequences $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ in X and for all convergent infinite sequences $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ in Y with the property that $(\mathbf{x}_j, \mathbf{y}_j) \in G$ for all positive integers j.

Proof Suppose that G is closed in $X \times Y$. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be an infinite sequence in X converging to some point $\mathbf{p}$ of X and let $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ be an infinite sequence in Y converging to a point $\mathbf{q}$ of Y, where $(\mathbf{x}_j, \mathbf{x}_j) \in G$ for all positive integers j. We must prove that $(\mathbf{p}, \mathbf{q}) \in G$. Now the infinite sequence consisting of the ordered pairs $(\mathbf{x}_j, \mathbf{y}_j)$ converges in $X \times Y$ to $(\mathbf{p}, \mathbf{q})$ (see Lemma 4.4). Now every infinite sequence contained in G that converges to a point of $X \times Y$ must converge to a point of G, because G is closed in $X \times Y$ (see Lemma 2.16). It follows that $(\mathbf{p}, \mathbf{q}) \in G$.

Now suppose that G is not closed in $X \times Y$. Then the complement of G in $X \times Y$ is not open, and therefore there exists a point $(\mathbf{p}, \mathbf{q})$ of $X \times Y$ that does not belong to G though every open ball of positive radius about the point $(\mathbf{p}, \mathbf{q})$ intersects G. It follows that, given any positive integer j, the open ball of radius $1/j$ about the point $(\mathbf{p}, \mathbf{q})$ intersects G and therefore there exist $\mathbf{x}_j \in X$ and $\mathbf{y}_j \in Y$ for which $|\mathbf{x}_j - \mathbf{p}| < 1/j$, $|\mathbf{y}_j - \mathbf{q}| < 1/j$ and $(\mathbf{x}_j, \mathbf{y}_j) \in G$. Then $\lim_{j\to+\infty} \mathbf{x}_j = \mathbf{p}$ and $\lim_{j\to+\infty} \mathbf{y}_j = \mathbf{q}$ and therefore

$$(\lim_{j\to\infty} \mathbf{x}_j,\ \lim_{j\to\infty} \mathbf{y}_j) \notin G.$$

The result follows. ∎

Definition Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a function from X and Y. The *graph* $\mathrm{Graph}(\varphi)$ of the function φ is the subset of $\mathbb{R}^n \times \mathbb{R}^m$ defined so that

$$\mathrm{Graph}(\varphi) = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \mathbf{x} \in X \text{ and } \mathbf{y} = \varphi(\mathbf{x})\}.$$

Definition Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi: X \rightrightarrows Y$ be a correspondence between X and Y. The *graph* $\mathrm{Graph}(\Phi)$ of the correspondence Φ is the subset of $\mathbb{R}^n \times \mathbb{R}^m$ defined so that

$$\mathrm{Graph}(\Phi) = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \mathbf{x} \in X \text{ and } \mathbf{y} \in \Phi(\mathbf{x})\}.$$

Lemma 4.7 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi: X \to Y$ be a function from X to Y. Suppose that $\varphi: X \to Y$ is continuous. Then the graph* $\mathrm{Graph}(\varphi)$ *of the function φ is closed in $X \times Y$.*

Proof Let $\psi: X \times Y \to Y$ be the function defined such that

$$\psi(\mathbf{x}, \mathbf{y}) = \mathbf{y} - \varphi(\mathbf{x})$$

for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$. Then $\mathrm{Graph}(\varphi) = \psi^{-1}(\{\mathbf{0}\})$, and $\{\mathbf{0}\}$ is closed in $\mathbb{R}^m$. It follows that $\mathrm{Graph}(\varphi)$ is closed in $X \times Y$ (see Corollary 2.18). ∎

Example Let $f: \mathbb{R} \to \mathbb{R}$ be defined such that

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0; \\ 0 & \text{if } x \le 0. \end{cases}$$

Then the graph $\mathrm{Graph}(f)$ of the function f satisfies $\mathrm{Graph}(f) = Z \cup H$, where

$$Z = \{(x, y) \in \mathbb{R}^2 : x \le 0 \text{ and } y = 0\}$$

and

$$H = \{(x, y) \in \mathbb{R}^2 : x \ge 0 \text{ and } xy = 1\}.$$

Each of the sets Z and H is a closed set in $\mathbb{R}^2$. It follows that $\mathrm{Graph}(f)$ is a closed set in $\mathbb{R}^2$. However the function $f: \mathbb{R} \to \mathbb{R}$ is not continuous at 0.

Lemma 4.8 *Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$, let S be a non-empty subset of X, and let*

$$d(\mathbf{x}, S) = \inf\{|\mathbf{x} - \mathbf{s}| : \mathbf{s} \in S\}$$

for all $\mathbf{x} \in X$. Then the function sending $\mathbf{x}$ to $d(\mathbf{x}, S)$ for all $\mathbf{x} \in X$ is a continuous function on X.

Proof Let $f(\mathbf{x}) = d(\mathbf{x}, S) = \inf\{|\mathbf{x} - \mathbf{s}| : \mathbf{s} \in S\}$ for all $\mathbf{x} \in X$.

Let $\mathbf{x}$ and $\mathbf{x}'$ be points of X. It follows from the Triangle Inequality that

$$f(\mathbf{x}) \le |\mathbf{x} - \mathbf{s}| \le |\mathbf{x} - \mathbf{x}'| + |\mathbf{x}' - \mathbf{s}|$$

for all $\mathbf{s} \in S$, and therefore

$$|\mathbf{x}' - \mathbf{s}| \geq f(\mathbf{x}) - |\mathbf{x} - \mathbf{x}'|$$

for all $\mathbf{s} \in S$. Thus $f(\mathbf{x}) - |\mathbf{x} - \mathbf{x}'|$ is a lower bound for the quantities $|\mathbf{x}' - \mathbf{s}|$ as $\mathbf{s}$ ranges over the set S, and therefore cannot exceed the greatest lower bound of these quantities.

It follows that

$$f(\mathbf{x}') = \inf\{|\mathbf{x}' - \mathbf{s}| : \mathbf{s} \in S\} \geq f(\mathbf{x}) - |\mathbf{x} - \mathbf{x}'|,$$

and thus

$$f(\mathbf{x}) - f(\mathbf{x}') \leq |\mathbf{x} - \mathbf{x}'|.$$

Interchanging $\mathbf{x}$ and $\mathbf{x}'$, it follows that

$$f(\mathbf{x}') - f(\mathbf{x}) \leq |\mathbf{x} - \mathbf{x}'|.$$

Thus

$$|f(\mathbf{x}) - f(\mathbf{x}')| \leq |\mathbf{x} - \mathbf{x}'|$$

for all $\mathbf{x}, \mathbf{x}' \in X$. It follows that the function $f: X \to \mathbb{R}$ is continuous, as required. ∎

The multidimensional Heine-Borel Theorem (Theorem 3.3) ensures that a subset of a Euclidean space is compact if and only if it is both closed and bounded.

Proposition 4.9 *Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$, let V be a subset of X that is open in X, and let K be a compact subset of $\mathbb{R}^n$ satisfying $K \subset V$. Then there exists some positive real number ε with the property that $B_X(K, \varepsilon) \subset V$, where $B_X(K, \varepsilon)$ denotes the subset of X consisting of those points of X that lie within a distance less than ε of some point of K.*

Proof using the Bolzano-Weierstrass Theorem Suppose that the proposition were false. Then there would exist infinite sequences $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ and $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \ldots$ such that $\mathbf{x}_j \in K$, $\mathbf{w}_j \in X \setminus V$ and $|\mathbf{w}_j - \mathbf{x}_j| < 1/j$ for all positive integers j. The set K is both closed and bounded in $\mathbb{R}^n$. The multidimensional Bolzano-Weierstrass Theorem (Theorem 2.5) would then ensure the existence of a subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converging to some point $\mathbf{q}$ of K. Moreover $\lim\limits_{j \to +\infty} (\mathbf{w}_j - \mathbf{x}_j) = \mathbf{0}$, and therefore

$$\lim_{j \to \infty} \mathbf{w}_{k_j} = \lim_{j \to \infty} \mathbf{x}_{k_j} = \mathbf{q}.$$

But $\mathbf{w}_j \in X \setminus V$. Moreover $X \setminus V$ is closed in X, and therefore any sequence of points in $X \setminus V$ that converges in X must converge to a point of $X \setminus V$ (see Lemma 2.16). It would therefore follow that $\mathbf{q} \in K \cap (X \setminus V)$. But this is impossible, because $K \subset V$ and therefore $K \cap (X \setminus V) = \emptyset$. Thus a contradiction would follow were the proposition false. The result follows. ∎

Proof using the Heine-Borel Theorem It follows from the multidimensional Heine-Borel Theorem (Theorem 3.3) that the set K is compact, and thus every open cover of K has a finite subcover. Given point $\mathbf{x}$ of K let $\varepsilon_{\mathbf{x}}$ be a positive real number with the property that

$$B_X(\mathbf{x}, 2\varepsilon_{\mathbf{x}}) \subset V,$$

where

$$B_X(\mathbf{x}, r) = \{\mathbf{x}' \in X : |\mathbf{x}' - \mathbf{x}| < r\}$$

for all positive integers r. The collection of open balls $B_X(\mathbf{x}, \varepsilon_{\mathbf{x}})$ determined by the points $\mathbf{x}$ of K covers K. By compactness this open cover of K has a finite subcover. Therefore there exist points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ of K such that

$$K \subset B(\mathbf{x}_1, \varepsilon_{\mathbf{x}_1}) \cup B(\mathbf{x}_2, \varepsilon_{\mathbf{x}_2}) \cup \cdots \cup B(\mathbf{x}_k, \varepsilon_{\mathbf{x}_k}).$$

Let ε be the minimum of $\varepsilon_{\mathbf{x}_1}, \varepsilon_{\mathbf{x}_2}, \ldots, \varepsilon_{\mathbf{x}_k}$. If $\mathbf{x}$ is a point of K then $\mathbf{x} \in B_X(\mathbf{x}_j, \varepsilon_{\mathbf{x}_j}$ for some integer j between 1 and k. But it then follows from the Triangle Inequality that

$$B(\mathbf{x}, \varepsilon) \subset B_X(\mathbf{x}_j, 2\varepsilon_{\mathbf{x}_j}) \subset V.$$

It follows from this that

$$B_X(K, \varepsilon) \subset V,$$

as required. ∎

Proof using the Extreme Value Theorem Let $f\colon K \to \mathbb{R}$ be defined such that

$$f(\mathbf{x}) = \inf\{|\mathbf{z} - \mathbf{x}| : \mathbf{z} \in X \setminus V\}.$$

for all $\mathbf{x} \in K$. It follows from Lemma 4.8 that the function f is continuous on K.

Now $K \subset V$ and therefore, given any point $\mathbf{x} \in K$, there exists some positive real number δ such that the open ball of radius δ about the point $\mathbf{x}$ is contained in V, and therefore $f(\mathbf{x}) \geq \delta$. It follows that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in K$.

It follows from the Extreme Value Theorem for continuous real-valued functions on closed bounded subsets of Euclidean spaces (Theorem 2.20) that the function $f\colon K \to \mathbb{R}$ attains its minimum value at some point of K. Let that minimum value be ε. Then $f(\mathbf{x}) \geq \varepsilon > 0$ for all $\mathbf{x} \in K$, and therefore $|\mathbf{x} - \mathbf{x}| \geq \varepsilon > 0$ for all $\mathbf{x} \in K$ and $\mathbf{z} \in X \setminus V$. It follows that $B_X(K, \varepsilon) \subset V$, as required. ∎

Example Let

$$F = \{(x, y) \in \mathbb{R}^2 : x \geq 0, \;\; y \geq 0 \text{ and } xy \geq 1\}.$$

and let

$$V = \{(x, y) \in \mathbb{R}^2 : y > 0\}.$$

Note that if $(x, y) \in F$ then $x > 0$ and $y > 0$, because $xy = 1$. It follows that $F \subset V$. Also F is a closed set in $\mathbb{R}^2$ and V is an open set in $\mathbb{R}^2$. However F is not a compact subset of $\mathbb{R}^2$ because F is not bounded.

We now show that there does not exist any positive real number ε with the property that $B_{\mathbb{R}^2}(F, \varepsilon) \subset V$, where $B_{\mathbb{R}^2}(F, \varepsilon)$ denotes the set of points of $\mathbb{R}^2$ that lie within a distance ε of some point of F. Indeed let some positive real number ε be given, let x be a positive real number satisfying $x > 2\varepsilon^{-1}$, and let $y = x^{-1} - \frac{1}{2}\varepsilon$. Then $y < 0$, and therefore $(x, y) \notin V$. But $(x, y + \frac{1}{2}\varepsilon) \in F$, and therefore $(x, y) \in B_{\mathbb{R}^2}(F, \varepsilon)$. This shows that there does not exist any positive real number ε for which $B_{\mathbb{R}^2}(F, \varepsilon) \subset V$.

Proposition 4.10 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let K be a non-empty compact subset of Y, and let U be an subset in $X \times Y$ that is open in $X \times Y$. Let*

$$d_Y(\mathbf{y}, K) = \inf\{|\mathbf{y} - \mathbf{z}| : \mathbf{z} \in K\}$$

for all $\mathbf{y} \in Y$. Let $\mathbf{p}$ be a point of X with the property that $(\mathbf{p}, \mathbf{z}) \in U$ for all $\mathbf{z} \in K$. Then there exists some positive number δ such that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ and $d(\mathbf{y}, K) < \delta$.

Proof Let

$$\tilde{K} = \{(\mathbf{p}, \mathbf{z}) : \mathbf{z} \in K\}.$$

Then $\tilde{K}$ is a closed bounded subset of $\mathbb{R}^n \times \mathbb{R}^m$. It follows from Proposition 4.9 that there exists some positive real number ε such that

$$B_{X \times Y}(\tilde{K}, \varepsilon) \subset U$$

where $B_{X \times Y}(\tilde{K}, \varepsilon)$ denotes that subset of $X \times Y$ consisting of those points $(\mathbf{x}, \mathbf{y})$ of $X \times Y$ that lie within a distance ε of a point of $\tilde{K}$. Now a point

$(\mathbf{x}, \mathbf{y})$ of $X \times Y$ belongs to $B_{X\times Y}(\tilde{K}, \varepsilon)$ if and only if there exists some point $\mathbf{z}$ of K for which

$$|\mathbf{x} - \mathbf{p}|^2 + |\mathbf{y} - \mathbf{z}|^2 < \varepsilon^2.$$

Let $\delta = \varepsilon/\sqrt{2}$. If $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfy $|\mathbf{x} - \mathbf{p}| < \delta$ and $d_Y(\mathbf{y}, K) < \delta$ then there exists some point $\mathbf{z}$ of K for which $|\mathbf{y} - \mathbf{z}| < \delta$. But then

$$|\mathbf{x} - \mathbf{p}|^2 + |\mathbf{y} - \mathbf{z}|^2 < 2\delta^2 = \varepsilon^2,$$

and therefore $(\mathbf{x}, \mathbf{y}) \in U$, as required. ∎

Proposition 4.11 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that $\Phi(\mathbf{x})$ is closed in Y for every $\mathbf{x} \in X$. Suppose also that $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous. Then the graph* $\mathrm{Graph}(\Phi)$ *of $\Phi\colon X \rightrightarrows Y$ is closed in $X \times Y$.*

Proof Let $(\mathbf{p}, \mathbf{q})$ be a point of the complement $X \times Y \setminus \mathrm{Graph}(\Phi)$ of the graph $\mathrm{Graph}(\Phi)$ of Φ in $X \times Y$. Then $\Phi(\mathbf{p})$ is closed in Y and $\mathbf{q} \notin \Phi(\mathbf{p})$. It follows that there exists some positive real number δ_Y such that $|\mathbf{y} - \mathbf{q}| > \delta_Y$ for all $\mathbf{y} \in \Phi(\mathbf{p})$.

Let

$$V = \{\mathbf{y} \in Y : |\mathbf{y} - \mathbf{q}| > \delta_Y\}$$

and

$$W = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V\}.$$

Then V is open in Y and $\Phi(\mathbf{p}) \subset V$. Now the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous. It therefore follows from the definition of upper hemicontinuity that the subset W of X is open in X. Moreover $\mathbf{p} \in W$. It follows that there exists some positive real number δ_X such that $\mathbf{x} \in W$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta_X$. Then $\Phi(\mathbf{x}) \subset V$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta_X$.

Let δ be the minimum of δ_X and δ_Y, and let $(\mathbf{x}, \mathbf{y})$ be a point of $X \times Y$ whose distance from the point $(\mathbf{p}, \mathbf{q})$ is less than δ. Then $|\mathbf{x} - \mathbf{p}| < \delta_X$ and therefore $\Phi(\mathbf{x}) \subset V$. Also $\mathbf{y} - \mathbf{q}| < \delta_Y$, and therefore $\mathbf{y} \notin V$. It follows that $\mathbf{y} \notin \Phi(\mathbf{x})$, and therefore $(\mathbf{x}, \mathbf{y}) \notin \mathrm{Graph}(\Phi)$. We conclude from this that the complement of $\mathrm{Graph}(\Phi)$ is open in $X \times Y$. It follows that $\mathrm{Graph}(\Phi)$ itself is closed in $X \times Y$, as required. ∎

Proposition 4.12 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that the graph* $\mathrm{Graph}(\Phi)$ *of the correspondence Φ is closed in $X \times Y$. Suppose also that Y is a compact subset of $\mathbb{R}^m$. Then the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous.*

Proof using Proposition 4.10 Let $\mathbf{p}$ be a point of X, let V be an open set satisfying $\Phi(\mathbf{p}) \subset V$, and let $K = Y \setminus V$. The compact set Y is closed and bounded in $\mathbb{R}^m$. Also K is closed in Y. It follows that K is a closed bounded subset of $\mathbb{R}^m$ (see Lemma 2.23). Let U be the complement of $\mathrm{Graph}(\Phi)$ in $X \times Y$. Then U is open in $X \times Y$, because $\mathrm{Graph}(\Phi)$ is closed in $X \times Y$. Also $(\mathbf{p}, \mathbf{y}) \in U$ for all $\mathbf{y} \in K$, because $\Phi(\mathbf{p}) \cap K = \emptyset$. It follows from Proposition 4.10 that there exists some positive number δ such that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in K$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus if $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then $\mathbf{y} \notin \Phi(\mathbf{x})$ for all $\mathbf{y} \in K$, and therefore $\Phi(\mathbf{x}) \subset V$, where $V = Y \setminus K$. Thus the correspondence Φ is upper hemicontinuous at $\mathbf{p}$, as required. ∎

Proof using the Bolzano-Weierstrass Theorem Let V be a subset of Y that is open in Y, and let $\mathbf{p}$ be a point of X for which $\Phi(\mathbf{p}) \subset V$. Let $F = Y \setminus V$. Then the set F is a subset of Y that is closed in Y, and $\Phi(\mathbf{p}) \cap F = \emptyset$. Now Y is a closed bounded subset of $\mathbb{R}^m$, because it is compact (Theorem 3.3). It follows that F is closed in $\mathbb{R}^m$ (Lemma 2.23).

Suppose that there did not exist any positive number δ such that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Then there would exist an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X converging to the point $\mathbf{p}$ with the property that $\Phi(\mathbf{x}_j) \cap F \neq \emptyset$ for all positive integers j. There would then exist an infinite sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ of elements of Y such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j) \cap F$ for all positive integers j. Then $(\mathbf{x}_j, \mathbf{y}_j) \in \mathrm{Graph}(\Phi)$ for all positive integers j. Moreover the infinite sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ would be bounded, because the set Y is bounded.

It would therefore follow from the multidimensional Bolzano-Weierstrass Theorem (Theorem 2.5) that there would exist a convergent subsequence

$$\mathbf{y}_{k_1}, \mathbf{y}_{k_2}, \mathbf{y}_{k_3}, \ldots$$

of the sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$. Let $\mathbf{q} = \lim\limits_{j \to +\infty} \mathbf{y}_{k_j}$. Then $\mathbf{q} \in F$, because the set F is closed in Y and $\mathbf{y}_{k_j} \in F$ for all positive integers j (see Lemma 2.16). Similarly $(\mathbf{p}, \mathbf{q}) \in \mathrm{Graph}(\Phi)$, because the set $\mathrm{Graph}(\Phi)$ is closed in $X \times Y$, $(\mathbf{x}_{k_j}, \mathbf{y}_{k_j}) \in \mathrm{Graph}(\Phi)$ for all positive integers j, and

$$(\mathbf{p}, \mathbf{q}) = \lim_{j \to +\infty} (\mathbf{x}_{k_j}, \mathbf{y}_{k_j}).$$

But were there to exist $(\mathbf{p}, \mathbf{q}) \in X \times Y$ for which $\mathbf{q} \in F$ and $(\mathbf{p}, \mathbf{q}) \in \mathrm{Graph}(\Phi)$, it would follow that $\mathbf{q} \in \Phi(\mathbf{p}) \cap F$. But this is impossible, because $\Phi(\mathbf{p}) \cap F = \emptyset$. Thus a contradiction would arise were there to exist an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X for which $\Phi(\mathbf{x}_j) \cap F \neq \emptyset$ and

$\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$. Therefore no such infinite sequence can exist, and therefore there must exist some positive real number δ such that $\Phi(\mathbf{x}) \subset V$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$. We conclude that

$$\{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V\}$$

is open in X. The result follows. ∎

Corollary 4.13 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi: X \to Y$ be a function from X to Y. Suppose that the graph* $\mathrm{Graph}(\varphi)$ *of the function φ is closed in $X \times Y$. Suppose also that Y is a compact subset of $\mathbb{R}^m$. Then the function $\varphi: X \to Y$ is continuous.*

Proof Let $\Phi: X \rightrightarrows Y$ be the correspondence defined such that $\Phi(\mathbf{x}) = \{\varphi(\mathbf{x})\}$ for all $\mathbf{x} \in X$. Then $\mathrm{Graph}(\Phi) = \mathrm{Graph}(\varphi)$, and therefore $\mathrm{Graph}(\Phi)$ is closed in $X \times Y$. The subset Y of $\mathbb{R}^m$ is compact. It therefore follows from Proposition 4.12 that the correspondence Φ is upper hemicontinuous. It then follows from Lemma 4.3 that the function $\varphi: X \to Y$ is continuous, as required. ∎

4.3 Compact-Valued Upper Hemicontinuous Correspondences

Lemma 4.14 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi: X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that $\Phi: X \rightrightarrows Y$ is upper hemicontinuous. Then*

$$\{\mathbf{x} \in X : \Phi(\mathbf{x}) \neq \emptyset\}$$

is closed in X.

Proof Given any open set V in Y, let

$$\Phi^+(V) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V\}.$$

It follows from the upper hemicontinuity of Φ that $\Phi^+(V)$ is open in X for all open sets V in Y (see Lemma 4.1). Now the empty set $\emptyset$ is open in Y. It follows that $\Phi^+(\emptyset)$ is open in X. But

$$\Phi^+(\emptyset) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset \emptyset\} = \{\mathbf{x} \in X : \Phi(\mathbf{x}) = \emptyset\}.$$

It follows that the set of point $\mathbf{x}$ in X for which $\Phi(\mathbf{x}) = \emptyset$ is open in X, and therefore the set of points $\mathbf{x} \in X$ for which $\Phi(\mathbf{x}) \neq \emptyset$ is closed in X, as required. ∎

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Given any subset S of X, we define the *image* $\Phi(S)$ of S under the correspondence Φ to be the subset of Y defined such that

$$\Phi(S) = \bigcup_{\mathbf{x} \in S} \Phi(\mathbf{x})$$

Lemma 4.15 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is compact-valued and upper hemicontinuous. Let K be a compact subset of X. Then $\Phi(K)$ is a compact subset of Y.*

Proof Let $\mathcal{V}$ be collection of open sets in Y that covers $\Phi(K)$. Given any point $\mathbf{p}$ of K, there exists a finite subcollection $\mathcal{W}_{\mathbf{p}}$ of $\mathcal{V}$ that covers the compact set $\Phi(\mathbf{p})$. Let $U_{\mathbf{p}}$ be the union of the open sets belonging to this subcollection $\mathcal{W}_{\mathbf{p}}$. Then $\Phi(\mathbf{p}) \subset U_{\mathbf{p}}$. Now it follows from the upper hemicontinuity of $\Phi\colon X \rightrightarrows Y$ that there exists an open set $N_{\mathbf{p}}$ in X such that $\Phi(\mathbf{x}) \subset U_{\mathbf{p}}$ for all $\mathbf{x} \in N_{\mathbf{p}}$. Moreover, given any $\mathbf{p} \in K$, the finite collection $\mathcal{W}_{\mathbf{p}}$ of open sets in Y covers $\Phi(N_{\mathbf{p}})$.

It then follows from the compactness of K that there exist points

$$\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k$$

of K such that

$$K \subset N_{\mathbf{p}_1} \cup N_{\mathbf{p}_2} \cup \cdots \cup N_{\mathbf{p}_k}.$$

Let

$$\mathcal{W} = \mathcal{W}_{\mathbf{p}_1} \cup \mathcal{W}_{\mathbf{p}_2} \cup \cdots \cup \mathcal{W}_{\mathbf{p}_k}.$$

Then $\mathcal{W}$ is a finite subcollection of $\mathcal{V}$ that covers $\Phi(K)$. The result follows. ∎

Proposition 4.16 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a compact-valued correspondence from X to Y. Let $\mathbf{p}$ be a point of X for which $\Phi(\mathbf{p})$ is non-empty. Then the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$ if and only if, given any positive real number ε, there exists some positive real number δ such that*

$$\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, where $B_Y(\Phi(\mathbf{p}), \varepsilon)$ denotes the subset of Y consisting of all points of Y that lie within a distance ε of some point of $\Phi(\mathbf{p})$.

Proof Let $\Phi: X \rightrightarrows Y$ is a compact-valued correspondence, and let $\mathbf{p}$ be a point of X for which $\Phi(\mathbf{p}) \neq \emptyset$.

First suppose that, given any positive real number ε, there exists some positive real number δ such that

$$\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. We must prove that $\Phi: X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$.

Let V be an open set in Y that satisfies $\Phi(\mathbf{p}) \subset V$. Now $\Phi(\mathbf{p})$ is a compact subset of Y, because $\Phi: X \to Y$ is compact-valued. It follows that there exists some positive real number ε such that $B_Y(\Phi(\mathbf{p}), \varepsilon) \subset V$ (see Proposition 4.9). There then exists some positive number δ such that

$$\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon) \subset V$$

whenever $|\mathbf{x} - \mathbf{p}| < \delta$. Thus $\Phi: X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$.

Conversely suppose that the correspondence $\Phi: X \rightrightarrows Y$ is upper hemicontinuous at the point $\mathbf{p}$. Now $\Phi(\mathbf{p})$ is a non-empty subset of Y. Let some positive number ε be given. Then $B_Y(\Phi(\mathbf{p}), \varepsilon)$ is open in Y and $\Phi(\mathbf{p}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$. It follows from the upper hemicontinuity of Φ at $\mathbf{p}$ that there exists some positive number δ such that $\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \varepsilon)$ whenever $|\mathbf{x} - \mathbf{p}| < \delta$. The result follows. ∎

Proposition 4.17 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi: X \rightrightarrows Y$ be a correspondence from X to Y. Then the correspondence is both compact-valued and upper hemicontinuous at a point $\mathbf{p} \in X$ if and only if, given any infinite sequences*

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

in X and Y respectively, where $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim\limits_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$, there exists a subsequence of

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

which converges to a point of $\Phi(\mathbf{p})$.

Proof Throughout this proof, let us say that the correspondence Φ satisfies the *constrained convergent subsequence criterion* if (and only if), given any infinite sequences

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

in X and Y respectively, where $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim\limits_{j\to+\infty} \mathbf{x}_j = \mathbf{p}$, there exists a subsequence of

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

which converges to a point of $\Phi(\mathbf{p})$. We must prove that the correspondence $\Phi\colon X \rightrightarrows Y$ satisfies the constrained convergent subsequence criterion if and only if it is compact-valued and upper hemicontinuous.

Suppose first the the correspondence $\Phi\colon X \rightrightarrows Y$ satisfies the constrained convergent subsequence criterion. Applying this criterion when $\mathbf{x}_j = \mathbf{p}$ for all positive integers j, we conclude that every infinite sequence $(\mathbf{y}_j : j \in \mathbb{N})$ of points of $\Phi(\mathbf{p})$ has a convergent subsequence, and therefore $\Phi(\mathbf{x})$ is compact.

Let

$$D = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \neq \emptyset\}.$$

We show that D is closed in X. Let

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

be a sequence of points of D converging to some point of $\mathbf{p}$ of X. Then $\Phi(\mathbf{x}_j)$ is non-empty for all positive integers j, and therefore there exists an infinite sequence

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

of points of Y such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j. The constrained convergent subsequence criterion ensures that this infinite sequence in Y must have a subsequence that converges to some point of $\Phi(\mathbf{p})$. It follows that $\phi(\mathbf{p})$ is non-empty, and thus $\mathbf{p} \in D$.

Let $\mathbf{p}$ be a point of the complement of D. Then $\Phi(\mathbf{p}) = \emptyset$. There then exists $\delta > 0$ such that $\Phi(\mathbf{x}) = \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then $\Phi(\mathbf{x}) \subset V$ for all open sets V in Y. It follows that the correspondence Φ is upper hemicontinuous at those points $\mathbf{p}$ for which $\Phi(\mathbf{p}) = \emptyset$.

Now consider the situation in which $\Phi\colon X \rightrightarrows Y$ satisfies the constrained convergent subsequence criterion and $\mathbf{p}$ is some point of X for which $\Phi(\mathbf{p}) \neq \emptyset$. Let $K = \Phi(\mathbf{p})$. Then K is a compact non-empty subset of Y. Let V be an open set in Y that satisfies $\Phi(\mathbf{p}) \subset V$. Suppose that there did not exist any positive real number δ with the property that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It would then follow that there would exist infinite sequences

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

in X and Y respectively for which $|\mathbf{x}_j - \mathbf{p}| < 1/j$, $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ and $\mathbf{y}_j \notin V$.

Then $\lim\limits_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$, and thus the constrained convergent subsequence criterion satisfied by the correspondence Φ would ensure the existence of a subsequence

$$\mathbf{y}_{k_1}, \mathbf{y}_{k_2}, \mathbf{y}_{k_3}, \ldots$$

of $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ converging to some point $\mathbf{q}$ of $\Phi(\mathbf{p})$. But then $\mathbf{q} \notin V$, because $\mathbf{y}_{k_j} \notin V$ for all positive integers j, and the complement $Y \setminus V$ of V is closed in Y. But $\Phi(\mathbf{p}) \subset V$, and $\mathbf{q} \in \Phi(\mathbf{p})$, and therefore $\mathbf{q} \in V$. Thus a contradiction would arise were there not to exist a positive real number δ with the property that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus such a real number δ must exist, and thus the constrained convergent subsequence criterion ensures that the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$.

It remains to show that any compact-valued upper hemicontinuous correspondence $\Phi\colon X \rightrightarrows Y$ satisfies the constrained convergent subsequence criterion. Let $\Phi\colon X \rightrightarrows Y$ be compact-valued and upper hemicontinuous. It follows from Lemma 4.14 that

$$\{\mathbf{x} \in X : \Phi(\mathbf{x}) \neq \emptyset\}$$

is closed in X.

Let

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

be infinite sequences in X and Y respectively, where $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim\limits_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$. Then $\Phi(\mathbf{p})$ is non-empty, because

$$\{\mathbf{x} \in X : \Phi(\mathbf{x}) \neq \emptyset\}$$

is closed in X (see Lemma 4.14). Let $K = \Phi(\mathbf{p})$. Then K is compact, because $\Phi\colon X \rightrightarrows Y$ is compact-valued by assumption. For each integer j let $d(\mathbf{y}_j, K)$ denote the greatest lower bound on the distances from $\mathbf{y}_j$ to points of K. There then exists an infinite sequence

$$\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots$$

of points of K such that $|\mathbf{y}_j - \mathbf{z}_j| \leq 2d(\mathbf{y}_j, K)$. for all positive integers j.

Now the upper hemicontinuity of $\Phi\colon X \rightrightarrows Y$ ensures that $d(\mathbf{y}_j, K) \to 0$ as $j \to +\infty$. Indeed, given any positive real number ε, the set $B_Y(K, \varepsilon)$ of points of Y that lie within a distance ε of a point of K is an open set containing $\Phi(\mathbf{p})$. It follows from the upper hemicontinuity of Φ that there exists some positive real number δ such that $\Phi(\mathbf{x}) \subset B_Y(K, \varepsilon)$ whenever $|\mathbf{x} - \mathbf{p}| < \delta$. Now $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$. It follows that there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$. But then $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ and therefore $d(\mathbf{y}_j, K) < \varepsilon$ whenever $j \geq N$.

Now the compactness of K ensures that the infinite sequence

$$\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots$$

of points of K has a subsequence

$$\mathbf{z}_{k_1}, \mathbf{z}_{k_2}, \mathbf{z}_{k_3}, \ldots$$

that converges to some point $\mathbf{q}$ of K. Now $|\mathbf{y}_j - \mathbf{z}_j| \leq 2d(\mathbf{y}_j, K)$ for all positive integers j, and $d(\mathbf{y}_j, K) \to 0$ as $j \to +\infty$. It follows that $\mathbf{y}_{k_j} \to \mathbf{q}$ as $j \to +\infty$. Morever $\mathbf{q} \in \Phi(\mathbf{p})$. We have therefore verified that the constrained convergent subsequence criterion is satisfied by any correspondence $\Phi\colon X \rightrightarrows Y$ that is compact-valued and upper hemicontinuous. This completes the proof. ∎

Proposition 4.18 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is both upper hemicontinuous and compact-valued. Let U be an open set in $X \times Y$. Then*

$$\{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \textit{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X.

Proof using Proposition 4.10 Let

$$W = \{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\},$$

and let $\mathbf{p} \in W$. If $\Phi(\mathbf{p}) = \emptyset$ then it follows from Lemma 4.14 that there exists some positive real number δ such that $\Phi(\mathbf{x}) = \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Then $\mathbf{x} \in W$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$.

Suppose that $\Phi(\mathbf{p}) \neq 0$. Let $K = \Phi(\mathbf{p})$. Then K is a compact subset of Y, because the correspondence Φ is compact-valued. Also $(\mathbf{p}, \mathbf{y}) \in U$ for all $\mathbf{y} \in K$. It follows from Proposition 4.10 that there exists some positive real number δ_1 such that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_1$ and $d_Y(\mathbf{y}, K) < \delta_1$, where

$$d_Y(\mathbf{y}, K) = \inf\{|\mathbf{y} - \mathbf{z}| : \mathbf{z} \in K\}.$$

Let

$$V = \{\mathbf{y} \in Y : d_Y(\mathbf{y}, K) < \delta_1\}.$$

Then V is open in Y because the function sending $\mathbf{y} \in Y$ to $d(\mathbf{y}, K)$ is continuous on Y (see Lemma 4.8). Also $\Phi(\mathbf{p}) \subset V$. It follows from the upper hemicontinuity of the correspondence Φ that there exists some positive number δ_2 such that $\Phi(\mathbf{x}) \subset V$ whenever $|\mathbf{x} - \mathbf{p}| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then $\Phi(\mathbf{x}) \subset V$. But then $d(\mathbf{y}, K) < \delta_1$ for all $\mathbf{y} \in \Phi(\mathbf{x})$. Moreover $|\mathbf{x} - \mathbf{p}| < \delta_1$. It follows that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{y} \in \Phi(\mathbf{x})$, and therefore $\mathbf{x} \in W$. This shows that W is an open subset of X, as required. ∎

Proof using Proposition 4.17 Let

$$W = \{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\},$$

and let $\mathbf{p} \in W$. Suppose that there did not exist any strictly positive real number δ with the property that $\mathbf{x} \in W$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Then, given any positive real number δ, there would exist points $\mathbf{x}$ of X and $\mathbf{y}$ of Y such that $|\mathbf{x} - \mathbf{p}| < \delta$, $\mathbf{y} \in \Phi(\mathbf{x})$ and $(\mathbf{x}, \mathbf{y}) \notin U$. Therefore there would exist infinite sequences

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

and

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

in X and Y respectively such that $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ and $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ and $(\mathbf{x}_j, \mathbf{y}_j) \notin U$ for all positive integers j.

The correspondence $\Phi\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous. Proposition 4.17 would therefore ensure the existence of a subsequence

$$\mathbf{y}_{k_1}, \mathbf{y}_{k_2}, \mathbf{y}_{k_3}, \ldots$$

of Y converging to some point $\mathbf{q}$ of $\Phi(\mathbf{p})$. Now the complement of U in $X \times Y$ is closed in $X \times Y$, because U is open in $X \times Y$ and $(\mathbf{x}_j, \mathbf{y}_j) \notin U$. It would therefore follow that $(\mathbf{p}, \mathbf{q}) \notin U$ (see Proposition 4.6). But this gives rise to a contradiction, because $\mathbf{q} \in \Phi(\mathbf{p})$ and $(\mathbf{p}, \mathbf{y}) \in U$ for all $\mathbf{y} \in \Phi(\mathbf{p})$. In order to avoid the contradiction, there must exist some positive real number δ with the property that with the property that $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ and $\mathbf{y} \in \Phi(\mathbf{x})$. The result follows. ∎

Proof using Compactness (Heine-Borel) directly Let $\Phi: X \to Y$ be a compact-valued upper hemicontinuous correspondence, and let U be a subset of $X \times Y$ that is open in $X \times Y$. Let

$$W = \{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}.$$

We must prove that W is open in X.

Let $K = \Phi(\mathbf{p})$. Then, given any point $\mathbf{y}$ of K, there exists an open set $M_{\mathbf{p},\mathbf{y}}$ in X and an open set $V_{\mathbf{p},\mathbf{y}}$ in Y such that $M_{\mathbf{p},\mathbf{y}} \times V_{\mathbf{p},\mathbf{y}} \subset U$ (see Lemma 4.5). Now every open cover of K has a finite subcover, because K is compact. Therefore there exist points $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k$ of K such that

$$K \subset V_{\mathbf{p},\mathbf{y}_1} \cup V_{\mathbf{p},\mathbf{y}_2} \cup \cdots \cup V_{\mathbf{p},\mathbf{y}_k}.$$

Let

$$M_{\mathbf{p}} = M_{\mathbf{p},\mathbf{y}_1} \cap M_{\mathbf{p},\mathbf{y}_2} \cap \cdots \cap M_{\mathbf{p},\mathbf{y}_k}$$

and

$$V_{\mathbf{p}} = V_{\mathbf{p},\mathbf{y}_1} \cup V_{\mathbf{p},\mathbf{y}_2} \cup \cdots \cup V_{\mathbf{p},\mathbf{y}_k}.$$

Then

$$M_{\mathbf{p}} \times V_{\mathbf{p}} \subset \bigcup_{j=1}^{k} (M_{\mathbf{p}} \times V_{\mathbf{p},\mathbf{y}_j}) \subset \bigcup_{j=1}^{k} (M_{\mathbf{p},\mathbf{y}_j} \times V_{\mathbf{p},\mathbf{y}_j}) \subset U.$$

Now $M_{\mathbf{p}}$ is open in X, because it is the intersection of a finite number of subsets of X that are open in X. Also it follows from the upper hemicontinuity of the correspondence Φ that $\Phi^+(V_{\mathbf{p}})$ is open in X, where

$$\Phi^+(V_{\mathbf{p}}) = \{\mathbf{x} \in X : \Phi(\mathbf{x}) \subset V_{\mathbf{p}}\}$$

(see Lemma 4.1). Let $N_{\mathbf{p}} = M_{\mathbf{p}} \cap \Phi^+(V_{\mathbf{p}})$. Then $N_{\mathbf{p}}$ is open in X and $\mathbf{p} \in N_{\mathbf{p}}$. Now if $\mathbf{x} \in N_{\mathbf{p}}$ then $\mathbf{x} \in M_{\mathbf{p}}$ and $\Phi(\mathbf{x}) \subset V_{\mathbf{p}}$, and therefore $(\mathbf{x}, \mathbf{y}) \in U$ for all $\mathbf{y} \in \Phi(\mathbf{x})$. We have thus shown that $N_{\mathbf{p}} \subset W$ for all $\mathbf{p} \in W$, where

$$W = \{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}.$$

Thus W is the union of the subsets $N_{\mathbf{p}}$ as $\mathbf{p}$ ranges over the points of W. Moreover the set $N_{\mathbf{p}}$ is open in X for each $\mathbf{p} \in W$. It follows that W must itself be open in X. Indeed, given any point $\mathbf{p}$ of W, there exists some positive real number δ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} \subset N_{\mathbf{p}} \subset W.$$

The result follows. ∎

Remark The various proofs of Proposition 4.18 were presented in the contexts of correspondences between subsets of Eucldean spaces. All these proofs generalize easily so as to apply to correspondence between subsets of metric spaces. The last of the proofs can be generalized without difficulty so as to apply to correspondences between topological spaces. Inded the notion of *correspondences* between topological spaces is defined so that a correspondence $\Phi \colon X \rightrightarrows Y$ between topological spaces X and Y associates to each point of X a subset $\Phi(\mathbf{x})$ of Y. Such a correspondence is said to be upper hemicontinuous at a point p of X if, given any open subset V of Y for which $\Phi(p) \subset V$, there exists an open set $N(p)$ in X such that $\Phi(x) \subset V$ for all $x \in N$.

The last of the proofs of Proposition 4.18 presented above can be generalized to show that, given a compact-valued correspondence $\Phi \colon X \rightrightarrows Y$ between topological spaces X and Y, and given a subset U of Y, the set

$$\{x \in X : (x, y) \in U \text{ for all } y \in \Phi(x)\}$$

is open in X.

Remark It should be noted that other results proved in this section do not necessarily generalize to correspondences $\Phi \colon X \rightrightarrows Y$ mapping the topological space X into an arbitrary topological space Y. For example all closed-valued upper hemicontinuous correspondences between metric spaces have closed graphs. The appropriate generalization of this result states that any closed-valued upper hemicontinuous correspondence $\Phi \colon X \rightrightarrows Y$ from a topological space X to a regular topological space Y has a closed graph. To interpret this, one needs to know the definition of what is meant by saying that a topological space is *regular*. A topological space Y is said to be *regular* if, given any closed subset F of Y, and given any point p of the complement $Y \setminus F$ of F, there exist open sets V and W in Y such that $F \subset V$, $p \in W$ and $V \cap W = \emptyset$. Metric spaces are regular. Also compact Hausdorff spaces are regular.

Corollary 4.19 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi \colon X \rightrightarrows Y$ be a correspondence from X to Y that is both upper hemicontinuous and compact-valued. Let K be a compact subset of X. Then*

$$\{(\mathbf{x}, \mathbf{y}) \in X \times Y : \mathbf{x} \in K \textit{ and } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is a compact subset of $X \times Y$.

Proof Let $\mathcal{V}$ be an open cover of L where

$$L = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : \mathbf{x} \in K \text{ and } \mathbf{y} \in \Phi(\mathbf{x})\}$$

For each $\mathbf{p} \in K$ let

$$L_{\mathbf{p}} = \{(\mathbf{p}, \mathbf{y}) : \mathbf{y} \in \Phi(\mathbf{p})\} = \{(\mathbf{p}, \mathbf{y}) : (\mathbf{p}, \mathbf{y}) \in L\}.$$

Then $L_{\mathbf{p}}$ is a compact subset of $X \times Y$ for all $\mathbf{p} \in K$. (Indeed this set is the image of the compact set $\Phi(\mathbf{p})$ under the continuous function that sends each point $\mathbf{y}$ of $\Phi(\mathbf{p})$ to $(\mathbf{p}, \mathbf{y})$, and any continuous function maps compact sets to compact sets.) It follows that, for each point $\mathbf{p}$ of K, there is some finite subcollection $\mathcal{W}_{\mathbf{p}}$ of $\mathcal{V}$ that covers $L_{\mathbf{p}}$.

Let $U_{\mathbf{p}}$ be the union of the sets belonging to the collection $\mathcal{W}_{\mathbf{p}}$. Then $U_{\mathbf{p}}$ is an open subset of $X \times Y$. Let

$$N_{\mathbf{p}} = \{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U_{\mathbf{p}} \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

for all $\mathbf{p} \in K$. It then follows from Proposition 4.18 that that $N_{\mathbf{p}}$ is open in X for all $\mathbf{p} \in K$. Moreover the definition of $N_{\mathbf{p}}$ ensures that

$$\{(\mathbf{x}, \mathbf{y}) \in L : \mathbf{x} \in N_{\mathbf{p}}\}$$

is covered by the finite subcollection $\mathcal{W}_{\mathbf{p}}$ of the given open cover $\mathcal{V}$.

It then follows from the compactness of K that there exist points

$$\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k$$

of K such that

$$K \subset N_{\mathbf{p}_1} \cup N_{\mathbf{p}_2} \cup \cdots \cup N_{\mathbf{p}_k}.$$

Let

$$\mathcal{W} = \mathcal{W}_{\mathbf{p}_1} \cup \mathcal{W}_{\mathbf{p}_2} \cup \cdots \cup \mathcal{W}_{\mathbf{p}_k}.$$

Then $\mathcal{W}$ is a finite subcollection of $\mathcal{V}$ that covers L. The result follows. ∎

4.4 A Criterion characterizing Lower Hemicontinuity

Proposition 4.20 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A correspondence $\Phi: X \rightrightarrows Y$ is lower hemicontinuous at a point $\mathbf{p}$ of X if and only if given any infinite sequence*

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

in X for which $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$ and given any point $\mathbf{q}$ of $\Phi(\mathbf{p})$, there exists an infinite sequence

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

of points of F such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim_{j \to +\infty} \mathbf{y}_j = \mathbf{q}$.

Proof First suppose that $\Phi\colon X \to Y$ is lower hemicontinuous at some point $\mathbf{p}$ of X. Let $\mathbf{q} \in \Phi(\mathbf{p})$, and let some positive number ε be given. Then the open ball $B_Y(\mathbf{q}, \varepsilon)$ in Y of radius ε centred on the point $\mathbf{q}$ is an open set in Y. It follows from the lower hemicontinuity of $\Phi\colon X \to Y$ that there exists some positive real number δ such that $\Phi(\mathbf{x}) \cap B_Y(\mathbf{q}, \varepsilon)$ is non-empty whenever $|\mathbf{x} - \mathbf{p}| < \delta$. Then, given any point $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ there exists some $\mathbf{y} \in \Phi(\mathbf{x})$ that satisfies $|\mathbf{y} - \mathbf{q}| < \varepsilon$. In particular, given any positive integer s, there exists some positive integer δ_s such that, given any point $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta_s$, there exists some $\mathbf{y} \in \Phi(\mathbf{x})$ that satisfies $|\mathbf{y} - \mathbf{q}| < 1/s$.

Now $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$. It follows that there exist positive integers $k(1), k(2), k(3), \ldots$, where

$$k(1) < k(2) < k(3) < \cdots$$

such that $|\mathbf{x}_j - \mathbf{p}| < \delta_s$ for all positive integers j satisfying $j \geq k(s)$. There then exists an infinite sequence

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $|\mathbf{y}_j - \mathbf{q}| < 1/s$ for all positive integers j and s satisfying $k(s) \leq j < k(s+1)$. Then $\lim\limits_{j \to +\infty} \mathbf{y}_j = \mathbf{q}$. We have thus shown that if $\Phi\colon X \to Y$ is lower hemicontinuous at the point $\mathbf{p}$, if $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is a sequence in X converging to the point $\mathbf{p}$, and if $\mathbf{q} \in \Phi(\mathbf{p})$, then there exists an infinite sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ in Y such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integer j and $\lim\limits_{j \to +\infty} \mathbf{y}_j = \mathbf{q}$.

Next suppose that the correspondence $\Phi\colon X \rightrightarrows Y$ is not lower hemicontinuous at $\mathbf{p}$. Then there exists an open set V in Y such that $\Phi(\mathbf{p}) \cap V$ is non-empty but there does not exist any positive real number δ with the property that $\Phi(\mathbf{x}) \cap V \neq \emptyset$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{p} - \mathbf{x}| < \delta$. Let $\mathbf{q} \in \Phi(\mathbf{p})$. There then exists an infinite sequence

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$$

converging to the point $\mathbf{p}$ with the property that $\Phi(\mathbf{x}_j) \cap V = \emptyset$ for all positive integers j. It is not then possible to construct an infinite sequence

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$$

such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim\limits_{j \to +\infty} \mathbf{y}_j = \mathbf{q}$. The result follows. ∎

4.5 Intersections of Correspondences

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi: X \rightrightarrows Y$ and $\Psi: X \to Y$ be correspondences between X and Y. The *intersection* $\Phi \cap \Psi$ of the correspondences Φ and Ψ is defined such that

$$(\Phi \cap \Psi)(\mathbf{x}) = \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$$

for all $\mathbf{x} \in X$.

Proposition 4.21 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $\Phi: X \rightrightarrows Y$ and $\Psi: X \rightrightarrows Y$ be correspondences from X to Y, where the correspondence $\Phi: X \rightrightarrows Y$ is compact-valued and upper hemicontinuous and the correspondence $\Psi: X \rightrightarrows Y$ has closed graph. Let $\Phi \cap \Psi: X \rightrightarrows Y$ be the correspondence defined such that*

$$(\Phi \cap \Psi)(\mathbf{x}) = \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$$

for all $\mathbf{x} \in X$. Then the correspondence Let $\Phi \cap \Psi: X \rightrightarrows Y$ is compact-valued and upper hemicontinuous.

Proof Let

$$W = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : \mathbf{y} \not\in \Psi(\mathbf{x})\}.$$

Then W is the complement of the graph $\mathrm{Graph}(\Psi)$ of Ψ in $X \times Y$. The graph of Ψ is closed in $X \times Y$, by assumption. It follows that W is open in $X \times Y$.

Let $\mathbf{x} \in X$. The subset $\Psi(\mathbf{x})$ of Y is closed in Y, because the graph of the correspondence Ψ is closed. It follows from the compactness of $\Phi(\mathbf{x})$ that $\Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$ is a closed subset of the compact set $\Phi(\mathbf{x})$, and must therefore be compact. Thus the correspondence $\Phi \cap \Psi$ is compact-valued.

Now let $\mathbf{p}$ be a point of X, and let V be any open set in Y for which $\Phi(\mathbf{p}) \cap \Psi(\mathbf{p}) \subset V$. In order to prove that $\Phi \cap \Psi$ is upper hemicontinuous we must show that there exists some positive real number δ such that $\Phi(\mathbf{x}) \cap \Psi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Let

$$U = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : \text{either } \mathbf{y} \in V \text{ or else } \mathbf{y} \not\in \Psi(\mathbf{x})\}.$$

Then U is the union of the subsets $X \times V$ and W of $X \times Y$, where both these subsets are open in $X \times Y$. It follows that U is open in $X \times Y$. Moreover if $\mathbf{y} \in \Phi(\mathbf{p})$ then either $\mathbf{y} \in \Phi(\mathbf{p}) \cap \Psi(\mathbf{p})$, in which case $\mathbf{y} \in V$, or else $\mathbf{y} \not\in \Psi(\mathbf{p})$. It follows that $(\mathbf{p}, \mathbf{y}) \in U$ for all $\mathbf{y} \in \Phi(\mathbf{p})$.

Now it follows from Proposition 4.18 that

$$\{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X. Therefore there exists some positive real number δ such that $(\mathbf{x}, \mathbf{y}) \in U$ for all $(\mathbf{x}, \mathbf{y}) \in X \times Y$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$ and $\mathbf{y} \in \Phi(\mathbf{x})$. Now if $(\mathbf{x}, \mathbf{y})$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ and $\mathbf{y} \in \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$ then $(\mathbf{x}, \mathbf{y}) \in U$ but $(\mathbf{x}, \mathbf{y}) \notin W$. It follows from the definition of U that $\mathbf{y} \in V$. Thus $\Phi(\mathbf{x}) \cap \Psi(\mathbf{x}) \subset V$ whenever $|\mathbf{x} - \mathbf{p}| < \delta$. The result follows. ∎

4.6 Berge's Maximum Theorem

Lemma 4.22 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is both upper hemicontinuous and compact-valued. Let $f\colon X \times Y \to \mathbb{R}$ be a continuous real-valued function on $X \times Y$, and let c be a real number. Then*

$$\{\mathbf{x} \in X : f(\mathbf{x}, \mathbf{y}) < c \textit{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X.

Proof Let

$$U = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : f(\mathbf{x}, \mathbf{y}) < c\}.$$

It follows from the continuity of the function f that U is open in $X \times Y$. It then follows from Proposition 4.18 that

$$\{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X. The result follows. ∎

Lemma 4.23 *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is lower hemicontinuous. Let $f\colon X \times Y \to \mathbb{R}$ be a continuous real-valued function on $X \times Y$, and let c be a real number. Then*

$$\{\mathbf{x} \in X : \textit{ there exists } \mathbf{y} \in \Phi(\mathbf{x}) \textit{ for which } f(\mathbf{x}, \mathbf{y}) > c\}$$

is open in X.

Proof Let

$$U = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : f(\mathbf{x}, \mathbf{y}) > c\},$$

and let

$$W = \{\mathbf{x} \in X : \text{ there exists } \mathbf{y} \in \Phi(\mathbf{x}) \text{ for which } f(\mathbf{x}, \mathbf{y}) > c\},$$

Let $\mathbf{p} \in W$. Then there exists $\mathbf{y} \in \Phi(\mathbf{p})$ for which $(\mathbf{p}, \mathbf{y}) \in U$. There then exist subsets W_X of X and W_Y of Y, where W_X is open in X and W_Y is open in Y, such that $\mathbf{p} \in W_X$, $\mathbf{y} \in W_Y$ and $W_X \times W_Y \subset U$ (see Lemma 4.5). There then exists some positive real number δ_1 such that $\mathbf{x} \in W_X$ whenever $|\mathbf{x} - \mathbf{p}| < \delta_1$.

Now $\Phi(\mathbf{p}) \cap W_Y \neq \emptyset$, because $\mathbf{y} \in \Phi(\mathbf{p}) \cap W_Y$. It follows from the lower hemicontinuity of the correspondence Φ that there exists some positive real number δ_2 such that $\Phi(\mathbf{x}) \cap W_Y \neq \emptyset$ whenever $|\mathbf{x} - \mathbf{p}| < \delta_2$.

Let δ be the minimum of δ_1 and δ_2. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then there exists $\mathbf{y} \in \Phi(\mathbf{x})$ for which $\mathbf{y} \in W_Y$. But then $(\mathbf{x}, \mathbf{y}) \in W_X \times W_Y$ and therefore $(\mathbf{x}, \mathbf{y}) \in U$, and thus $f(\mathbf{x}, \mathbf{y}) > c$. The result follows. ∎

Theorem 4.24 (Berge's Maximum Theorem) *Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $f: X \times Y \to \mathbb{R}$ be a continuous real-valued function on $X \times Y$, and let $\Phi: X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that $\Phi(\mathbf{x})$ is both non-empty and compact for all $\mathbf{x} \in X$ and that the correspondence $\Phi: X \to Y$ is both upper hemicontinuous and lower hemicontinuous. Let the real-valued function $m: X \to \mathbb{R}$ be defined on X such that*

$$m(\mathbf{x}) = \sup\{f(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in \Phi(\mathbf{x})\}$$

for all $\mathbf{x} \in X$, and let the correspondence $M: X \rightrightarrows Y$ be defined such that

$$M(\mathbf{x}) = \{\mathbf{y} \in \Phi(\mathbf{x}) : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}$$

for all $\mathbf{x} \in X$. Then $m: X \to \mathbb{R}$ is continuous, $M(\mathbf{x})$ is a non-empty compact subset of Y for all $\mathbf{x} \in X$, and the correspondence $M: X \rightrightarrows Y$ is upper hemicontinuous.

Proof Let $\mathbf{x} \in X$. Then $\Phi(\mathbf{x})$ is a non-empty compact subset of Y. It is thus a closed bounded subset of $\mathbb{R}^m$. It follows from the Extreme Value Theorem (Theorem 2.20) that there exists at least one point $\mathbf{y}^*$ of $\Phi(\mathbf{x})$ with the property that $f(\mathbf{x}, \mathbf{y}^*) \geq f(\mathbf{x}, \mathbf{y})$ for all $\mathbf{y} \in \Phi(\mathbf{x})$. Then $m(\mathbf{x}) = f(\mathbf{x}, \mathbf{y}^*)$ and $\mathbf{y}^* \in M(\mathbf{x})$. Moreover

$$M(\mathbf{x}) = \{\mathbf{y} \in \Phi(\mathbf{x}) : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}.$$

It follows from the continuity of f that the set $M(\mathbf{x})$ is closed in Y (see Corollary 2.18). It is thus a closed subset of the compact set $\Phi(\mathbf{x})$ and must therefore itself be compact.

Let some positive number ε be given. Then $f(\mathbf{p}, \mathbf{y}) < m(\mathbf{p}) + \varepsilon$ for all $\mathbf{y} \in \Phi(\mathbf{p})$. It follows from Lemma 4.22 that

$$\{\mathbf{x} \in X : f(\mathbf{x}, \mathbf{y}) < m(\mathbf{p}) + \varepsilon \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X, and thus there exists some positive real number δ_1 such that $f(\mathbf{x},\mathbf{y}) < m(\mathbf{p}) + \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x}-\mathbf{p}| < \delta_1$ and $\mathbf{y} \in \Phi(\mathbf{x})$ Then $m(\mathbf{x}) < m(\mathbf{p}) + \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x}-\mathbf{p}| < \delta_1$.

The correspondence $\Phi\colon X \rightrightarrows Y$ is also lower hemicontinuous. It therefore follows from Lemma 4.23 that there exists some positive real number δ_2 such that, given any $\mathbf{x} \in X$ satisfying $|\mathbf{x}-\mathbf{p}| < \delta_2$, there exists some $\mathbf{y} \in \Phi(\mathbf{x})$ for which $f(\mathbf{x},\mathbf{y}) > m(\mathbf{p}) - \varepsilon$. It follows that $m(\mathbf{x}) > m(\mathbf{p}) - \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x}-\mathbf{p}| < \delta_2$.

Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and

$$m(\mathbf{p}) - \varepsilon < m(\mathbf{x}) < m(\mathbf{p}) + \varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x}-\mathbf{p}| < \delta$. Thus the function $m\colon X \to \mathbb{R}$ is continuous on X.

It only remains to prove that the correspondence $M\colon X \rightrightarrows Y$ is upper hemicontinuous. Let

$$\Psi(\mathbf{x}) = \{\mathbf{y} \in Y : f(\mathbf{x},\mathbf{y}) = m(\mathbf{x})\}$$

for all $\mathbf{x} \in X$. Then

$$\mathrm{Graph}(\Psi) = \{(\mathbf{x},\mathbf{y}) \in X \times Y : f(\mathbf{x},\mathbf{y}) = m(\mathbf{x})\}$$

Thus $\mathrm{Graph}(\Psi)$ is the preimage of zero under the continuous real-valued function that sends $(\mathbf{x},\mathbf{y}) \in X \times Y$ to $f(\mathbf{x},\mathbf{y}) - m(\mathbf{x})$. It follows that $\mathrm{Graph}(\Psi)$ is a closed subset of $X \times Y$.

Now $M(\mathbf{x}) = \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$ for all $\mathbf{x} \in X$, where the correspondence Φ is compact-valued and upper hemicontinuous and the correspondence Ψ has closed graph. It follows from Proposition 4.21 that the correspondence M must itself be both compact-valued and upper hemicontinuous. This completes the proof of Berge's Maximum Theorem. ∎

We describe another proof of the Berge Maximum Theorem using the characterization of compact-valued upper hemicontinuous correspondences using sequences established in Proposition 4.17 and the characterization of lower hemicontinuous correspondences using sequences established in Proposition 4.20. First we introduce some terminology.

Definition Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Let $(\mathbf{x}_j : j \in \mathbb{N})$ be a sequence of points of the domain X of the correspondence. We say that an infinite sequence $(\mathbf{y}_j : j \in \mathbb{N})$ in the codomain of the correspondence is a *companion sequence* for $(\mathbf{x}_j)$ with respect to the correspondence Φ if $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j.

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\Phi: X \rightrightarrows Y$ be a correspondence from X to Y. Then the continuity properties of $\Phi: X \rightrightarrows Y$ can be characterized in terms of companion sequences with respect to Φ as follows:—

- the correspondence $\Phi: X \rightrightarrows Y$ is compact-valued and upper hemicontinuous at a point $\mathbf{p}$ of X if and only if, given any infinite sequence $(\mathbf{x}_j : j \in \mathbb{N})$ in X converging to the point $\mathbf{p}$, and given any companion sequence $(\mathbf{y}_j : j \in \mathbb{N})$ in Y, that companion sequence has a subsequence that converges to a point of $\Phi(\mathbf{p})$ (Proposition 4.17);

- the correspondence $\Phi: X \rightrightarrows Y$ is lower hemicontinuous at a point $\mathbf{p}$ of X if and only if, given any infinite sequence $(\mathbf{x}_j : j \in \mathbb{N})$ in X converging to the point $\mathbf{p}$, and given any point $\mathbf{q}$ of $\Phi(\mathbf{p})$, there exists a companion sequence $(\mathbf{y}_j : j \in \mathbb{N})$ in Y converging to the point $\mathbf{q}$. (Proposition 4.20).

Proof of Theorem 4.24 using Companion Sequences Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $f: X \times Y \to \mathbb{R}$ be a continuous real-valued function on $X \times Y$, and let $\Phi: X \rightrightarrows Y$ be a correspondence from X to Y that is both upper and lower hemicontinuous and that also has the property that $\Phi(\mathbf{x})$ is non-empty and compact for all $\mathbf{x} \in X$. Let

$$m(\mathbf{x}) = \sup\{f(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in \Phi(\mathbf{x})\}$$

for all $\mathbf{x} \in X$, and let the correspondence $M: X \rightrightarrows Y$ be defined such that

$$M(\mathbf{x}) = \{\mathbf{y} \in \Phi(\mathbf{x}) : f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}$$

for all $\mathbf{x} \in X$. We must prove that $m: X \to \mathbb{R}$ is continuous, $M(\mathbf{x})$ is a non-empty compact subset of Y for all $\mathbf{x} \in X$, and the correspondence $M: X \rightrightarrows Y$ is upper hemicontinuous.

It follows from the continuity of $f: X \times Y \to \mathbb{R}$ that $M(\mathbf{x})$ is closed in $\Phi(\mathbf{x})$ for all $\mathbf{x} \in X$. It also follows from the Extreme Value Theorem (Theorem 2.20) that $M(\mathbf{x})$ is non-empty for all $\mathbf{x}$.

Let $(\mathbf{x}_j, j \in \mathbb{N})$ be a sequence in X which converges to a point $\mathbf{p}$ of X, and let $(\mathbf{y}_j^* : j \in \mathbb{N})$ be a companion sequence of $(\mathbf{x}_j)$ with respect to the correspondence M. Then, for each positive integer j, $\mathbf{y}_j^* \in \Phi(\mathbf{x}_j)$ and

$$f(\mathbf{x}_j, \mathbf{y}_j^*) \geq f(\mathbf{x}_j, \mathbf{y})$$

for all $\mathbf{y} \in \Phi(\mathbf{x}_j)$. Now the correspondence Φ is compact-valued and upper hemicontinuous. It follows from Proposition 4.17 that there exists a subsequence of $(\mathbf{y}_j^* : j \in \mathbb{N})$ that converges to an element $\mathbf{q}$ of $\Phi(\mathbf{q})$. Let that subsequence be the sequence $(\mathbf{y}^*_{k_j} : j \in \mathbb{N})$ whose members are

$$\mathbf{y}^*_{k_1}, \mathbf{y}^*_{k_2}, \mathbf{y}^*_{k_3}, \ldots,$$

where $k_1 < k_2 < k_3 < \cdots$. Then $\mathbf{q} = \lim_{j \to +\infty} \mathbf{y}^*_{k_j}$.

We show that $\mathbf{q} \in M(\mathbf{p})$. Let $\mathbf{r} \in \Phi(\mathbf{p})$. The correspondence $\Phi: X \to Y$ is lower hemicontinuous. It follows that there exists a companion sequence $(\mathbf{z}_j : j \in N)$ to $(\mathbf{x}_j : j \in N)$ with respect to the correspondence Φ that converges to $\mathbf{r}$ (Proposition 4.20). Then

$$\lim_{j \to +\infty} \mathbf{y}^*_{k_j} = \mathbf{q} \quad \text{and} \quad \lim_{j \to +\infty} \mathbf{z}_{k_j} = \mathbf{r}.$$

It follows from the continuity of $f: X \times Y \to \mathbb{R}$ that

$$\lim_{j \to +\infty} f(\mathbf{x}_{k_j}, \mathbf{y}^*_{k_j}) = f(\mathbf{p}, \mathbf{q}) \quad \text{and} \quad \lim_{j \to +\infty} f(\mathbf{x}_{k_j}, \mathbf{z}_{k_j}) = f(\mathbf{p}, \mathbf{r}).$$

Now

$$f(\mathbf{x}_{k_j}, \mathbf{y}^*_{k_j}) \geq f(\mathbf{x}_{k_j}, \mathbf{z}_{k_j})$$

for all positive integers j, because $\mathbf{y}^*_{k_j} \in M(\mathbf{x}_{k_j})$. It follows that

$$f(\mathbf{p}, \mathbf{q}) = \lim_{j \to +\infty} f(\mathbf{x}_{k_j}, \mathbf{y}^*_{k_j}) \geq \lim_{j \to +\infty} f(\mathbf{x}_{k_j}, \mathbf{z}_{k_j}) = f(\mathbf{p}, \mathbf{r}).$$

Thus $f(\mathbf{p}, \mathbf{q}) \geq f(\mathbf{p}, \mathbf{r})$ for all $\mathbf{r} \in \Phi(\mathbf{p})$. It follows that $\mathbf{q} \in M(\mathbf{p})$.

We have now shown that, given any sequence $(\mathbf{x}_j : j \in \mathbb{R})$ in X converging to the point $\mathbf{p}$, and given any companion sequence $(\mathbf{y}^*_j : j \in \mathbb{R})$ with respect to the correspondence M, there exists a subsequence of $(\mathbf{y}^*_j : j \in \mathbb{R})$ that converges to a point of $M(\mathbf{x})$. It follows that the correspondence $M: X \to Y$ is compact-valued and upper hemicontinuous at the point $\mathbf{p}$ (Proposition 4.17).

It remains to show that the function $m: X \to \mathbb{R}$ is continuous at the point $\mathbf{p}$, where $m(\mathbf{x}) = f(\mathbf{x}, \mathbf{y}^*)$ for all $\mathbf{x} \in X$ and $\mathbf{y}^* \in M(\mathbf{x})$. Let $(\mathbf{x}_j : j \in \mathbb{R})$ be an infinite sequence converging to the point $\mathbf{p}$, and let $v_j = m(\mathbf{x}_j)$ for all positive integers j. Then there exists an infinite sequence Let $(\mathbf{y}^*_j : j \in \mathbb{R})$ in Y that is a companion sequence to $(\mathbf{x}_j)$ with respect to the correspondence M. Then $\mathbf{y}^*_j \in M(\mathbf{x}_j)$ and therefore $v_j = f(\mathbf{x}_j, \mathbf{y}^*_j)$ for all positive integers j. Now the correspondence $M: X \rightrightarrows Y$ has been shown to be compact-valued and upper hemicontinuous. There therefore exists a subsequence $(\mathbf{y}^*_{k_j} : j \in \mathbb{N})$ of $(\mathbf{y}_j)$ that converges to a point $\mathbf{q}$ of $M(\mathbf{p})$. It then follows from the continuity of the function $f: X \times Y \to \mathbb{R}$ that

$$\lim_{j \to +\infty} m(\mathbf{x}_{k_j}) = \lim_{j \to +\infty} v_{k_j} = \lim_{j \to +\infty} f(\mathbf{x}_{k_j}, \mathbf{y}^*_{k_j}) = f(\mathbf{p}, \mathbf{q}) = m(\mathbf{p}).$$

Now the result just proved can be applied with any subsequence of $(\mathbf{x}_j : j \in \mathbb{N})$ in place of the original sequence. It follows that *every subsequence* of of $(v_j : j \in \mathbb{R})$ itself has a subsequence that converges to $m(\mathbf{p})$.

Let some positive real number ε be given. Suppose that there did not exist any positive integer N with the property that $|v_j - m(\mathbf{p})| < \varepsilon$ whenever $j \geq N$. Then there would exist infinitely many positive integers j for which $|v_j - m(\mathbf{p})| \geq \varepsilon$. It follows that there would exist some subsequence

$$v_{l_1}, v_{l_2}, v_{l_3}, \ldots$$

of $v_1, v_2, v_3, \ldots$ with the property that $|v_{l_j} - m(\mathbf{p})| \geq \varepsilon$ for all positive integers j. This subsequence would not in turn contain any subsequences converging to the point $m(\mathbf{p})$.

But we have shown that every subsequence of $(v_j : j \in \mathbb{N})$ contains a subsequence converging to $m(\mathbf{p})$. It follows that there must exist some positive integer N with the property that $|v_j - m(\mathbf{p})| < \varepsilon$ whenever $j \geq N$. We conclude from this that $\lim\limits_{j \to +\infty} m(\mathbf{x}_j) = m(\mathbf{p})$.

We have shown that if $(\mathbf{x}_j : j \in \mathbb{N})$ is an infinite sequence in X and if $\lim\limits_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$ then $\lim\limits_{j \to +\infty} m(\mathbf{x}_j) = m(\mathbf{p})$. It follows that the function $m: X \to \mathbb{R}$ is continuous at $\mathbf{p}$. This completes the proof of Berge's Maximum Theorem. ∎

Remark In 1959, the French mathematician Claude Berge published a book entitled *Espaces topologiques: fonctions multivoques* (Dunod, Paris, 1959). This book was subsequently translated into English by E.M. Patterson, and the translation was published with the title *Topological spaces, including a treatment of multi-valued functions, vector spaces and convexity* (Oliver and Boyd, Edinburgh and London, 1963).

Claude Berge had completed his Ph.D. at the University of Paris in 1953, supervised by the differential geometer and mathematical physicist *André Lichnerowicz*. His thesis was entitled *Sur une théorie ensembliste des jeux alternatifs*, and a paper of that name was published by him (*J. Math. Pures Appl.* **32** (1953), 129–184). He subsequently published *Théorie Générale des Jeux à N Personnes* (Gauthier Villars, Paris, 1957). The title translates as "General theory of n-person games".

Claude Berge was Professor at the Institute of Statistics at the University of Paris from 1957 to 1964, and subsequently directed the International Computing Center in Rome. Following his early work in game theory, his research developed in the fields of combinatorics and graph theory.

The preface of the 1959 book, *Espaces topologiques: fonctions multivoques*, includes a passage translated by E.M. Patterson as follows:—

> In Set Topology, with which we are concerned in this book, we study sets in topological spaces and topological vector spaces; whenever these sets are colletions of n-tuples or classes of functions, we recover well-known results of classical analysis.
>
> But the role of topology does not stop there; the majority of text-books seem to ignore certain problems posed by the calculus of probabilities, the decision functions of statistics, linear programming, cybernetics, economics; thus, in order to provide a topological tool which is of equal interest to the student of pure mathematics and the student of applied mathematics, we have felt it desirable to include a systematic devcelopment of the properties of *multi-valued functions.*

The following theorem is included in *Espaces topologiques* by Claude Berge (Chapter 6, Section 3, page 122):—

> **Théorème du maximum.** — Si $\varphi(y)$ est une fonction numérique continue dans Y, et si Γ est un application continue de X dans Y telle que $\Gamma x \neq \emptyset$ pour tout x,
>
> $$M(x) = \max\{\varphi(y)/y \in \Gamma x\}$$
>
> est une fonction numérique continue dans X, et
>
> $$\Phi x = \{y/y \in \Gamma x, \varphi(y) = M(x)\}$$
>
> est une application u.s.c. de X dans Y.

This theorem is translated by E.M. Patterson as follows (*Topological Spaces*, Claude Berge, translated by E.M. Patterson, Oliver and Boyd, Edinburgh, 1963, in Chapter 6, Section 3, page 116):—

> **Maximum Theorem** — If φ is a continuous numerical function in Y and Γ is a continuous mapping of X into Y such that, for each x, $\Gamma x \neq \emptyset$, then the numerical function M defined by
>
> $$M(x) = \max\{\varphi(y)/y \in \Gamma x\}$$
>
> is continuous in X and the mapping Φ defined by
>
> $$\Phi x = \{y/y \in \Gamma x, \varphi(y) = M(x)\}$$
>
> is an u.s.c. mapping of X into Y.

In this context X and Y are Hausdorff topological spaces. Indeed in Chapter 4, Section 5 of *Espaces topologiques*, Berge introduces the concept of a *separated* (or *Hausdorff*) space and then, after some discussion of separation properties, makes that statement translated by E.M. Patterson as follows:—

> In what follows all the topological spaces which we consider will be assumed to be separated.

It seems that, in the original statement, the objective function φ was required to be a continuous function on Y, but the first sentence of the proof of the "Maximum Theorem" notes that φ is continuous on $X \times Y$. A "mapping" in Berge is a correspondence. A mapping (or correspondence) is said by Berge to be "upper semi-continuous" when it is both compact-valued and upper hemicontinuous; a mapping is said by Berge to be "lower semi-continuous" when it is lower hemicontinuous.

Berge's proof of the *Théorème du maximum* is just one short paragraph, but requires the work of earlier theorems. We discuss his proof using the terminology adopted in these lectures. In Theorem 1 of Chapter 6, Section 4, Berge shows that if the correpondence $\Gamma\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous then, given any point x_0 of X, and given any positive real number ε, the function $M(x)$ equal to the maximum value of the objective function ϕ on $\Gamma(x)$ satisfies $M(x) \leq M(x_0) + \varepsilon$ throughout some open neighbourhood of the point x_0. (This result can be compared with Lemma 4.22 and the first proof of Theorem 4.24 presented in these notes.) In Theorem 2 of Chapter 6, Section 4, Berge shows that if the correspondence Γ is lower hemicontinuous then, given any point x_0 of X, and given any positive real number ε, the function $M(x)$ equal to the maximum value of the objective function ϕ on $\Gamma(x)$ satisfies $M(x) \geq M(x_0) - \varepsilon$ throughout some open neighbourhood of the point x_0.

(This result can be compared with Lemma 4.23 and the first proof of Theorem 4.24 presented in these notes.) These two results ensure that if Γ is compact-valued, everywhere non-empty and both upper and lower hemicontinuous then the function function M is continuous on X. In Theorem 7 of Chapter 6, Section 1, Berge had proved that the intersection of a compact-valued upper hemicontinuous correspondence and a correspondence with closed graph is compact valued and upper hemicontinuous (see Proposition 4.21 of these notes). Berge completes the proof of the *Théorème du maximum* by putting these results together in a fashion to obtain a proof (in the contexts of correspondences between Hausdorff topological spaces) similar in structure to the first proof of Theorem 4.24 presented in these notes.

The definitions of "upper-semicontinuous" and "lower-semicontinuous" mappings (i.e., correspondences) Given by Claude Berge at the beginning of Chapter VI are accompanied by a footnote translated by E.M. Patterson as follows (C. Berge, translated E.M. Patterson, *Topological Spaces*, *loc. cit.*, p. 109):—

> The two kinds of semi-continuity of a multivalued function were introduced independently by Kuratowski (*Fund. Math.* **18**, 1932, p.148) and Bouligand (*Ens. Math.*, 1932, p. 14). In general, the definitions given by different authors do not coincide whenever we deal with non-compact spaces (at least for upper semi-continuity, which is the more important from the point of view of applications). The definitions adopted here, which we have developed elsewhere (C. Berge, *Mém. Sc. Math.* 138), enable us to include the case when the image of a point x can be empty.

In 1959, the year in which Claude Berge published *Espaces topologiques*, Gérard Debreu published his influential monograph *Theory of value: an axiomatic analysis of economic equilibrium* (Cowles Foundation Monographs 17, 1959). Section 1.8 of Debreu's monograph discusses "continuous correspondences", developing the theory of correspondences φ from S to T, where S is a subset of $\mathbb{R}^m$ and T is a compact subset of $\mathbb{R}^n$. Debreu also requires correspondences to be non-empty-valued. In consequence of these conventions, closed-valued correspondences from S to T must necessarily be compact-valued. Also a correspondence from S to T is upper hemicontinuous if and only if its graph is closed (see Propositions 4.11 4.12 of these notes).

In the notes to Chapter 1 of the *Theory of Value*, Debreu notes that "a study of the *continuity of correspondences* from a topological space to a topological space will be found in C. Berge [1], Chapter 6". The reference is to *Espace Topologiques.*

According to Debreu, the correspondence φ is *upper semicontinuous* at the point x^0 if the following condition is satisfied:

$$\text{“}x^q \to x^0,\ y^q \in \varphi(x^q),\ y^q \to y^0\text{” implies “}y^0 \in \varphi(x^0)\text{”.}$$

This condition is satisfied at each point of the domain of a correspondence if and only if that correspondence has closed graph. Thus Debreu's definition is in accordance with the definition of *upper hemicontinuity* for those correspondences, and only those correspondences, where the codomain of the correspondence is a compact subset of a Euclidean space. Indeed Debreu notes the following in Section 1.8 of the *Theory of Value*:—

"(1) *The correspondence φ is upper semicontinuous on S if and only if its graph is closed in $S \times T$.*"

Again according to Debreu, the correspondence φ is *lower semicontinuous* at the point x^0 if the following condition is satisfied:

"$x^q \to x^0$, $y^0 \in \varphi(x^0)$" implies "there is (y^q) such that $y^q \in \varphi(x^q)$, $y^q \to y^0$".

This condition is satisfied at each point of the domain of a correspondence if and only if that correspondence is lower hemicontinuous (in accordance with the definitions adopted in those notes, see Proposition 4.20 of these notes).

A correspondence from S to T is said by Debreu to be continuous if it is both upper semicontinuous and lower semicontinuous according to his definitions.

Debreu discusses Berge's Maximum Theorem, in the context of a correspondence φ from a subset S of $\mathbb{R}^m$ to a compact subset T of $\mathbb{R}^n$, as follows (*Theory of Value*, Section 1.8, page 19):—

> The interest of these concepts for economics lies, in particular, in the interpretations of an element x of S as the environment of a certain agent, of T as the set of actions *a priori* available to him, and of $\varphi(x)$ (assumed here to be closed for every x in S) as the subset of T to which his choice is actually restricted by the environment x. Let f be a continuous *real-valued function on $S \times T$*, and interpret $f(x, y)$ as the gain for that agent when his environment is x and his action y. *Given x*, one is interested in the *elements of $\varphi(x)$ which maximize f* (now *a function of y alone*) on $\varphi(x)$; they form a set $\mu(x)$. What can be said about the continuity of the correspondence μ from S to T?
>
> One is also interested in $g(x)$, *the value of the maximum of f on $\phi(x)$ for a given x*. What can be said about the continuity of the real-valued function g on S? An answer to these two questions is given by the following result (the proof of the continuity of g should not be attempted).
> (4) If f is continuous on $S \times T$, and if φ is continuous at $x \in S$, then μ is upper semicontinuous at x, and g is continuous on x.

(Note that, because the set T is compact and Debreu requires correspondences to be non-empty valued, the conventions adopted by Debreu in his *Theory of Value* ensure that if φ is an "upper semicontinuous" correspondence from a set S to a compact set T, where S and T are subsets of

Euclidean spaces, then $\varphi(x)$ will necessarily be a non-empty compact subset of T.)

The book *Infinite dimensional analysis: a hitchhiker's guide* by Charalambos D. Aliprantis and Kim C. Border (2nd edition, Springer-Verlag, 1999) discusses the theory of continuous correspondences between topological spaces (Chapter 16). Berge's Maximum Theorem is stated and proved, in the context of correspondences between topological spaces, as Theorem 16.31 (p. 539). The definitions of upper hemicontinuity and lower hemicontinuity for correspondences are consistent with the definitions adopted in these lecture notes. These definitions are accompanied by the following footnote:—

> J. C. Moore [...] identifies five slightly different definitions of upper semicontinuity in use by economists, and points out some of the differences for compositions, etc. T. Ichiishi [...] and E. Klein and A. C. Thompson [...] also give other notions of continuity.

The book *Mathematical Methods and Models for Economists* by Angel de la Fuente (Cambridge University Press, 2000) includes a section on continuity of correspondences between subsets of Euclidean spaces (Chapter 2, Section 11). The definitions of upper and lower hemicontinuity adopted there are consistent with those given in these lecture notes. The sequential characterization of compact-valued upper hemicontinuous correspondences in terms of companion sequences (Proposition 4.17 of these lecture notes) is stated and proved as Theorem 11.2 of Chapter 2 of Angel de la Fuente's textbook. Similarly the sequencial characterization of lower hemicontinuous correspondences in terms of companion sequences Proposition 4.20 is stated and proved as Theorem 11.3 of that textbook.

Theorem 11.6 in Chapter 2 of that textbook covers the result that a closed-valued upper hemicontinuous correspondence has a closed graph (see Proposition 4.11) and the result that a correspondence with closed graph whose codomain is compact is upper hemicontinuous (see Proposition 4.12). The result that the intersection of a compact-valued upper hemicontinuous correspondence and a correspondence with closed graph is compact-valued and upper hemicontinuous (see Proposition 4.21) is Theorem 11.7 in Chapter 2 of the textbook by Angel de la Fuente. Berge's Maximal Theorem is Theorem 2.1 in Chapter 7 of that textbook. The proof is based on the use of the sequential characterizations of upper and lower hemicontinuity in terms of existence and properties of companion sequences.