Module MA3484: Annual Examination 2019
Worked solutions

David R. Wilkins

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Module Website

The module website, with online lecture notes, problem sets. etc. are located at

`http://www.maths.tcd.ie/~dwilkins/Courses/MA3484/`

Note

Previous examinations in this module have include a transportation problem similar to question 1, and a simplex method problem similar to question 3, part (b). In the case of question 3 they will have been told that, in previous years, the simplex method problem could be solved by any one of three methods developed in the lecture notes. The structure of these questions is intended to simply present the bare bones of a problem of the relevant type, without constraining students to a particular method of solution.

1. [Seen similar.]

[Note: the following worked solution reproduces the detail of a large number of small steps to explain, demonstrate and validate the reasoning. A solution with this amount of detail would not be expected in a solution written out during the course of an examination.]

We now find a basic optimal solution to a transportation problem with 4 suppliers and 3 recipients. We find an initial basic feasible solution using the Minimum Cost Method, and then continue to find a basic optimal solution using a form of the Simplex Method adapted to the Transportation Problem.

The supply vector is $(12, 8, 14, 16)$. The demand vector is $(18, 19, 13)$. The components of both the supply vector and the demand vector add up to 50.

The costs are as specified in the following cost matrix:—

$$\begin{pmatrix} 4 & 2 & 5 \\ 10 & 8 & 9 \\ 7 & 6 & 10 \\ 11 & 12 & 10 \end{pmatrix}.$$

We find an initial basic feasible solution using the Minimum Cost Method, in which we select a cell with minimum cost for which $x_{i,j}$ is as yet undetermined, then fill in the cell with the minimum of the supply s_i and demand d_j. Then either one column, or one row is completed with zeros. This reduces to a transportation-type problem of smaller size, and the Minimum Cost Method is applied recursively until the initial basic feasible solution has been found.

The cell $(1, 2)$ is first selected, as being that with minimum cost, and $(1, 2)$ becomes the first selected element of the basis B The corresponding variable $x_{1,2}$ is increased to its maximum possible value, which is 12, and row 1 is padded out with zeros in cells $(1, 1)$ and $(1, 3)$.

Next the cell $(3, 2)$ is selected, as being the undetermined cell with minimum cost, and $(3, 2)$ is added to the basis B. The corresponding variable $x_{3,2}$ is increased to its maximum possible value, which is 7, and column 2 is padded out with zeros in cells $(2, 2)$ and $(4, 2)$.

Next the cell $(3, 1)$ is selected, as being the undetermined cell with minimum cost, and $(3, 1)$ is added to the basis B. The corresponding variable $x_{3,1}$ is increased to its maximum possible value, which is 7, and row 3 is padded out with a zero in cell $(3, 3)$.

Next the cell $(2, 3)$ is selected, as being the undetermined cell with minimum cost, and $(2, 3)$ is added to the basis B. The corresponding variable $x_{2,3}$ is increased to its maximum possible value, which is 8, and row 2 is padded out with a zero in cells $(2, 1)$.

Next the cell $(4, 3)$ is selected, as being the undetermined cell with minimum cost, and $(4, 3)$ is added to the basis B. The corresponding variable $x_{4,3}$ is increased to its maximum possible value, which is 5.

Finally the cell $(4, 1)$ is selected, as being the undetermined cell with minimum cost, and $(4, 1)$ is added to the basis B. The corresponding variable $x_{4,1}$ is given its only possible value, which is 11.

The completed tableau after determining the initial basic solution by the Minimum Cost Method is as follows:—

$c_{i,j} \searrow x_{i,j}$				s_i
	4 0	2 • 12	5 0	12
	10 0	8 0	9 • 8	8
	7 • 7	6 • 7	10 0	14
	11 • 11	12 0	10 • 5	16
d_j	18	19	13	50

Our initial basic feasible solution is thus specified by the 4×3 matrix X, where

$$X = \begin{pmatrix} 0 & 12 & 0 \\ 0 & 0 & 8 \\ 7 & 7 & 0 \\ 11 & 0 & 5 \end{pmatrix}.$$

The initial basis B for the transportation problem is as follows:—

$$B = \{(1, 2), (3, 2), (3, 1), (2, 3), (4, 3), (4, 1)\}.$$

The basis has six elements as expected. (The number of basis elements should be $m+n-1$, where m is the number of suppliers and n is the number of recipients.)

The cost of the initial basic feasible solution is 358, as

$$\begin{aligned} & 2\times 12+9\times 8+7\times 7+6\times 7+11\times 11+10\times 5 \\ = \quad & 24+72+49+42+121+50=358. \end{aligned}$$

The next stage is to determine real numbers u_i and v_j for $i = 1,2,3,4$ and $j = 1,2,3$ to satisfy the following conditions: $c_{i,j} = v_j - u_i + q_{i,j}$ for all i and j; $q_{i,j} = 0$

The solution is exemplified in the following tableau:

$c_{i,j} \searrow q_{i,j}$							u_i
	4		2	$\bullet$	5		0
		1		0		3	
	10		8		9	$\bullet$	-7
		0		-1		0	
	7	$\bullet$	6	$\bullet$	10		-4
		0		0		4	
	11	$\bullet$	12		10	$\bullet$	-8
		0		2		0	
v_j	3		2		2		

The initial basic feasible solution is not optimal because one of the quantities $q_{i,j}$ is negative. Indeed $q_{2,2} = -1$, We therefore seek to bring $(2,2)$ into the basis.

The procedure for achieving this requires us to determine a 4×3 matrix Y satisfying the following conditions:—

- $y_{2,2} = 1$;
- $y_{i,j} = 0$ when $(i,j) \notin B \cup \{(2,2)\}$;
- all rows and columns of the matrix Y sum to zero.

The unique solution is evident from the following tableau with those coefficients $y_{i,j}$ of the matrix Y that correspond to cells in the current basis (marked with the $\bullet$ symbol), so that all rows sum to zero and all columns sum to zero:—

$y_{i,j}$	1		2		3		
1			0	$\bullet$			0
2			1	$\circ$	-1	$\bullet$	0
3	1	$\bullet$	-1	$\bullet$			0
4	-1	$\bullet$	0	$\bullet$	1		0
	0		0		0		0

The following 4×3 matrix Y therefore satisfies our requirements:—

$$Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Now $X + \lambda Y$ is a feasible solution of the given transportation problem for all values of λ for which the coefficients are all non-negative. Now

$$X + \lambda Y = \begin{pmatrix} 0 & 12 & 0 \\ 0 & \lambda & 8-\lambda \\ 7+\lambda & 7-\lambda & 0 \\ 11-\lambda & 0 & 5+\lambda \end{pmatrix}.$$

We can increase λ, decreasing the cost by λ up to $\lambda = 7$. This gives us a new basic feasible solution, which we take to be the current basic feasible solution.

Let X now denote the current basic feasible solution after the first iteration, and let B now denote the associated basis. Then

$$X = \begin{pmatrix} 0 & 12 & 0 \\ 0 & 7 & 1 \\ 14 & 0 & 0 \\ 4 & 0 & 12 \end{pmatrix}.$$

and

$$B = \{(1,2), (2,2), (2,3), (3,1), (4,1), (4,3)\}.$$

We next compute the numbers u_i and v_j and $q_{i,j}$ so that $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$ and $c_{i,j} = v_j - u_i + q_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$.

We following tableau exhibits the values of $q_{i,j}$ corresponding to the basis obtained at the first iteration:

$c_{i,j} \searrow q_{i,j}$				u_i
	4 0	2 $\bullet$ 0	5 2	0
	10 0	8 $\bullet$ 0	9 $\bullet$ 0	-6
	7 $\bullet$ 0	6 1	10 4	-3
	11 $\bullet$ 0	12 3	10 $\bullet$ 0	-7
v_j	4	2	3	

The fact that all $q_{i,j}$ are non-negative ensures that the current feasible solution is optimal.

Indeed let $\overline{x}_{i,j}$ be the components of a feasible solution to the problem. Then

$$\sum_{i=1}^{4}\sum_{j=1}^{3} c_{i,j}\overline{x}_{i,j} = \sum_{j=1}^{3} v_j d_j - \sum_{i=1}^{4} u_i s_i + \sum_{i=1}^{4}\sum_{j=1}^{3} q_{i,j}\overline{x}_{i,j}.$$

The last summand is always non-negative, and is equal to zero for the current feasible solution.

The cost of the this optimal solution is 351, as

$$\begin{aligned} 2 \times 12 + 8 \times 7 + 9 \times 1 + 7 \times 14 + 11 \times 4 + 10 \times 12 \\ = \quad 24 + 56 + 9 + 98 + 44 + 120 = 351. \end{aligned}$$

2. [Bookwork.] Let m and n be positive integers, and let let the $m \times n$ matrix X represent a feasible solution of a transportation problem with supply vector $\mathbf{s}$, demand vector $\mathbf{d}$ and cost matrix C, where C is an $m \times n$ matrix with real coefficients. Then $s_i \geq 0$ for $i = 1, 2, \ldots, m$ and $d_j \geq 0$ for $j = 1, 2, \ldots, n$, where

$$\mathbf{s} = (s_1, s_2, \ldots, s_m), \quad \mathbf{d} = (d_1, d_2, \ldots, d_n).$$

Also $x_{i,j} \geq 0$ for all i and j, $\sum_{j=1}^{n} x_{i,j} = s_i$ for $i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} x_{i,j} = d_j$ for $j = 1, 2, \ldots, n$. The cost of the feasible solution X is then $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}$, where $c_{i,j}$ is the coefficient in the ith row and jth column of the cost matrix C.

If the feasible solution X is itself basic then there is nothing to prove. Suppose therefore that X is not a basic solution. We show that there then exists a feasible solution $\overline{X}$ with fewer non-zero components than the given feasible solution.

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let

$$K = \{(i, j) \in I \times J : x_{i,j} > 0\}.$$

Because X is not a basic solution to the Transportation Problem, there does not exist any basis B for the transportation problem satisfying $K \subset B$. Therefore there exists a non-zero $m \times n$ matrix Y whose coefficients $y_{i,j}$ satisfy the following conditions:—

- $\sum_{j=1}^{n} y_{i,j} = 0$ for $i = 1, 2, \ldots, m$;
- $\sum_{i=1}^{m} y_{i,j} = 0$ for $j = 1, 2, \ldots, n$;
- $y_{i,j} = 0$ when $(i, j) \notin K$.

We can assume without loss of generality that $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} y_{i,j} \geq 0$, where the quantities $c_{i,j}$ are the coefficients of the cost matrix C, because otherwise we can replace Y with $-Y$.

Let $Z_\lambda = X - \lambda Y$ for all real numbers λ, and let $z_{i,j}(\lambda)$ denote the coefficient $(Z_\lambda)_{i,j}$ in the ith row and jth column of the matrix Z_λ. Then $z_{i,j}(\lambda) = x_{i,j} - \lambda y_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Moreover

- $\sum_{j=1}^{n} z_{i,j}(\lambda) = s_i$;
- $\sum_{i=1}^{m} z_{i,j}(\lambda) = d_j$;
- $z_{i,j}(\lambda) = 0$ whenever $(i,j) \notin K$;
- $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j} z_{i,j}(\lambda) \leq \sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j} x_{i,j}$ whenever $\lambda \geq 0$.

Now the matrix Y is a non-zero matrix whose rows and columns all sum to zero. It follows that at least one of its coefficients must be strictly positive. Thus there exists at least one ordered pair (i,j) belonging to the set K for which $y_{i,j} > 0$. Let

$$\lambda_0 = \text{minimum} \left\{ \frac{x_{i,j}}{y_{i,j}} : (i,j) \in K \text{ and } y_{i,j} > 0 \right\}.$$

Then $\lambda_0 > 0$. Moreover if $0 \leq \lambda < \lambda_0$ then $x_{i,j} - \lambda y_{i,j} > 0$ for all $(i,j) \in K$, and if $\lambda > \lambda_0$ then there exists at least one element (i_0, j_0) of K for which $x_{i_0,j_0} - \lambda y_{i_0,j_0} < 0$. It follows that $x_{i,j} - \lambda_0 y_{i,j} \geq 0$ for all $(i,j) \in K$, and $x_{i_0,j_0} - \lambda_0 y_{i_0,j_0} = 0$.

Thus Z_{λ_0} is a feasible solution of the given transportation problem whose cost does not exceed that of the given feasible solution X. Moreover Z_{λ_0} has fewer non-zero components than the given feasible solution X.

If Z_{λ_0} is itself a basic feasible solution, then we have found the required basic feasible solution whose cost does not exceed that of the given feasible solution. Otherwise we can iterate the process until we arrive at the required basic feasible solution whose cost does not exceed that of the given feasible solution.

3. (a) [Unseen.] A vector $\mathbf{x}$ of the form $(0, x_2, 0, x_4, 0)$ satisfies the matrix equation $A\mathbf{x} = \mathbf{b}$, where A is the constraint matrix and $\mathbf{b}$ is the target vector, if and only if

$$\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \end{pmatrix}.$$

But then

$$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} -4 & 3 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} 9 \\ 5 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} -21 \\ 35 \end{pmatrix}.$$

This is not a feasible solution. Thus there cannot exist a basic feasible solution corresponding to the basis $\{2, 4\}$.

(b) [Seen similar.]

Note: *There are several ways of organizing the calculation using tableaux. Any method that arrives at and verifies the optimal solution is acceptable.*

The problem is to maximize $\mathbf{c}^T\mathbf{x}$ subject to constraints $A\mathbf{x} = \mathbf{b}$, and $\mathbf{x} \geq \mathbf{0}$, where

$$A = \begin{pmatrix} 1 & 2 & 2 & 3 & 5 \\ 2 & 5 & 1 & 4 & 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 9 \\ 5 \end{pmatrix}$$

and

$$\mathbf{c}^T = \begin{pmatrix} 13 & 24 & 8 & 32 & 25 \end{pmatrix}, \quad \mathbf{x}^T = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix}.$$

We have an initial solution $\mathbf{x} = (0, 0, 2, 0, 1)$ with initial basis $B = \{3, 5\}$ and initial cost 41. We find $\mathbf{p} \in \mathbb{R}^2$ to satisfy the matrix equation

$$\begin{pmatrix} 8 & 25 \end{pmatrix} = \begin{pmatrix} c_3 & c_5 \end{pmatrix} = \mathbf{p}^T M_B,$$

where

$$M_B = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}, \quad M_B^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix},$$

and thus

$$\mathbf{p}^T = \begin{pmatrix} c_3 & c_5 \end{pmatrix} M_B^{-1} = \begin{pmatrix} 8 & 25 \end{pmatrix} \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 10 \end{pmatrix}.$$

Then

$$\begin{aligned}
\mathbf{c}^T - \mathbf{p}^T A &= \begin{pmatrix} 13 & 24 & 8 & 32 & 25 \end{pmatrix} - \begin{pmatrix} -1 & 10 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 3 & 5 \\ 2 & 5 & 1 & 4 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 13 & 24 & 8 & 32 & 25 \end{pmatrix} - \begin{pmatrix} 19 & 48 & 8 & 37 & 25 \end{pmatrix} \\
&= \begin{pmatrix} -6 & -24 & 0 & -5 & 0 \end{pmatrix}
\end{aligned}$$

Let $\overline{\mathbf{x}}$ be a feasible solution, where $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4, \overline{x}_5)$. Then $A\overline{\mathbf{x}} = \mathbf{b}$ and $\overline{x}_j \geq 0$ for $j = 1, 2, 3, 4, 5$. Then

$$\begin{aligned}
\mathbf{c}^T\overline{\mathbf{x}} &= \mathbf{p}^T A\overline{\mathbf{x}} + \mathbf{q}^T\overline{\mathbf{x}} = \mathbf{p}^T\mathbf{b} + \mathbf{q}^T\overline{\mathbf{x}} = 41 + \mathbf{q}^T\overline{\mathbf{x}} \\
&= 41 - 6\overline{x}_1 - 24\overline{x}_2 - 5\overline{x}_4,
\end{aligned}$$

where $\mathbf{q}^T = \mathbf{c}^T - \mathbf{p}^T A$. The initial basic feasible solution cannot be optimal, because the vector $\mathbf{q}$ has some negative coefficients. Cost reduces at the fastest rate if $\overline{x}_2$ is increased. We therefore look for a basis that includes 2. Now

$$\mathbf{a}^{(2)} = t_{3,2}\mathbf{a}^{(3)} + t_{5,2}\mathbf{a}^{(5)} = M_B \begin{pmatrix} t_{3,2} \\ t_{5,2} \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} t_{3,2} \\ t_{5,2} \end{pmatrix} = M_B^{-1}\mathbf{a}^{(2)} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -19 \\ 8 \end{pmatrix}.$$

Thus

$$\mathbf{a}^{(2)} + 19\mathbf{a}^{(3)} - 8\mathbf{a}^{(5)} = \mathbf{0}.$$

It follows that

$$\begin{pmatrix} 0 & \lambda & 2 + 19\lambda & 0 & 1 - 8\lambda \end{pmatrix}$$

is a feasible solution of the problem whenever all components are non-negative. We obtain another basic solution on determining λ such that $1 - 8\lambda = 0$. We find that $\lambda = \frac{1}{8}$ and the new basic feasible solution is

$$\begin{pmatrix} 0 & \frac{1}{8} & \frac{35}{8} & 0 & 0 \end{pmatrix}$$

We now let this row vector represent the current basic solution. The current cost is then 38 and the current basis B is given by $B = \{2, 3\}$. Now let M_B now consist of the 2nd and 3rd columns of the matrix A. We find that

$$M_B = \begin{pmatrix} 2 & 2 \\ 5 & 1 \end{pmatrix}, \quad M_B^{-1} = \frac{1}{8} \begin{pmatrix} -1 & 2 \\ 5 & -2 \end{pmatrix}.$$

We then let

$$\mathbf{p}^T = \begin{pmatrix} c_2 & c_3 \end{pmatrix} M_B^{-1} = \frac{1}{8} \begin{pmatrix} 24 & 8 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{c}^T - \mathbf{p}^T A &= \begin{pmatrix} 13 & 24 & 8 & 32 & 25 \end{pmatrix} - \begin{pmatrix} 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 3 & 5 \\ 2 & 5 & 1 & 4 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 13 & 24 & 8 & 32 & 25 \end{pmatrix} - \begin{pmatrix} 10 & 24 & 8 & 22 & 22 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 & 0 & 10 & 3 \end{pmatrix} \end{aligned}$$

Because all components of this last row vector are non-negative, the argument presented at the end of the first iteration now demonstrates that the current basic feasible solution is optimal.

4. (a) [Bookwork.] The constraints satisfied by the vectors $\mathbf{x}$ and $\mathbf{p}$ ensure that

$$\mathbf{c}^T\mathbf{x} - \mathbf{p}^T\mathbf{b} = (\mathbf{c}^T - \mathbf{p}^T A)\mathbf{x} + \mathbf{p}^T(A\mathbf{x} - \mathbf{b}).$$

But $\mathbf{x} \geq \mathbf{0}$, $\mathbf{p} \geq \mathbf{0}$, $A\mathbf{x} - \mathbf{b} \geq \mathbf{0}$ and $\mathbf{c}^T - \mathbf{p}^T A \geq \mathbf{0}$. It follows that $\mathbf{c}^T\mathbf{x} - \mathbf{p}^T\mathbf{b} \geq 0$. and therefore $\mathbf{c}^T\mathbf{x} \geq \mathbf{p}^T\mathbf{b}$. Moreover $\mathbf{c}^T\mathbf{x} - \mathbf{p}^T\mathbf{b} = 0$ if and only if $(\mathbf{c}^T - \mathbf{p}^T A)_j(\mathbf{x})_j = 0$ for $j = 1, 2, \ldots, n$ and $(\mathbf{p})_i(A\mathbf{x} - \mathbf{b})_i = 0$, and therefore $\mathbf{c}^T\mathbf{x} = \mathbf{p}^T\mathbf{b}$ if and only if the complementary slackness conditions are satisfied.

(b) [Standard definition.] A subset C of $\mathbb{R}^m$ is said to be a *convex cone* in $\mathbb{R}^m$ if $\lambda\mathbf{v} + \mu\mathbf{w} \in C$ for all $\mathbf{v}, \mathbf{w} \in C$ and for all real numbers λ and μ satisfying $\lambda \geq 0$ and $\mu \geq 0$.

(c) [Bookwork.] Let $\mathbf{v}$ and $\mathbf{w}$ be elements of C. Then there exist non-negative real numbers $s_1, s_2, \ldots, s_n$ and $t_1, t_2, \ldots, t_n$ such that

$$\mathbf{v} = \sum_{j=1}^{n} s_j\mathbf{a}^{(j)} \quad \text{and} \quad \mathbf{w} = \sum_{j=1}^{n} t_j\mathbf{a}^{(j)}.$$

Let λ and μ be non-negative real numbers. Then

$$\lambda\mathbf{v} + \mu\mathbf{w} = \sum_{j=1}^{n} (\lambda s_j + \mu t_j)\mathbf{a}^{(j)},$$

and $\lambda s_j + \mu t_j \geq 0$ for $j = 1, 2, \ldots, n$. It follows that $\lambda\mathbf{v} + \mu\mathbf{w} \in C$, as required.

(d) [Bookwork.] Let $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}$ be the vectors in $\mathbb{R}^m$ determined by the columns of the matrix A, so that $(\mathbf{a}^{(j)})_i = (A)_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, and let

$$C = \left\{ \sum_{j=1}^{n} x_j\mathbf{a}^{(j)} : x_j \geq 0 \text{ for } j = 1, 2, \ldots, n \right\}.$$

Then C is a closed convex cone in $\mathbb{R}^m$. Moreover

$$C = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{x} \geq \mathbf{0}\}.$$

Thus $\mathbf{b} \in C$ if and only if there exists $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{b} = A\mathbf{x}$ and $\mathbf{x} \geq \mathbf{0}$. Therefore statement (i) in the statement of Farkas' Lemma is true if and only if $\mathbf{b} \in C$.

If $\mathbf{b} \notin C$ then it follows from a result stated on the examination paper that there exists a linear functional $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ such that $\varphi(\mathbf{v}) \geq 0$ for all $\mathbf{v} \in C$ and $\varphi(\mathbf{b}) < 0$. Then there exists $\mathbf{y} \in \mathbb{R}^m$ with the property that $\varphi(\mathbf{v}) = \mathbf{y}^T\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^m$. Now $A\mathbf{x} \in C$ for all $\mathbf{x} \in \mathbb{R}^n$ satisfying $\mathbf{x} \geq \mathbf{0}$. It follows that $\mathbf{y}^T A\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$ satisfying $\mathbf{x} \geq \mathbf{0}$. In particular $(\mathbf{y}^T A)_i = \mathbf{y}^T A\mathbf{e}^{(i)} \geq 0$ for $i = 1, 2, \ldots, m$, where $\mathbf{e}^{(i)}$ is the vector in $\mathbb{R}^m$ whose ith component is equal to 1 and whose other components are zero. Thus if $\mathbf{b} \notin C$ then there exists $\mathbf{y} \in \mathbb{R}^m$ for which $\mathbf{y}^T A \geq \mathbf{0}$ and $\mathbf{y}^T\mathbf{b} < 0$.

Conversely suppose that there exists $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}^T A \geq 0$ and $\mathbf{y}^T\mathbf{b} < 0$. Then $\mathbf{y}^T A\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$ satisfying $\mathbf{x} \geq \mathbf{0}$, and therefore $\mathbf{y}^T\mathbf{v} \geq 0$ for all $\mathbf{v} \in C$. But $\mathbf{y}^T\mathbf{b} < 0$. It follows that $\mathbf{b} \notin C$. Thus statement (ii) in the statement of Farkas's Lemma is true if and only if $\mathbf{b} \notin C$. The result follows.