Module MA3484: Methods of Mathematical Economics (Mathematical Programming) Hilary Term 2019 Part I (Sections 1, 2 and 3)

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1 Mathematical Programming Problems

1.1 A Furniture Retailing Problem

A retail business is planning to devote a number of retail outlets to the sale of armchairs and sofas.

The retail prices of armchairs and sofas are determined by fierce competition in the furniture retailing business. Armchairs sell for €700 and sofas sell for €1000.

However

- the amount of floor space (and warehouse space) available for stocking the sofas and armchairs is limited;
- the amount of capital available for purchasing the initial stock of sofas and armchairs is limited;
- market research shows that the ratio of armchairs to sofas in stores should neither be too low nor too high.

Specifically:

- there are 1000 square metres of floor space available for stocking the initial purchase of sofas and armchairs;
- each armchair takes up 1 square metre;
- each sofa takes up 2 square metres;
- the amount of capital available for purchasing the initial stock of armchairs and sofas is €351,000;
- the wholesale price of an armchair is €400;
- the wholesale price of a sofa is €600;
- market research shows that between 4 and 9 armchairs should be in stock for each 3 sofas in stock.

We suppose that the retail outlets are stocked with x armchairs and y sofas.

The armchairs (taking up 1 sq. metre each) and the sofas (taking up 2 sq. metres each) cannot altogether take up more than 1000 sq. metres of floor space. Therefore

$$x + 2y \leq 1000 \quad \text{(Floor space constraint).}$$

The cost of stocking the retail outlets with armchairs (costing €400 each) and sofas (costing €600 each) cannot exceed the available capital of €351000. Therefore

$$4x + 6y \leq 3510 \quad \text{(Capital constraint)}.$$

Consumer research indicates that x and y should satisfy

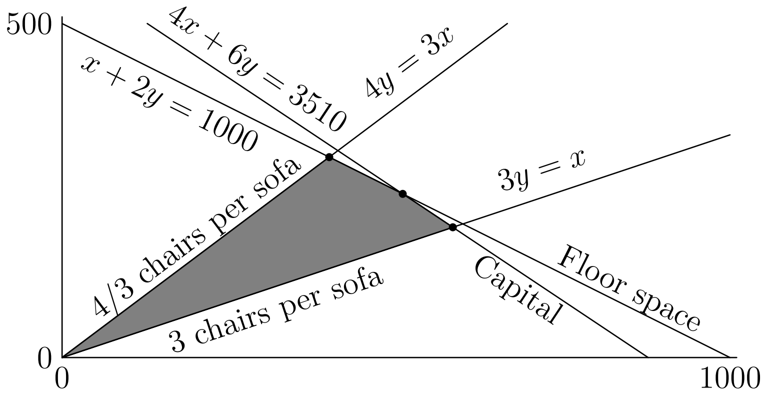
$$4y \leq 3x \leq 9y \quad \text{(Armchair/Sofa ratio)}.$$

An ordered pair (x, y) of real numbers is said to specify a *feasible solution* to the linear programming problem if this pair of values meets all the relevant constraints.

An ordered pair (x, y) constitutes a feasible solution to the the Furniture Retailing problem if and only if all the following constraints are satisfied:

$$\begin{array}{rcl} x - 3y & \leq & 0; \\ 4y - 3x & \leq & 0; \\ x + 2y & \leq & 1000; \\ 4x + 6y & \leq & 3510; \\ x & \geq & 0; \\ y & \geq & 0; \end{array}$$

The feasible region for the Furniture Retailing problem is depicted below:



We identify the *vertices* (or *corners*) of the feasible region for the Furniture Retailing problem. There are four of these:

- there is a vertex at $(0, 0)$;
- there is a vertex at $(400, 300)$ where the line $4y = 3x$ intersects the line $x + 2y = 1000$;

- there is a vertex at $(510, 245)$ where the line $x + 2y = 1000$ intersects the line $4x + 6y = 3510$;
- there is a vertex at $(585, 195)$ where the line $3y = x$ intersects the line $4x + 6y = 3510$.

These vertices are identified by inspection of the graph that depicts the constraints that determine the feasible region.

The furniture retail business obviously wants to confirm that the business will make a profit, and will wish to determine how many armchairs and sofas to purchase from the wholesaler to maximize expected profit.

There are fixed costs for wages, rental etc., and we assume that these are independent of the number of armchairs and sofas sold.

The *gross margin* on the sale of an armchair or sofa is the difference between the wholesale and retail prices of that item of furniture.

Armchairs cost €400 wholesale and sell for €700, and thus provide a gross margin of €300.

Sofas cost €600 wholesale and sell for €1000, and thus provide a gross margin of €400.

In a typical linear programming problem, one wishes to determine not merely *feasible* solutions to the problem. One wishes to determine an *optimal* solution that maximizes some *objective function* amongst all feasible solutions to the problem.

The objective function for the Furniture Retailing problem is the gross profit that would accrue from selling the furniture in stock. This gross profit is the difference between the cost of purchasing the furniture from the wholesaler and the return from selling that furniture.

This objective function is thus $f(x, y)$, where

$$f(x, y) = 300x + 400y.$$

We should determine the maximum value of this function on the feasible region.

Because the objective function $f(x, y) = 300x + 400y$ is linear in x and y, its maximum value on the feasable region must be achieved at one of the vertices of the region.

Clearly this function is not maximized at the origin $(0, 0)$!

Now the remaining vertices of the feasible region are at $(400, 300)$, $(510, 245)$ and $(585, 195)$, and

$$\begin{aligned} f(400, 300) &= 240,000, \\ f(510, 245) &= 251,000, \end{aligned}$$

$$f(585, 195) = 253,500.$$

It follows that the objective function is maximized at $(585, 195)$.

The furniture retail business should therefore use up the available capital, stocking 3 armchairs for every sofa, despite the fact that this will not utilize the full amount of floor space available.

A linear programming problem may be presented as follows:

> given real numbers c_i, $A_{i,j}$ and b_j for
> $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$,
>
> find real numbers $x_1, x_2, \ldots, x_n$ so as to
>
> maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$
>
> subject to constraints
>
> $x_j \geq 0$ for $j = 1, 2, \ldots, n$, and
>
> $A_{i,1}x_1 + A_{i,2}x_2 + \cdots + A_{i,n}x_n \leq b_i$ for $i = 1, 2, \ldots, m$.

The furniture retailing problem may be presented in this form with $n = 2$, $m = 4$,

$$(c_1, c_2) = (300, 400),$$

$$A = \begin{pmatrix} 1 & -3 \\ -3 & 4 \\ 1 & 2 \\ 4 & 6 \end{pmatrix}, \quad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1000 \\ 3510 \end{pmatrix}.$$

Here A represents the $m \times n$ whose coefficient in the ith row and jth column is $A_{i,j}$.

Linear programming problems may be presented in matrix form. We adopt the following notational conventions with regard to transposes, row and column vectors and vector inequalities:—

- vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$ are represented as column vectors;
- we denote by M^{T} the $n \times m$ matrix that is the transpose of an $m \times n$ matrix M;
- in particular, given $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, where b and c are represented as column vectors, we denote by b^{T} and c^{T} the corresponding row vectors obtained on transposing the column vectors representing b and c;

- given vectors u and v in $\mathbb{R}^n$ for some positive integer n, we write $u \leq v$ (and $v \geq u$) if and only if $u_j \leq v_j$ for $j = 1, 2, \ldots, n$.

Linear programming problems formulated as above may be presented in matrix notation as follows:—

Given an $m \times n$ matrix A with real coefficients,

and given column vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$,

find $x \in \mathbb{R}^n$ so as to

maximize $c^{\mathrm{T}} x$

subject to constraints $Ax \leq b$ and $x \geq 0$.

1.2 A Transportation Problem concerning Dairy Produce

The *Transportation Problem* is a well-known problem and important example of a linear programming problem.

Discussions of the general problem are to be found in textbooks in the following places:—

- Chapter 8 of *Linear Programming: 1 Introduction*, by George B. Danzig and Mukund N. Thapa (Springer, 1997);
- Section 18 of Chapter I of *Methods of Mathematical Economics* by Joel N. Franklin (SIAM 2002).

We discuss an example of the Transportation Problem of Linear Programming, as it might be applied to optimize transportation costs in the dairy industry.

A food business has milk-processing plants located in various towns in a small country. We shall refer to these plants as *dairies*. Raw milk is supplied by numerous farmers with farms located throughout that country, and is transported by milk tanker from the farms to the dairies. The problem is to determine the catchment areas of the dairies so as to minimize transport costs.

We suppose that there are m farms, labelled by integers from 1 to m that supply milk to n dairies, labelled by integers from 1 to n. Suppose that, in a given year, the ith farm has the capacity to produce and supply a s_i litres of milk for $i = 1, 2, \ldots, n$, and that the jth dairy needs to receive at least d_j litres of milk for $j = 1, 2, \ldots, n$ to satisfy the business obligations.

The quantity $\sum_{i=1}^{m} s_i$ then represents that *total supply* of milk, and the quantity $\sum_{j=1}^{n} d_j$ represents the *total demand* for milk.

We suppose that $x_{i,j}$ litres of milk are to be transported from the ith farm to the jth dairy, and that $c_{i,j}$ represents the cost per litre of transporting this milk.

Then the total cost of transporting milk from the farms to the dairies is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}.$$

The quantities $x_{i,j}$ of milk to be transported from the farms to the dairies should then be determined for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ so as to minimize the total cost of transporting milk.

However the ith farm can supply no more than s_i litres of milk in a given year, and that jth dairy requires at least d_j litres of milk in that year. It follows that the quantities $x_{i,j}$ of milk to be transported between farms and dairy are constrained by the requirements that

$$\sum_{j=1}^{n} x_{i,j} \leq s_i \quad \text{for } i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} x_{i,j} \geq d_j \quad \text{for } j = 1, 2, \ldots, n.$$

Suppose that the requirements of supply and demand are satisfied. Then

$$\sum_{j=1}^{n} d_j \leq \sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j} \leq \sum_{i=1}^{m} s_i.$$

Thus the total supply must equal or exceed the total demand.

If it is the case that $\sum_{j=1}^{n} x_{i,j} < s_i$ for at least one value of i then $\sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j} < \sum_{i=1}^{m} s_i$. Similarly if it is the case that $\sum_{i=1}^{m} x_{i,j} > d_j$ for at least one value of j then $\sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j} > \sum_{j=1}^{n} d_j$.

It follows that if total supply equals total demand, so that

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j,$$

then

$$\sum_{j=1}^{n} x_{i,j} = s_i \quad \text{for } i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} x_{i,j} = d_j \quad \text{for } j = 1, 2, \ldots, n.$$

The following report, published in 2006, describes a study of milk transport costs in the Irish dairy industry:

> Quinlan C., Enright P., Keane M., O'Connor D. 2006. *The Milk Transport Cost Implications of Alternative Dairy Factory Location.* Agribusiness Discussion Paper No. 47. Dept of Food Business and Development. University College, Cork.

The report is available at the following URL

> http://www.ucc.ie/en/media/academic/
> foodbusinessanddevelopment/paper47.pdf

The problem was investigated using commercial software that implements standard linear programming algorithms for the solution of forms of the Transportation Problem.

The description of the methodology used in the study begins as follows:

> A transportation model based on linear programming was developed and applied the Irish dairy industry to meet the study objectives. In such transportation models, transportation costs are treated as a direct linear function of the number of units shipped. The major assumptions are:
>
> 1. The items to be shipped are homogenous (i.e., they are the same regardless of their source or destination.
> 2. The shipping cost per unit is the same regardless of the number of units shipped.
> 3. There is only one route or mode of transportation being used between each source and each destination, Stevenson, (1993).
>
> **Sources and Destinations**

In 2004 there were about 25,000 dairy farmers in the Irish Republic. Hence identifying the location and size of each individual dairy farm as sources for the transportation model was beyond available resources. An alternative approach based on rural districts was adopted. There are 156 rural districts in the state and data for dairy cow numbers by rural district from the most recent livestock census was available from the Central Statistics Office (CSO). These data were converted to milk equivalent terms using average milk yield estimates.

Typical seasonal milk supply patterns were also assumed. In this way an estimate of milk availability throughout the year by rural district was derived and this could then be further converted to milk tanker loads, depending on milk tanker size.

The following is quoted from the conclusions of that report:—

A major report on the strategic development of the Irish dairy-processing sector proposed processing plant rationalization, 'Strategic Development Plan for the Irish Dairy Processing Sector' Prospectus, (2003). It was recommended that in the long term the number of plants processing butter, milk powder, casein and whey products in Ireland should be reduced to create four major sites for these products, with a limited number of additional sites for cheese and other products. It was estimated that savings from processing plant economies of scale would amount to €20m per annum, Prospectus (2003).

However, there is an inverse relationship between milk transport costs and plant size. Thus the optimum organisation of the industry involves a balancing of decreasing average plant costs against the increasing transport costs. In this analysis, the assumed current industry structure of 23 plants was reduced in a transportation modelling exercise firstly to 12 plants, then 9 plants and finally 6 plants and the increase in total annual milk transport costs for each alternative was calculated. Both a 'good' location and a 'poor location' 6 plant option were considered. The estimated milk transport costs for the different alternatives were; 4.60 cent per gallon for 23 plants; 4.85 cent per gallon for 12 plants; 5.04 cent per gallon for 9 plants; 5.24 cent per gallon for 6 plants ('good' location) and 5.75 cent per gallon ('poor' location) respectively.

In aggregate terms the results showed that milk transport costs would increase by €3, €5, €7 and €13 million per annum if processing plants were reduced from 23 to 12 to 9 to 6 (good location) and 6 (poor location) respectively. As the study of processing plant rationalization did not consider cheese plant rationalization in detail, it was inferred that the estimated saving from economies of scale of €20 million per annum was associated with between 6 and 12 processing sites. Excluding the 6 plant (poor location) option, the additional milk transport cost of moving to this reduced number of sites was estimated to be of the order of *5* million per annum. This represents about 25 per cent of the estimated benefits from economies of scale arising from processing plant rationalization.

The transportation model also facilitated a comparison of current milk catchment areas of processing plants with optimal catchment areas, assuming no change in number of processing plants. It was estimated that if dairies were to collect milk on an optimal basis, there would be an 11% reduction from the current (2005) milk transport costs.

In the "benchmark" model 23 plants were required to stay open at peak to accommodate milk supply and it was initially assumed that all 23 remained open throughout the year with the same catchment areas. However, due to seasonality in milk supply, it is not essential that all 23 plants remain open outside the peak.

Two options were analysed. The first involved allowing the model to determine the least cost transport pattern outside the peak i.e. a relaxation of the constraint of fixed catchment areas throughout the year, with all plants available for milk intake. Further modest reductions in milk transport costs were realisable in this case. The second option involved keeping only the bigger plants open outside the peak period. A modest increase in milk transport costs was estimated for this option due to tankers having to travel longer distances outside the peak period.

The analysis of milk transport costs in the Irish Dairy Industry is a significant topic in the Ph.D. thesis of the first author of the 2006 report from which the preceding quotation was taken:

Quinlan, Carrie, Brigid, 2013. *Optimisation of the food dairy coop supply chain.* PhD Thesis, University College Cork.

which is available at the following URL:

http://cora.ucc.ie/bitstream/handle/
10468/1197/QuinlanCB_PhD2013.pdf

The Transportation Problem, with equality of total supply and total demand, can be expressed generally in the following form. Some commodity is supplied by m suppliers and is transported from those suppliers to n recipients. The ith supplier can supply at most s_i units of the commodity, and the jth recipient requires at least d_j units of the commodity. The cost of transporting a unit of the commodity from the ith supplier to the jth recipient is $c_{i,j}$.

The total transport cost is then

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}x_{i,j}.$$

where $x_{i,j}$ denote the number of units of the commodity transported from the ith supplier to the jth recipient.

The Transportation Problem can then be presented as follows:

determine $x_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$

so as minimize $\sum_{i,j} c_{i,j}x_{i,j}$

subject to the constraints

$x_{i,j} \geq 0$ for all i and j,

$\sum_{j=1}^{n} x_{i,j} \leq s_i$ and $\sum_{i=1}^{m} x_{i,j} \geq d_j$, where

$\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j$.

2 Finite-Dimensional Vector Spaces

2.1 Real Vector Spaces

Definition A *real vector space* consists of a set V on which there is defined an operation of vector addition, yielding an element $\mathbf{v}+\mathbf{w}$ of V for each pair $\mathbf{v}, \mathbf{w}$ of elements of V, and an operation of multiplication-by-scalars that yields an element $\lambda\mathbf{v}$ of V for each $\mathbf{v} \in V$ and for each real number λ. The operation of vector addition is required to be commutative and associative. There must exist a zero element $\mathbf{0}_V$ of V that satisfies $\mathbf{v} + \mathbf{0}_V = \mathbf{v}$ for all $\mathbf{v} \in V$, and, for each $\mathbf{v} \in V$ there must exist an element $-\mathbf{v}$ of V for which $\mathbf{v}+(-\mathbf{v}) = \mathbf{0}_V$. The following identities must also be satisfied for all $\mathbf{v}, \mathbf{w} \in V$ and for all real numbers λ and μ:

$$(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}, \quad \lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w},$$

$$\lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}, \quad 1\mathbf{v} = \mathbf{v}.$$

Let n be a positive integer. The set $\mathbb{R}^n$ consisting of all n-tuples of real numbers is then a real vector space, with addition and multiplication-by-scalars defined such that

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots x_n + y_n)$$

and

$$\lambda(x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$$

for all $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \mathbb{R}$ and for all real numbers λ.

The set $M_{m,n}(\mathbb{R})$ of all $m \times n$ matrices is a real vector space with respect to the usual operations of matrix addition and multiplication of matrices by real numbers.

2.2 Linear Dependence and Bases

Elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ of a real vector space V are said to be *linearly dependent* if there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$, not all zero, such that

$$\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 + \cdots + \lambda_m\mathbf{u}_m = \mathbf{0}_V.$$

If elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ of real vector space V are not linearly dependent, then they are said to be *linearly independent*.

Elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ of a real vector space V are said to *span* V if, given any element $\mathbf{v}$ of V, there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\mathbf{v} = \lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 + \cdots + \lambda_n\mathbf{u}_n$.

A vector space is said to be *finite-dimensional* if there exists a finite subset of V whose members span V.

Elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ of a finite-dimensional real vector space V are said to constitute a *basis* of V if they are linearly independent and span V.

Lemma 2.1 *Elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ of a real vector space V constitute a basis of V if and only if, given any element $\mathbf{v}$ of V, there exist uniquely-determined real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that*

$$\mathbf{v} = \lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_n \mathbf{u}_n.$$

Proof Suppose that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is a basis of V. Let $\mathbf{v}$ be an element V. The requirement that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ span V ensures that there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$\mathbf{v} = \lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_n \mathbf{u}_n.$$

If $\mu_1, \mu_2, \ldots, \mu_n$ are real numbers for which

$$\mathbf{v} = \mu_1 \mathbf{u}_1 + \mu_2 \mathbf{u}_2 + \cdots + \mu_n \mathbf{u}_n,$$

then

$$(\mu_1 - \lambda_1)\mathbf{u}_1 + (\mu_2 - \lambda_2)\mathbf{u}_2 + \cdots + (\mu_n - \lambda_n)\mathbf{u}_n = \mathbf{0}_V.$$

It then follows from the linear independence of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ that $\mu_i - \lambda_i = 0$ for $i = 1, 2, \ldots, n$, and thus $\mu_i = \lambda_i$ for $i = 1, 2, \ldots, n$. This proves that the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ are uniquely-determined.

Conversely suppose that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is a list of elements of V with the property that, given any element $\mathbf{v}$ of V, there exist uniquely-determined real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$\mathbf{v} = \lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_n \mathbf{u}_n.$$

Then $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ span V. Moreover we can apply this criterion when $\mathbf{v} = 0$. The uniqueness of the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ then ensures that if

$$\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_n \mathbf{u}_n = \mathbf{0}_V$$

then $\lambda_i = 0$ for $i = 1, 2, \ldots, n$. Thus $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are linearly independent. This proves that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is a basis of V, as required. ∎

Proposition 2.2 *Let V be a finite-dimensional real vector space, let*

$$\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$$

be elements of V that span V, and let K be a subset of $\{1, 2, \ldots, n\}$. Suppose either that $K = \emptyset$ or else that those elements $\mathbf{u}_i$ for which $i \in K$ are linearly independent. Then there exists a basis of V whose members belong to the list $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ which includes all the vectors $\mathbf{u}_i$ for which $i \in K$.

Proof We prove the result by induction on the number of elements in the list $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ of vectors that span V. The result is clearly true when $n = 1$. Thus suppose, as the induction hypothesis, that the result is true for all lists of elements of V that span V and that have fewer than n members.

If the elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are linearly independent, then they constitute the required basis. If not, then there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, not all zero, such that

$$\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_n \mathbf{u}_n = \mathbf{0}_V.$$

Now there cannot exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, not all zero, such that both $\sum_{i=1}^{n} \lambda_i \mathbf{u}_i = \mathbf{0}_V$ and also $\lambda_i = 0$ whenever $i \neq K$. Indeed, in the case where $K = \emptyset$, this conclusion follows from the requirement that the real numbers λ_i cannot all be zero, and, in the case where $K \neq \emptyset$, the conclusion follows from the linear independence of those $\mathbf{u}_i$ for which $i \in K$. Therefore there must exist some integer i satisfying $1 \leq i \leq n$ for which $\lambda_i \neq 0$ and $i \notin K$.

Without loss of generality, we may suppose that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are ordered so that $n \notin K$ and $\lambda_n \neq 0$. Then

$$\mathbf{u}_n = -\sum_{i=1}^{n-1} \frac{\lambda_i}{\lambda_n} \mathbf{u}_i.$$

Let $\mathbf{v}$ be an element of V. Then there exist real numbers $\mu_1, \mu_2, \ldots, \mu_n$ such that $\mathbf{v} = \sum_{i=1}^{n} \mu_i \mathbf{u}_i$, because $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ span V. But then

$$\mathbf{v} = \sum_{i=1}^{n-1} \left(\mu_i - \frac{\mu_n \lambda_i}{\lambda_n} \right) \mathbf{u}_i.$$

We conclude that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-1}$ span the vector space V. The induction hypothesis then ensures that there exists a basis of V consisting of members of this list that includes the linearly independent elements $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$, as required. ∎

Corollary 2.3 *Let V be a finite-dimensional real vector space, and let*

$$\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$$

be elements of V that span the vector space V. Then there exists a basis of V whose elements are members of the list $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

Proof This result is a restatement of Proposition 2.2 in the special case where the set K in the statement of that proposition is the empty set. ∎

2.3 Dual Spaces

Definition Let V be a real vector space. A *linear functional* $\varphi\colon V \to \mathbb{R}$ on V is a linear transformation from the vector space V to the field $\mathbb{R}$ of real numbers.

Given linear functionals $\varphi\colon V \to \mathbb{R}$ and $\psi\colon V \to \mathbb{R}$ on a real vector space V, and given any real number λ, we define $\varphi + \psi$ and $\lambda\varphi$ to be the linear functionals on V defined such that $(\varphi + \psi)(\mathbf{v}) = \varphi(\mathbf{v}) + \psi(\mathbf{v})$ and $(\lambda\varphi)(\mathbf{v}) = \lambda\,\varphi(\mathbf{v})$ for all $\mathbf{v} \in V$.

The set V^* of linear functionals on a real vector space V is itself a real vector space with respect to the algebraic operations of addition and multiplication-by-scalars defined above.

Definition Let V be a real vector space. The *dual space* V^* of V is the vector space whose elements are the linear functionals on the vector space V.

Now suppose that the real vector space V is finite-dimensional. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of V, where $n = \dim V$. Given any $\mathbf{v} \in V$ there exist uniquely-determined real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\mathbf{v} = \sum_{j=1}^{n} \lambda_j \mathbf{u}_j$. It follows that there are well-defined functions $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ from V to the field $\mathbb{R}$ defined such that

$$\varepsilon_i \left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j \right) = \lambda_i$$

for $i = 1, 2, \ldots, n$ and for all real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. These functions are linear transformations, and are thus linear functionals on V.

Lemma 2.4 *Let V be a finite-dimensional real vector space, let*

$$\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$$

be a basis of V, and let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be the linear functionals on V defined such that

$$\varepsilon_i \left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j \right) = \lambda_i$$

for $i = 1, 2, \ldots, n$ and for all real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ constitute a basis of the dual space V^* of V. Moreover $\varphi = \sum_{i=1}^{n} \varphi(\mathbf{u_i})\varepsilon_i$ for all $\varphi \in V^*$.

Proof Let $\mu_1, \mu_2, \ldots, \mu_n$ be real numbers with the property that $\sum_{i=1}^{n} \mu_i \varepsilon_i = \mathbf{0}_{V^*}$. Then

$$0 = \left(\sum_{i=1}^{n} \mu_i \varepsilon_i \right) (\mathbf{u}_j) = \sum_{i=1}^{n} \mu_i \varepsilon_i(\mathbf{u}_j) = \mu_j$$

for $j = 1, 2, \ldots, n$. Thus the linear functionals $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ on V are linearly independent elements of the dual space V^*.

Now let $\varphi: V \to \mathbb{R}$ be a linear functional on V, and let $\mu_i = \varphi(\mathbf{u}_i)$ for $i = 1, 2, \ldots, n$. Now

$$\varepsilon_i(\mathbf{u}_j) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

It follows that

$$\begin{aligned} \left(\sum_{i=1}^{n} \mu_i \varepsilon_i \right) \left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j \right) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \mu_i \lambda_j \varepsilon_i(\mathbf{u}_j) = \sum_{j=1}^{n} \mu_j \lambda_j \\ &= \sum_{j=1}^{n} \lambda_j \varphi(\mathbf{u}_j) = \varphi \left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j \right) \end{aligned}$$

for all real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

It follows that

$$\varphi = \sum_{i=1}^{n} \mu_i \varepsilon_i = \sum_{i=1}^{n} \varphi(\mathbf{u}_i) \varepsilon_i.$$

We conclude from this that every linear functional on V can be expressed as a linear combination of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$. Thus these linear functionals span V^*. We have previously shown that they are linearly independent. It follows that they constitute a basis of V^*. Moreover we have verified that $\varphi = \sum_{i=1}^{n} \varphi(\mathbf{u}_i) \varepsilon_i$ for all $\varphi \in V^*$, as required. ∎

Definition Let V be a finite-dimensional real vector space, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of V. The corresponding *dual basis* of the dual space V^* of V consists of the linear functionals $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ on V, where

$$\varepsilon_i \left(\sum_{j=1}^{n} \lambda_j \mathbf{u}_j \right) = \lambda_i$$

for $i = 1, 2, \ldots, n$ and for all real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Corollary 2.5 *Let V be a finite-dimensional real vector space, and let V^* be the dual space of V. Then $\dim V^* = \dim V$.*

Proof We have shown that any basis of V gives rise to a dual basis of V^*, where the dual basis of V has the same number of elements as the basis of V to which it corresponds. The result follows immediately from the fact that the dimension of a finite-dimensional real vector space is the number of elements in any basis of that vector space. ∎

Let V be a real-vector space, and let V^* be the dual space of V. Then V^* is itself a real vector space, and therefore has a dual space V^{**}. Now each element $\mathbf{v}$ of V determines a corresponding linear functional $E_{\mathbf{v}}: V^* \to \mathbb{R}$ on V^*, where $E_{\mathbf{v}}(\varphi) = \varphi(\mathbf{v})$ for all $\varphi \in V^*$. It follows that there exists a function $\iota: V \to V^{**}$ defined so that $\iota(\mathbf{v}) = E_{\mathbf{v}}$ for all $\mathbf{v} \in V$. Then $\iota(\mathbf{v})(\varphi) = \varphi(\mathbf{v})$ for all $\mathbf{v} \in V$ and $\varphi \in V^*$.

Now

$$\iota(\mathbf{v}+\mathbf{w})(\varphi) = \varphi(\mathbf{v}+\mathbf{w}) = \varphi(\mathbf{v}) + \varphi(\mathbf{w}) = (\iota(\mathbf{v}) + \iota(\mathbf{w}))(\varphi)$$

and

$$\iota(\lambda\mathbf{v})(\varphi) = \varphi(\lambda\mathbf{v}) = \lambda\varphi(\mathbf{v}) = (\lambda\iota(\mathbf{v}))(\varphi)$$

for all $\mathbf{v}, \mathbf{w} \in V$ and $\varphi \in V^*$ and for all real numbers λ. It follows that $\iota(\mathbf{v}+\mathbf{w}) = \iota(\mathbf{v}) + \iota(\mathbf{w})$ and $\iota(\lambda\mathbf{v}) = \lambda\iota(\mathbf{v})$ for all $\mathbf{v}, \mathbf{w} \in V$ and for all real numbers λ. Thus $\iota: V \to V^{**}$ is a linear transformation.

Proposition 2.6 *Let V be a finite-dimensional real vector space, and let $\iota: V \to V^{**}$ be the linear transformation defined such that $\iota(\mathbf{v})(\varphi) = \varphi(\mathbf{v})$ for all $\mathbf{v} \in V$ and $\varphi \in V^*$. Then $\iota: V \to V^{**}$ is an isomorphism of real vector spaces.*

Proof Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis of V, let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be the dual basis of V^*, where

$$\varepsilon_i(\mathbf{u}_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and let $\mathbf{v} \in V$. Then there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\mathbf{v} = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i$.

Suppose that $\iota(\mathbf{v}) = \mathbf{0}_{V^{**}}$. Then $\varphi(\mathbf{v}) = E_{\mathbf{v}}(\varphi) = 0$ for all $\varphi \in V^*$. In particular $\lambda_i = \varepsilon_i(\mathbf{v}) = 0$ for $i = 1, 2, \ldots, n$, and therefore $\mathbf{v} = \mathbf{0}_V$. We conclude that $\iota: V \to V^{**}$ is injective.

Now let $F: V^* \to \mathbb{R}$ be a linear functional on V^*, let $\lambda_i = F(\varepsilon_i)$ for $i = 1, 2, \ldots, n$, let $\mathbf{v} = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i$, and let $\varphi \in V^*$. Then $\varphi = \sum_{i=1}^{n} \varphi(\mathbf{u}_i)\varepsilon_i$ (see

Lemma 2.4), and therefore

$$\begin{aligned}\iota(\mathbf{v})(\varphi) &= \varphi(\mathbf{v}) = \sum_{i=1}^{n} \lambda_i \varphi(\mathbf{u}_i) = \sum_{i=1}^{n} F(\varepsilon_i)\varphi(\mathbf{u}_i) \\ &= F\left(\sum_{i=1}^{n} \varphi(\mathbf{u}_i)\varepsilon_i\right) = F(\varphi).\end{aligned}$$

Thus $\iota(\mathbf{v}) = F$. We conclude that the linear transformation $\iota: V \to V^{**}$ is surjective. We have previously shown that this linear transformation is injective. There $\iota: V \to V^{**}$ is an isomorphism between the real vector spaces V and V^{**} as required. ∎

The following corollary is an immediate consequence of Proposition 2.6.

Corollary 2.7 *Let V be a finite-dimensional real vector space, and let V^* be the dual space of V. Then, given any linear functional $F: V^* \to \mathbb{R}$, there exists some $\mathbf{v} \in V$ such that $F(\varphi) = \varphi(\mathbf{v})$ for all $\varphi \in V^*$.*

3 The Transportation Problem

3.1 The General Transportation Problem

The Transportation Problem can be expressed in the following form. Some commodity is supplied by m suppliers and is transported from those suppliers to n recipients. The ith supplier can supply at most s_i units of the commodity, and the jth recipient requires at least d_j units of the commodity. The cost of transporting a unit of the commodity from the ith supplier to the jth recipient is $c_{i,j}$.

The total transport cost is then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}.$$

where $x_{i,j}$ denote the number of units of the commodity transported from the ith supplier to the jth recipient.

The Transportation Problem can then be presented as follows:

> *determine* $x_{i,j}$ *for* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$ *so as minimize* $\sum_{i,j} c_{i,j} x_{i,j}$ *subject to the constraints* $x_{i,j} \geq 0$ *for all* i *and* j, $\sum_{j=1}^{n} x_{i,j} \leq s_i$ *and* $\sum_{i=1}^{m} x_{i,j} \geq d_j$, *where* $s_i \geq 0$ *for all* i, $d_j \geq 0$ *for all* i, *and* $\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j$.

The quantities $s_1, s_2, \ldots, s_m$ representing the quantities of the transported commodity supplied by the suppliers are the components of an m-dimensional vector $(s_1, s_2, \ldots, s_m)$. We refer to this vector as the *supply vector* for the transportation problem.

The quantities $d_1, d_2, \ldots, d_n$ representing the quantities of the transported commodity demanded by the recipients are the components of an n-dimensional vector $(d_1, d_2, \ldots, d_n)$. We refer to this vector as the *demand vector* for the transportation problem.

The quantities $c_{i,j}$ that represent the cost of transporting the commodity from the ith supplier to the jth recipient are the components of an $m \times n$ matrix. We refer to this matrix as the *cost matrix* for the transportation problem.

3.2 Transportation Problems where Supply equals Demand

Consider a transportation problem with m suppliers and n recipients. The following proposition shows that a solution to the transportation problem can only exist if total supply of the relevant commodity exceeds total demand for that commodity.

Proposition 3.1 *Let $s_1, s_2, \ldots, s_m$ and $d_1, d_2, \ldots, d_n$ be non-negative real numbers. Suppose that there exist non-negative real numbers $x_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ that satisfy the inequalities*

$$\sum_{j=1}^{n} x_{i,j} \leq s_i \quad \text{and} \quad \sum_{i=1}^{m} x_{i,j} \geq d_j.$$

Then

$$\sum_{j=1}^{n} d_j \leq \sum_{i=1}^{m} s_i.$$

Moreover if it is the case that

$$\sum_{j=1}^{n} d_j = \sum_{i=1}^{m} s_i.$$

then

$$\sum_{j=1}^{n} x_{i,j} = s_i \quad \text{for } i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} x_{i,j} = d_j \quad \text{for } j = 1, 2, \ldots, n.$$

Proof The inequalities satisfied by the non-negative real numbers $x_{i,j}$ ensure that

$$\sum_{j=1}^{n} d_j \leq \sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j} \leq \sum_{i=1}^{m} s_i.$$

Thus the total supply must equal or exceed the total demand.

Now $s_i - \sum\limits_{j=1}^{n} x_{i,j} \geq 0$ for $i = 1, 2, \ldots, m$. It follows that if $s_i > \sum_{j=1}^{n} x_{i,j}$ for at least one value of i then $\sum\limits_{i=1}^{m} s_i > \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} x_{i,j}$. Similarly $\sum\limits_{i=1}^{m} x_{i,j} - d_j \geq 0$

for $j = 1, 2, \ldots, n$. It follows that if it is the case that $\sum_{i=1}^{m} x_{i,j} > d_j$ for at least one value of j then $\sum_{i=1}^{m}\sum_{j=1}^{n} x_{i,j} > \sum_{j=1}^{n} d_j$.

It follows that if total supply equals total demand, so that

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j,$$

then

$$\sum_{j=1}^{n} x_{i,j} = s_i \quad \text{for } i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} x_{i,j} = d_j \quad \text{for } j = 1, 2, \ldots, n,$$

as required. ∎

We analyse the Transportation Problem in the case where total supply equals total demand. The optimization problem in this case can then be stated as follows:—

determine $x_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ so as minimize $\sum_{i,j} c_{i,j}x_{i,j}$ subject to the constraints $x_{i,j} \geq 0$ for all i and j, $\sum_{j=1}^{n} x_{i,j} = s_i$ and $\sum_{i=1}^{m} x_{i,j} = d_j$, where $s_i \geq 0$ and $d_j \geq 0$ for all i and j, and $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$.

Definition A *feasible* solution to a transportation problem (with equality of total supply and total demand) is represented by real numbers $x_{i,j}$, where

- $x_{i,j} \geq 0$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$;
- $\sum_{j=1}^{n} x_{i,j} = s_i$ for $= 1, 2, \ldots, m$;
- $\sum_{i=1}^{m} x_{i,j} = d_j$ for $j = 1, 2, \ldots, n$.

Definition A feasible solution $(x_{i,j})$ of a transportation problem is said to be *optimal* if it minimizes cost amongst all feasible solutions of that transportation problem.

3.3 Bases for the Transportation Problem

Definition Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, where m and n are positive integers. Then a subset B of $I \times J$ is said to be a *basis* for the transportation problem with m suppliers and n recipients if, given any vectors $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{z} \in \mathbb{R}^n$ satisfying $\sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j$, there exists a unique $m \times n$ matrix X with real coefficients satisfying the following properties:—

(i) $\sum_{j=1}^{n} (X)_{i,j} = (\mathbf{y})_i$ for $i = 1, 2, \ldots, m$;

(ii) $\sum_{i=1}^{m} (X)_{i,j} = (\mathbf{z})_j$ for $j = 1, 2, \ldots, n$;

(iii) $(X)_{i,j} = 0$ unless $(i, j) \in B$.

Lemma 3.2 *Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, where m and n are positive integers. and let*

$$B = \{(i, j) \in I \times J : i = m \text{ or } j = n\}.$$

Then B is a basis for a transportation problem with m suppliers and n recipients.

Proof The result can readily be verified when $m = 1$ or $n = 1$. We therefore restrict attention to cases where $m > 1$ and $n > 1$.

Let

$$B = \{(i, j) \in I \times J : i = m \text{ or } j = n\},$$

where $m > 1$ and $n > 1$. Then, given any vectors $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{z} \in \mathbb{R}^n$ that satisfy $\sum_{i=1}^{m} y_i = \sum_{j=1}^{n} z_j$, there exists a unique $m \times n$ matrix X with real coefficients with all the following properties:

(i) $\sum_{j=1}^{n} (X)_{i,j} = y_i$ for $i = 1, 2, \ldots, m$;

(ii) $\sum_{i=1}^{m} (X)_{i,j} = z_j$ for $j = 1, 2, \ldots, n$;

(iii) $(X)_{i,j} = 0$ unless $(i, j) \in B$.

This matrix X has coefficients as follows: $X_{i,j} = 0$ if $i < m$ and $j < n$; $X_{i,n} = y_i$ for $i < m$; $X_{m,j} = z_j$ for $j < n$; $X_{m,n} = w$, where

$$w = y_m - \sum_{j=1}^{n-1} z_j = z_n - \sum_{i=1}^{m-1} y_i.$$

This matrix X is thus of the form

$$X = \begin{pmatrix} 0 & 0 & \dots & 0 & y_1 \\ 0 & 0 & \dots & 0 & y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & y_{m-1} \\ z_1 & z_2 & \dots & z_{n-1} & w \end{pmatrix},$$

where

$$w = y_m - \sum_{j=1}^{n-1} z_j = z_n - \sum_{i=1}^{m-1} y_i.$$

It follows from the definition of bases for transportation problems that the subset B of $I \times J$ is a basis for a transportation problem with m suppliers and n recipients. This completes the proof. ∎

We now introduce some notation for use in discussion of the theory of transportation problems.

For each integer i between 1 and m, let $\mathbf{e}^{(i)}$ denote the m-dimensional vector whose ith component is equal to 1 and whose other components are zero. For each integer j between 1 and n, let $\hat{\mathbf{e}}^{(j)}$ denote the n-dimensional vector whose jth component is equal to 1 and whose other components are zero. Thus

$$(\mathbf{e}^{(i)})_k = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases} \quad \text{and} \quad (\hat{\mathbf{e}}^{(j)})_\ell = \begin{cases} 1 & \text{if } j = \ell; \\ 0 & \text{if } j \neq \ell. \end{cases}$$

Moreover $\mathbf{y} = \sum_{i=1}^{m} (\mathbf{y})_i \mathbf{e}^{(i)}$ for all $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{z} = \sum_{j=1}^{n} (\mathbf{z})_j \hat{\mathbf{e}}^{(j)}$ for all $\mathbf{z} \in \mathbb{R}^n$. Also, for each ordered pair (i, j) of integers with $1 \leq i \leq m$ and $1 \leq j \leq n$, let $E^{(i,j)}$ denote the $m \times n$ matrix that has a single non-zero coefficient equal to 1 located in the ith row and jth column of the matrix. Thus

$$(E^{(i,j)})_{k,\ell} = \begin{cases} 1 & \text{if } k = i \text{ and } j = \ell; \\ 0 & \text{if } k \neq i \text{ or } j \neq \ell. \end{cases}$$

Moreover

$$X = \sum_{i=1}^{m} \sum_{j=1}^{n} (X)_{i,j} E^{(i,j)}$$

for all $m \times n$ matrices X with real coefficients.

We let $\rho\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^m$ and $\sigma\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}^n$ be the linear transformations defined such that $(\rho(X))_i = \sum_{j=1}^{n} (X)_{i,j}$ for $i = 1, 2, \ldots, m$ and $(\sigma(X))_j = \sum_{i=1}^{m} (X)_{i,j}$ for $j = 1, 2, \ldots, n$. Then $\rho(E^{(i,j)}) = \mathbf{e}^{(i)}$ for $i = 1, 2, \ldots, m$ and $\sigma(E^{(i,j)}) = \hat{\mathbf{e}}^{(j)}$ for $j = 1, 2, \ldots, n$.

A feasible solution of the transportation problem with given supply vector $\mathbf{s}$, demand vector $\mathbf{d}$ and cost matrix C is represented by an $m \times n$ matrix X satisfying the following three conditions:—

- The coefficients of X are all non-negative;
- $\rho(X) = \mathbf{s}$;
- $\sigma(X) = \mathbf{d}$.

The cost functional $f\colon M_{m,n}(\mathbb{R}) \to \mathbb{R}$ is defined so that

$$f(X) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{i,j} (X)_{i,j} = \operatorname{trace}(C^T X)$$

for all $X \in M_{m,n}(\mathbb{R})$, where C is the cost matrix and $c_{i,j} = (C)_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

A feasible solution $\hat{X}$ of the Transportation problem is optimal if and only if $f(\hat{X}) \leq f(X)$ for all feasible solutions X of that problem.

Lemma 3.3 *Let X be an $m \times n$ matrix, let $\rho(X) \in \mathbb{R}^m$ and $\sigma(X) \in \mathbb{R}^n$ be defined so that $(\rho(X))_i = \sum_{j=1}^{n} (X)_{i,j}$ for $i = 1, 2, \ldots, m$ and $(\sigma(X))_j = \sum_{i=1}^{m} (X)_{i,j}$ for $j = 1, 2, \ldots, n$, and let*

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j \right\}.$$

Then $(\rho(X), \sigma(X)) \in W$.

Proof Summing the components of the vectors $\rho(X)$ and $\sigma(X)$, we find that

$$\sum_{i=1}^{m}(\rho(X))_i = \sum_{i=1}^{m}\sum_{j=1}^{n}(X)_{i,j} = \sum_{j=1}^{n}(\sigma(X))_j.$$

Thus $(\rho(X), \sigma(X)) \in W$, as required. ∎

Given a subset K of $I \times J$, where $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, we denote by M_K the vector subspace of the space $M_{m,n}(\mathbb{R})$ of $m \times n$ matrices with real coefficients defined such that

$$M_K = \{X \in M_{m,n}(\mathbb{R}) : (X)_{i,j} = 0 \text{ unless } (i,j) \in K\}.$$

The definition of bases for transportation problems then ensures that a subset B of $I \times J$ is a basis for a transportation problem with m suppliers and n recipients if and only if the linear transformation $\theta_B \colon M_B \to W$ is an isomorphism of vector spaces, where

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m}(\mathbf{y})_i = \sum_{j=1}^{n}(\mathbf{z})_j \right\},$$

and $\theta_B(X) = (\rho(X), \sigma(X))$ for all $X \in M_B$, where $(\rho(X))_i = \sum_{j=1}^{n}(X)_{i,j}$ for $i = 1, 2, \ldots, m$ and $(\sigma(X))_j = \sum_{i=1}^{m}(X)_{i,j}$ for $j = 1, 2, \ldots, n$.

Proposition 3.4 *A basis for a transportation problem with m suppliers and n recipients has $m + n - 1$ elements.*

Proof Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$ and, for all $(i,j) \in I \times J$, let $E^{(i,j)}$ denote the $m \times n$ matrix defined so that

$$(E^{(i,j)})_{k,\ell} = \begin{cases} 1 & \text{if } k = i \text{ and } j = \ell; \\ 0 & \text{if } k \neq i \text{ or } j \neq \ell. \end{cases}$$

Let B be a basis for the transportation problem with m suppliers and n recipients. Then the $m \times n$ matrices $E^{(i,j)}$ for which $(i,j) \in B$ constitute a basis of the vector space M_B where

$$M_B = \{X \in M_{m,n}(\mathbb{R}) : (X)_{i,j} = 0 \text{ unless } (i,j) \in B\}.$$

It follows that the dimension of the vector space M_B is equal to the number of elements in the basis B.

Let

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^m (\mathbf{y})_i = \sum_{j=1}^n (\mathbf{z})_j \right\},$$

and let $\theta_B \colon M_B \to W$ be defined so that $\theta_B(X) = (\rho(X), \sigma(X))$ for all $X \in M_B$, where $\rho(X)_i = \sum_{j=1}^n (X)_{i,j}$ for $i = 1, 2, \ldots, m$, and $\sigma(X)_j = \sum_{i=1}^m (X)_{i,j}$ for $j = 1, 2, \ldots, n$. Now the definition of bases for transportation problems ensures that $\theta \colon M_B \to W$ is an isomorphism of vector spaces. Therefore $\dim M_B = \dim W$. It follows that any two bases for a transportation problem with m suppliers and n recipients have the same number of elements.

Lemma 3.2 showed that

$$\{(i,j) \in I \times J : i = m \text{ or } j = n\}$$

is a basis for a transportation problem with m suppliers and n recipients. This basis has $m + n - 1$ elements. It follows that $\dim W = m + n - 1$, and therefore every basis for a transportation problem with m suppliers and n recipients has $m + n - 1$ elements, as required. ∎

Proposition 3.5 *Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, where m and n are positive integers, and let K be a subset of $I \times J$. Suppose that, given any vectors $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{z} \in \mathbb{R}^n$ satisfying $\sum_{i=1}^m (\mathbf{y})_i = \sum_{j=1}^n (\mathbf{z})_j$, there exists an $m \times n$ matrix X with real coefficients belonging to M_K with the following properties:*

(i) $\sum_{j=1}^n (X)_{i,j} = y_i$ *for* $i = 1, 2, \ldots, m$;

(ii) $\sum_{i=1}^m (X)_{i,j} = z_j$ *for* $j = 1, 2, \ldots, n$;

(iii) $(X)_{i,j} = 0$ *unless* $(i,j) \in K$.

Then there exists a basis B for the transportation problem satisfying $B \subset K$.

Proof First we define bases for the vector spaces involved in the proof. For each integer i between 1 and m, let $\mathbf{e}^{(i)} \in \mathbb{R}^m$ be defined such that

$$(\mathbf{e}^{(i)})_k = \begin{cases} 1 & \text{if } i = k; \\ 0 & \text{if } i \neq k. \end{cases}$$

For each integer j between 1 and n, let $\hat{\mathbf{e}}^{(j)} \in \mathbb{R}^n$ be defined such that

$$(\hat{\mathbf{e}}^{(j)})_\ell = \begin{cases} 1 & \text{if } j = \ell; \\ 0 & \text{if } j \neq \ell. \end{cases}$$

For each ordered pair (i, j) of integers with $1 \leq i \leq m$ and $1 \leq j \leq n$, let $E^{(i,j)} \in M_n(\mathbb{R})$ be defined such that

$$(E^{(i,j)})_{k,\ell} = \begin{cases} 1 & \text{if } k = i \text{ and } j = \ell; \\ 0 & \text{if } k \neq i \text{ or } j \neq \ell. \end{cases}$$

Let M_K denote the vector subspace of the space $M_{m,n}(\mathbb{R})$ of $m\times n$ matrices with real coefficients defined such that

$$M_K = \{X \in M_{m.n}(\mathbb{R}) : (X)_{i,j} = 0 \text{ unless } (i,j) \in K\},$$

let

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^m (\mathbf{y})_i = \sum_{j=1}^n (\mathbf{z})_j \right\},$$

and let $\theta_K \colon M_K \to W$ be the linear transformation defined so that $\theta_K(X) = (\rho(X), \sigma(X))$ for all $X \in M_{m,n}(\mathbb{R})$, where $\rho(X)_i = \sum_{j=1}^n (X)_{i,j}$ for $i = 1, 2, \ldots, m$ and $\sigma(X)_j = \sum_{i=1}^m (X)_{i,j}$ for $j = 1, 2, \ldots, n$. Then

$$X = \sum_{(i,j)\in K} (X)_{i,j} E^{(i,j)}$$

for all $X \in M_K$, and therefore

$$\theta_K(X) = \sum_{(i,j)\in K} (X)_{i,j} \theta(E^{(i,j)}) = \sum_{(i,j)\in K} (X)_{i,j} (\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)})$$

for all $X \in M_K$. The conditions of the proposition ensure that that the ordered pairs $(\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)})$ of basis vectors for which (i, j) belongs to K span the vector space W. It then follows from standard linear algebra that there exists a subset B of K such that those ordered pairs $(\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)})$ for which (i, j) belongs to B constitute a basis for the vector space W (see Corollary 2.3).

Thus, given any ordered pair $(\mathbf{y}, \mathbf{z})$ of vectors belonging to W, there exist uniquely determined real numbers $x_{i,j}$ for all $(i, j) \in B$ such that

$$(\mathbf{y}, \mathbf{z}) = \sum_{(i,j)\in B} x_{i,j} (\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)}).$$

Let $X \in M_B$ be the $m \times n$ matrix defined such that $(X)_{i,j} = x_{i,j}$ for all $(i,j) \in B$ and $(X)_{i,j} = 0$ for all $(i,j) \in (I \times J) \setminus B$. Then X is the unique $m \times n$ matrix with the properties that $\rho(X) = \mathbf{y}$, $\sigma(X) = \mathbf{z}$ and $X_{(i,j)} = 0$ unless $(i,j) \in B$. It follows that the subset B of K is the required basis for the transportation problem. ∎

Proposition 3.6 *Let m and n be positive integers, let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let K be a subset of $I \times J$. Suppose that there is no basis B of the transportation problem for which $K \subset B$. Then there exists a non-zero $m \times n$ matrix Y with real coefficients which satisfies the following conditions:*

- $\sum_{j=1}^{n} (Y)_{i,j} = 0$ *for* $i = 1, 2, \ldots, m$;
- $\sum_{i=1}^{m} (Y)_{i,j} = 0$ *for* $j = 1, 2, \ldots, n$;
- $(Y)_{i,j} = 0$ *when* $(i,j) \notin K$.

Proof For each integer i between 1 and m, let $\mathbf{e}^{(i)} \in \mathbb{R}^m$ be defined such that

$$(\mathbf{e}^{(i)})_k = \begin{cases} 1 & \text{if } i = k; \\ 0 & \text{if } i \neq k. \end{cases}$$

For each integer j between 1 and n, let $\hat{\mathbf{e}}^{(j)} \in \mathbb{R}^n$ be defined such that

$$(\hat{\mathbf{e}}^{(j)})_\ell = \begin{cases} 1 & \text{if } j = \ell; \\ 0 & \text{if } j \neq \ell. \end{cases},$$

and let

$$W = \left\{ (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{i=1}^{m} (\mathbf{y})_i = \sum_{j=1}^{n} (\mathbf{z})_j \right\}.$$

Now follows from Proposition 2.2 that if the elements $(\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)})$ for which $(i,j) \in K$ were linearly independent then there would exist a subset B of $I \times J$ satisfying $K \subset B$ such that the elements $(\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)})$ for which $(i,j) \in B$ would constitute a basis of W. It would then follow that, given any ordered pair $(\mathbf{y}, \mathbf{z})$ of vectors belonging to W, there would exist a unique $m \times n$ matrix X with real coefficients with the properties that $\sum_{j=1}^{m} (X)_{i,j} = (\mathbf{y})_i$ for $i = 1, 2, \ldots, m$, $\sum_{i=1}^{n} (X)_{i,j} = (\mathbf{z})_i$ for $j = 1, 2, \ldots, n$, and $(X)_{i,j} = 0$ unless $(i,j) \in B$. The subset B of $I \times J$ would thus be a basis for the transportation problem. But the subset K is not contained in any basis for the Transportation Problem. It follows that the elements $(\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)})$ for which

$(i, j) \in K$ must be linearly dependent. Therefore there exists a non-zero $m \times n$ matrix Y with real coefficients such that $(Y)_{i,j} = 0$ when $(i, j) \notin K$ and

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (Y)_{i,j} (\mathbf{e}^{(i)}, \hat{\mathbf{e}}^{(j)}) = (\mathbf{0}, \mathbf{0}).$$

But then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (Y)_{i,j} \mathbf{e}^{(i)} = \mathbf{0} \quad \text{and} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} (Y)_{i,j} \hat{\mathbf{e}}^{(j)} = \mathbf{0},$$

and therefore

$$\sum_{j=1}^{n} (Y)_{i,j} = 0 \quad \text{for } i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} (Y)_{i,j} = 0 \quad \text{for } j = 1, 2, \ldots, n.$$

Also $(Y)_{i,j} = 0$ unless $(i, j) \in K$. The result follows. ▮

3.4 Basic Feasible Solutions of Transportation Problems

Consider the transportation problem with m suppliers and n recipients, where the ith supplier can provide at most s_i units of some given commodity, where $s_i \geq 0$, and the jth recipient requires at least d_j units of that commodity, where $d_j \geq 0$. We suppose also that total supply equals total demand, so that

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j,$$

The cost of transporting the commodity from the ith supplier to the jth recipient is $c_{i,j}$.

Definition A feasible solution $(x_{i,j})$ of a transportation problem is said to be *basic* if there exists a basis B for that transportation problem such that $x_{i,j} = 0$ whenever $(i, j) \notin B$.

Example Consider a transportation problem where $m = n = 2$, $s_1 = 8$, $s_2 = 3$, $d_1 = 2$, $d_2 = 9$, $c_{1,1} = 2$, $c_{1,2} = 3$, $c_{2,1} = 4$ and $c_{2,2} = 1$.

A feasible solution takes the form of a 2×2 matrix

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$$

with non-negative components which satisfies the two matrix equations

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = \begin{pmatrix} 2 & 9 \end{pmatrix}.$$

A basic feasible solution will have at least one component equal to zero. There are four matrices with at least one zero component which satisfy the required equations. They are the following:—

$$\begin{pmatrix} 0 & 8 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 8 & 0 \\ -6 & 9 \end{pmatrix}, \quad \begin{pmatrix} 2 & 6 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} -1 & 9 \\ 3 & 0 \end{pmatrix}.$$

The first and third of these matrices have non-negative components. These two matrices represent basic feasible solutions to the problem, and moreover they are the only basic feasible solutions.

The costs associated with the components of the matrices are $c_{1,1} = 2$, $c_{1,2} = 3$, $c_{2,1} = 4$ and $c_{2,2} = 1$.

The cost of the basic feasible solution $\begin{pmatrix} 0 & 8 \\ 2 & 1 \end{pmatrix}$ is

$$8c_{1,2} + 2c_{2,1} + c_{2,2} = 24 + 8 + 1 = 33.$$

The cost of the basic feasible solution $\begin{pmatrix} 2 & 6 \\ 0 & 3 \end{pmatrix}$ is

$$2c_{1,1} + 6c_{1,2} + 3c_{2,2} = 4 + 18 + 3 = 25.$$

Now any 2×2 matrix $\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$ satisfying the two matrix equations

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = \begin{pmatrix} 2 & 9 \end{pmatrix}$$

must be of the form

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = \begin{pmatrix} \lambda & 8-\lambda \\ 2-\lambda & 1+\lambda \end{pmatrix}$$

for some real number λ.

But the matrix $\begin{pmatrix} \lambda & 8-\lambda \\ 2-\lambda & 1+\lambda \end{pmatrix}$ has non-negative components if and only if $0 \leq \lambda \leq 2$. It follows that the set of feasible solutions of this instance of the transportation problem is

$$\left\{ \begin{pmatrix} \lambda & 8-\lambda \\ 2-\lambda & 1+\lambda \end{pmatrix} : \lambda \in \mathbb{R} \text{ and } 0 \leq \lambda \leq 2 \right\}.$$

The costs associated with the components of the matrices are $c_{1,1} = 2$, $c_{1,2} = 3$, $c_{2,1} = 4$ and $c_{2,2} = 1$. Therefore, for each real number λ satisfying $0 \leq \lambda \leq 2$, the cost $f(\lambda)$ of the feasible solution $\begin{pmatrix} \lambda & 8-\lambda \\ 2-\lambda & 1+\lambda \end{pmatrix}$ is given by

$$f(\lambda) = 2\lambda + 3(8-\lambda) + 4(2-\lambda) + (1+\lambda) = 33 - 4\lambda.$$

Cost is minimized when $\lambda = 2$, and thus $\begin{pmatrix} 2 & 6 \\ 0 & 3 \end{pmatrix}$ is the optimal solution of this transportation problem. The cost of this optimal solution is 25.

Proposition 3.7 *Given any feasible solution of a transportation problem, there exists a basic feasible solution with whose cost does not exceed that of the given solution.*

Proof Let m and n be positive integers, and let let the $m \times n$ matrix X represent a feasible solution of a transportation problem with supply vector $\mathbf{s}$, demand vector $\mathbf{d}$ and cost matrix C, where C is an $m \times n$ matrix with real coefficients. Then $s_i \geq 0$ for $i = 1, 2, \ldots, m$ and $d_j \geq 0$ for $j = 1, 2, \ldots, n$, where

$$\mathbf{s} = (s_1, s_2, \ldots, s_m), \quad \mathbf{d} = (d_1, d_2, \ldots, d_n).$$

Also $x_{i,j} \geq 0$ for all i and j, $\sum_{j=1}^{n} x_{i,j} = s_i$ for $i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} x_{i,j} = d_j$ for $j = 1, 2, \ldots, n$. The cost of the feasible solution X is then $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}$, where $c_{i,j}$ is the coefficient in the ith row and jth column of the cost matrix C.

If the feasible solution X is itself basic then there is nothing to prove. Suppose therefore that X is not a basic solution. We show that there then exists a feasible solution $\overline{X}$ with fewer non-zero components than the given feasible solution.

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let

$$K = \{(i, j) \in I \times J : x_{i,j} > 0\}.$$

Because X is not a basic solution to the Transportation Problem, there does not exist any basis B for the transportation problem satisfying $K \subset B$. It therefore follows from Proposition 3.6 that there exists a non-zero $m \times n$ matrix Y whose coefficients $y_{i,j}$ satisfy the following conditions:—

- $\sum_{j=1}^{n} y_{i,j} = 0$ for $i = 1, 2, \ldots, m$;
- $\sum_{i=1}^{m} y_{i,j} = 0$ for $j = 1, 2, \ldots, n$;
- $y_{i,j} = 0$ when $(i, j) \notin K$.

We can assume without loss of generality that $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} y_{i,j} \geq 0$, where the quantities $c_{i,j}$ are the coefficients of the cost matrix C, because otherwise we can replace Y with $-Y$.

Let $Z_\lambda = X - \lambda Y$ for all real numbers λ, and let $z_{i,j}(\lambda)$ denote the coefficient $(Z_\lambda)_{i,j}$ in the ith row and jth column of the matrix Z_λ. Then $z_{i,j}(\lambda) = x_{i,j} - \lambda y_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Moreover

- $\sum_{j=1}^{n} z_{i,j}(\lambda) = s_i$;
- $\sum_{i=1}^{m} z_{i,j}(\lambda) = d_j$;
- $z_{i,j}(\lambda) = 0$ whenever $(i, j) \notin K$;
- $\displaystyle\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} z_{i,j}(\lambda) \leq \sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}$ whenever $\lambda \geq 0$.

Now the matrix Y is a non-zero matrix whose rows and columns all sum to zero. It follows that at least one of its coefficients must be strictly positive. Thus there exists at least one ordered pair (i, j) belonging to the set K for which $y_{i,j} > 0$. Let

$$\lambda_0 = \text{minimum} \left\{ \frac{x_{i,j}}{y_{i,j}} : (i, j) \in K \text{ and } y_{i,j} > 0 \right\}.$$

Then $\lambda_0 > 0$. Moreover if $0 \leq \lambda < \lambda_0$ then $x_{i,j} - \lambda y_{i,j} > 0$ for all $(i,j) \in K$, and if $\lambda > \lambda_0$ then there exists at least one element (i_0, j_0) of K for which $x_{i_0,j_0} - \lambda y_{i_0,j_0} < 0$. It follows that $x_{i,j} - \lambda_0 y_{i,j} \geq 0$ for all $(i,j) \in K$, and $x_{i_0,j_0} - \lambda_0 y_{i_0,j_0} = 0$.

Thus Z_{λ_0} is a feasible solution of the given transportation problem whose cost does not exceed that of the given feasible solution X. Moreover Z_{λ_0} has fewer non-zero components than the given feasible solution X.

If Z_{λ_0} is itself a basic feasible solution, then we have found the required basic feasible solution whose cost does not exceed that of the given feasible solution. Otherwise we can iterate the process until we arrive at the required basic feasible solution whose cost does not exceed that of the given feasible solution. ∎

A transportation problem has only finitely many basic feasible solutions. Indeed there are only finitely many bases for the problem, and any basis is associated with at most one basic feasible solution. Therefore there exists a basic feasible solution whose cost does not exceed the cost of any other basic feasible solution. It then follows from Proposition 3.7 that the cost of this basic feasible solution cannot exceed the cost of any other feasible solution of the given transportation problem. This basic feasible solution is thus a basic optimal solution of the Transportation Problem.

The transportation problem determined by the supply vector, demand vector and cost matrix has only finitely many basic feasible solutions, because there are only finitely many bases for the problem, and each basis can determine at most one basic feasible solution. Nevertheless the number of basic feasible solutions may be quite large.

But it can be shown that a transportation problem always has a basic optimal solution. It can be found using an algorithm that implements the Simplex Method devised by George B. Dantzig in the 1940s. This algorithm involves passing from one basis to another, lowering the cost at each stage, until one eventually finds a basis that can be shown to determine a basic optimal solution of the transportation problem.

3.5 The Northwest Corner Method

Example We discuss in detail how to find an initial basic feasible solution of a transportation problem with 4 suppliers and 5 recipients, using a method known as the *Northwest Corner Method*. This method does not make use of cost information.

The course of the calculation is determined by the supply vector $\mathbf{s}$ and the demand vector $\mathbf{d}$, where

$$\mathbf{s} = (9, 11, 4, 5), \quad \mathbf{d} = (6, 7, 5, 3, 8).$$

We need to fill in the entries in a tableau of the form

$x_{i,j}$	1	2	3	4	5	s_i
1	·	·	·	·	·	9
2	·	·	·	·	·	11
3	·	·	·	·	·	4
4	·	·	·	·	·	5
d_j	6	7	5	3	8	29

In the tableau just presented the labels on the left hand side identify the suppliers, the labels at the top identify the recipients, the numbers on the right hand side list the number of units that the relevant supplier must provide, and the numbers at the bottom identify the number of units that the relevant recipient must obtain. Number in the bottom right hand corner gives the common value of the total supply and the total demand.

The values in the individual cells must be non-zero, the rows must sum to the value on the left, and the columns must sum to the value on the bottom. The Northwest Corner Method is applied recursively. At each stage the undetermined cell in at the top left (the northwest corner) is given the maximum possible value allowable with the constraints. The remainder of either the first row or the first column must then be completed with zeros. This leads to a reduced tableau to be determined with either one fewer row or else one fewer column. One continues in this fashion, as exemplified in the solution of this particular problem, until the entire tableau has been completed.

The method will also determine a basis associated with the basic feasible solution determined by the Northwest Corner Method. This basis lists the cells that play the role of northwest corner at each stage of the method. At the first stage, the northwest corner cell is associated with supplier 1 and recipient 1. This cell is assigned a value equal to the mimimum of the corresponding column and row sums. Thus, this example, the northwest corner cell, is given the value 6, which is the desired column sum. The remaining cells in that row are given the value 0.

The tableau then takes the following form:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	·	·	·	·	9
2	0	·	·	·	·	11
3	0	·	·	·	·	4
4	0	·	·	·	·	5
d_j	6	7	5	3	8	29

The ordered pair $(1,1)$ commences the list of elements making up the associated basis.

At the second stage, one applies the Northwest Corner Method to the following reduced tableau:—

$x_{i,j}$	2	3	4	5	s_i
1	·	·	·	·	3
2	·	·	·	·	11
3	·	·	·	·	4
4	·	·	·	·	5
d_j	7	5	3	8	23

The required value for the first row sum of the reduced tableau has been reduced to reflect the fact that the values in the remaining undetermined cells of the first row must sum to the value 3.

The value 3 is then assigned to the northwest corner cell of the reduced tableau (as 3 is the maximum possible value for this cell subject to the constraints on row and column sums). The reduced tableau therefore takes the following form after the second stage:—

$x_{i,j}$	2	3	4	5	s_i
1	3	0	0	0	3
2	·	·	·	·	11
3	·	·	·	·	4
4	·	·	·	·	5
d_j	7	5	3	8	23

The main tableau at the completion of the second stage then stands as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	·	·	·	·	11
3	0	·	·	·	·	4
4	0	·	·	·	·	5
d_j	6	7	5	3	8	29

The list of ordered pairs representing the basis elements determined at the second stage then stands as follows:—

Basis: $(1, 1), (2, 1), \ldots$.

The reduced tableau for the third stage then stands as follows:—

$x_{i,j}$	2	3	4	5	s_i
2	·	·	·	·	11
3	·	·	·	·	4
4	·	·	·	·	5
d_j	4	5	3	8	20

Accordingly the northwest corner of the reduced tableau should be assigned the value 4, and the remaining elements of the first column should be assigned the value 0.

The reduced tableau at the completion of the third stage stands as follows:—

$x_{i,j}$	2	3	4	5	s_i
2	4	·	·	·	11
3	0	·	·	·	4
4	0	·	·	·	5
d_j	4	5	3	8	20

The main tableau and list of basis elements at the completion of the third stage then stand as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	·	·	·	11
3	0	0	·	·	·	4
4	0	0	·	·	·	5
d_j	6	7	5	3	8	29

Basis: $(1,1),(2,1),(2,2),\ldots$.

The reduced tableau at the completion of the fourth stage is as follows:—

$x_{i,j}$	3	4	5	s_i
2	5	·	·	7
3	0	·	·	4
4	0	·	·	5
d_j	5	3	8	16

The main tableau and list of basis elements at the completion of the fourth stage then stand as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	5	·	·	11
3	0	0	0	·	·	4
4	0	0	0	·	·	5
d_j	6	7	5	3	8	29

Basis: $(1,1),(2,1),(2,2),(2,3),\ldots$.

At the fifth stage the sum of the undetermined cells for the 2nd supplier must sum to 2. Therefore the main tableau and list of basis elements at the completion of the fifth stage then stand as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	5	2	0	11
3	0	0	0	·	·	4
4	0	0	0	·	·	5
d_j	6	7	5	3	8	29

Basis: $(1,1),(2,1),(2,2),(2,3),(2,4),\ldots$.

At the sixth stage the sum of the undetermined cells for the 4th recipient must sum to 1. Therefore the main tableau and list of basis elements at the completion of the sixth stage then stand as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	5	2	0	11
3	0	0	0	1	$\cdot$	4
4	0	0	0	0	$\cdot$	5
d_j	6	7	5	3	8	29

Basis: $(1,1), (2,1), (2,2), (2,3), (2,4), (3,4), \ldots$.

Two further stages suffice to complete the tableau. Moreover, at the completion of the eighth and final stage the main tableau and list of basis elements stand as follows:—

$x_{i,j}$	1	2	3	4	5	s_i
1	6	3	0	0	0	9
2	0	4	5	2	0	11
3	0	0	0	1	3	4
4	0	0	0	0	5	5
d_j	6	7	5	3	8	29

Basis: $(1,1), (2,1), (2,2), (2,3), (2,4), (3,4), (3,5), (4,5)$.

We now check that we have indeed obtained a basis B, where

$$B = \{(1,1), (2,1), (2,2), (2,3), (2,4), (3,4), (3,5), (4,5)\}.$$

If B is indeed a basis, then arbitrary values s_1, s_2, s_3, s_4 and d_1, d_2, d_3, d_4, d_5 should determine corresponding values of $x_{i,j}$ for $(i,j) \in B$, as indicated in the following tableau:—

$x_{i,j}$	1	2	3	4	5	
1	$x_{1,1}$	$x_{1,2}$				s_1
2		$x_{2,2}$	$x_{2,3}$	$x_{2,4}$		s_2
3				$x_{3,4}$	$x_{3,5}$	s_3
4					$x_{4,5}$	s_4
	d_1	d_2	d_3	d_4	d_5	

Now analysis of the Northwest Corner Method shows that, when successive elements of the set B are ordered by the stage of the method at which they are determined. Then the value of $x_{i',j'}$ for a given ordered pair $(i', j') \in B$ is determined by the values of the row sums s_i, the column sums d_j, together with the values $x_{i,j}$ for the ordered pairs (i, j) in the set B determined at earlier stages of the method.

In the specific numerical example that we have just considered, we find that the values of $x_{i,j}$ for ordered pairs (i, j) in the set B, where

$$B = \{(1, 1), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (3, 5), (4, 5)\},$$

are determined by solving, successively, the following equations:—

$$x_{1,1} = d_1, \quad x_{1,2} = s_1 - x_{1,1}, \quad x_{2,2} = d_2 - x_{1,2},$$

$$x_{2,3} = d_3, \quad x_{2,4} = s_2 - x_{2,3} - x_{2,2}, \quad x_{3,4} = d_4 - x_{2,4},$$

$$x_{3,5} = s_3 - x_{3,4}, \quad x_{4,5} = d_5 - x_{3,5},$$

It follows that the values of $x_{i,j}$ for $(i, j) \in B$ are indeed determined by s_1, s_2, s_3, s_4 and d_1, d_2, d_3, d_4, d_5.

Indeed we find that

$$\begin{aligned}
x_{1,1} &= d_1, \\
x_{1,2} &= s_1 - d_1, \\
x_{2,2} &= d_2 - s_1 + d_1, \\
x_{2,3} &= d_3, \\
x_{2,4} &= s_2 - d_3 - d_2 + s_1 - d_1, \\
x_{3,4} &= d_4 - s_2 + d_3 + d_2 - s_1 + d_1, \\
x_{3,5} &= s_3 - d_4 + s_2 - d_3 - d_2 + s_1 - d_1, \\
x_{4,5} &= d_5 - s_3 + d_4 - s_2 + d_3 + d_2 - s_1 + d_1.
\end{aligned}$$

Note that, in this specific example, the values of $x_{i,j}$ for ordered pairs (i, j) in the basis B are expressed as sums of terms of the form $\pm s_i$ and $\pm d_j$. Moreover the summands s_i all have the same sign, the summands d_j all have the same sign, and the sign of the terms s_i is opposite to the sign of the terms d_j. Thus, for example

$$x_{4,5} = (d_1 + d_2 + d_3 + d_4 + d_5) - (s_1 + s_2 + s_3).$$

This pattern is in fact a manifestation of a general result applicable to all instances of the Transportation Problem.

Remark The basic feasible solution produced by applying the Northwest Corner Method is just one amongst many basic feasible solutions. There are many others. Some of these may be obtained on applying the Northwest Corner Method after reordering the rows and columns (thus renumbering the suppliers and recipients).

It would take significant work to calculate all basic feasible solutions and then calculate the cost associated with each one.

3.6 The Minimum Cost Method for finding Basic Feasible Solutions

We discuss another method for finding an initial basic feasible solution of a transportation problem. This method is similar to the Northwest Corner Method, but takes account of the transport costs encoded in the cost matrix. The method is known as the *Minimum Cost Method*, on account of the method of selecting the cell of the tableau to be filled in at each stage in the application of the algorithm. The initial basic feasible solution obtained by this method is not necessarily optimal.

Example Let $c_{i,j}$ be the coefficient in the ith row and jth column of the cost matrix C, where

$$C = \begin{pmatrix} 8 & 4 & 16 \\ 3 & 7 & 2 \\ 13 & 8 & 6 \\ 5 & 7 & 8 \end{pmatrix}.$$

and let

$$s_1 = 13, \quad s_2 = 8, \quad s_3 = 11, \quad s_4 = 13,$$

$$d_1 = 19, \quad d_2 = 12, \quad d_3 = 14.$$

We seek to find non-negative real numbers $x_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ that minimize $\sum_{i=1}^{4} \sum_{j=1}^{3} c_{i,j} x_{i,j}$ subject to the following constraints:

$$\sum_{j=1}^{3} x_{i,j} = s_i \quad \text{for} \quad i = 1, 2, 3, 4,$$

$$\sum_{i=1}^{4} x_{i,j} = d_j \quad \text{for} \quad j = 1, 2, 3,$$

and $x_{i,j} \geq 0$ for all i and j.

For this problem the supply vector is $(13, 8, 11, 13)$ and the demand vector is $(19, 12, 14)$. The components of both the supply vector and the demand vector add up to 45.

In order to start the process of finding an initial basic solution for this problems, we set up a tableau that records the row sums (or supplies), the column sums (or demands) and the costs $c_{i,j}$ for the given problem, whilst leaving cells to be filled in with the values of the non-negative real numbers $x_{i,j}$ that will specify the initial basic feasible solution. The resultant tableau is structured as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	s_i
1	8 ?	4 ?	16 ?	13
2	3 ?	7 ?	2 ?	8
3	13 ?	8 ?	6 ?	11
4	5 ?	7 ?	8 ?	13
d_j	19	12	14	45

We apply the minimum cost method to find an initial basic solution.

The cell with lowest cost is the cell $(2, 3)$. We assign to this cell the maximum value possible, which is the minimum of s_2, which is 8, and d_3, which is 14. Thus we set $x_{2,3} = 8$. This forces $x_{2,1} = 0$ and $x_{2,2} = 0$. The pair $(2, 3)$ is added to the current basis. At the completion of the first stage the tableau is structured as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	s_i
1	8 ?	4 ?	16 ?	13
2	3 0	7 0	2 $\bullet$ 8	8
3	13 ?	8 ?	6 ?	11
4	5 ?	7 ?	8 ?	13
d_j	19	12	14	45

We enter a $\bullet$ symbol into the tableau in the relevant cell to indicate that $(1, 2)$ will be belong to the basis constructed by this method.

The next undetermined cell of lowest cost is $(1, 2)$. We assign to this cell the minimum of s_1, which is 13, and $d_2 - x_{2,2}$, which is 12. Thus we set $x_{1,2} = 12$. This forces $x_{3,2} = 0$ and $x_{4,2} = 0$. The pair $(1, 2)$ is added to the current basis. At the completion of this stage the tableau is structured as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	s_i
1	8 ?	4 • 12	16 ?	13
2	3 0	7 0	2 • 8	8
3	13 ?	8 0	6 ?	11
4	5 ?	7 0	8 ?	13
d_j	19	12	14	45

The next undetermined cell of lowest cost is $(4, 1)$. We assign to this cell the minimum of $s_4 - x_{4,2}$, which is 13, and $d_1 - x_{2,1}$, which is 19. Thus we set $x_{4,1} = 13$. This forces $x_{4,3} = 0$. The pair $(4, 1)$ is added to the current basis. At the completion of this stage the tableau is structured as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	s_i
1	8 ?	4 • 12	16 ?	13
2	3 0	7 0	2 • 8	8
3	13 ?	8 0	6 ?	11
4	5 • 13	7 0	8 0	13
d_j	19	12	14	45

The next undetermined cell of lowest cost is $(3, 3)$. We assign to this cell the minimum of $s_3 - x_{3,2}$, which is 11, and $d_3 - x_{2,3} - x_{4,3}$, which is 6 $(= 14 - 8)$. Thus we set $x_{3,3} = 6$. This forces $x_{1,3} = 0$. The pair $(3, 3)$ is added to the current basis. At the completion of this stage the tableau is structured as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	s_i
1	8 ?	4 • 12	16 0	13
2	3 0	7 0	2 • 8	8
3	13 ?	8 0	6 • 6	11
4	5 • 13	7 0	8 0	13
d_j	19	12	14	45

The next undetermined cell of lowest cost is $(1, 1)$. We assign to this cell the minimum of $s_1 - x_{1,2} - x_{1,3}$, which is 1, and $d_1 - x_{2,1} - x_{4,1}$, which is 6. Thus we set $x_{1,1} = 1$. The pair $(1, 1)$ is added to the current basis. At the completion of this stage the tableau is structured as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	s_i
1	8 • 1	4 • 12	16 0	13
2	3 0	7 0	2 • 8	8
3	13 ?	8 0	6 • 6	11
4	5 • 13	7 0	8 0	13
d_j	19	12	14	45

The final undetermined cell is $(3, 1)$. We assign to this cell the common value of $s_3 - x_{3,2} - x_{3,3}$ and $d_1 - x_{1,1} - x_{2,1} - x_{4,1}$, which is 5. Thus we set $x_{3,1} = 5$. The pair $(3, 1)$ is added to the current basis. At the completion of this final stage the tableau is structured as follows:—

$c_{i,j} \searrow x_{i,j}$	1	2	3	s_i
1	8 • 1	4 • 12	16 0	13
2	3 0	7 0	2 • 8	8
3	13 • 5	8 0	6 • 6	11
4	5 • 13	7 0	8 0	13
d_j	19	12	14	45

The initial basis is thus B where

$$B = \{(1,1),\ (1,2),\ (2,3),\ (3,1),\ (3,3),\ (4,1)\}.$$

The basic feasible solution is represented by the 6×5 matrix X, where

$$X = \begin{pmatrix} 1 & 12 & 0 \\ 0 & 0 & 8 \\ 5 & 0 & 6 \\ 13 & 0 & 0 \end{pmatrix}.$$

The cost of this initial feasible basic solution is

$$\begin{aligned} & 8 \times 1 + 4 \times 12 + 2 \times 8 + 13 \times 5 + 6 \times 6 \\ & \qquad + 5 \times 13 \\ = {} & 8 + 48 + 16 + 65 + 36 + 65 \\ = {} & 238. \end{aligned}$$

3.7 Effectiveness of the Minimum Cost Method

We now discuss the reasons why the Minimum Cost Method yields a feasible solution to a transportation problem that is a basic feasible solution.

Consider a transportation problem with m suppliers and n recipients, determined by a supply vector $\mathbf{s}$, a demand vector $\mathbf{d}$ and a cost matrix C, where

$$\mathbf{s} = (s_1, s_2, \ldots, s_m), \quad \mathbf{d} = (d_1, d_2, \ldots, d_n).$$

and where $\mathbf{d} \in \mathbb{R}^n$ and cost matrix C, We denote by $c_{i,j}$ the coefficient in the ith row and jth column of the matrix C.

The Minimum Cost Method determines a feasible solution to this transportation problem. A feasible solution is represented by an $m \times n$ matrix X whose coefficients $x_{i,j}$ satisfy the following conditions: $x_{i,j} \geq 0$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$; $\sum_{j=1}^{n} x_{i,j} = s_i$ for $i = 1, 2, \ldots, m$; $\sum_{i=1}^{m} x_{i,j} = d_j$ for $j = 1, 2, \ldots, n$. We must show that there exists a basis B such that the feasible solution determined by the Minimum Cost Method satisfies $x_{i,j} = 0$ when $(i, j) \not\in B$.

In applying the Minimum Cost Method, we begin by locating a coefficient of the cost matrix which does not exceed the other coefficients of this matrix. Renumbering the suppliers and recipients, if necessary, we may assume, without loss of generality, that $c_{i,j} \geq c_{m,n}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. The feasible solution with coefficients $x_{i,j}$ that results from application of the Minimum Cost Method then conforms to a structure specified in at least one of the two cases that are described immediately below:—

- in *Case I*, the following conditions are satisfied: $d_n \leq s_m$; $x_{m,n} = d_n$; $x_{i,n} = 0$ when $1 \leq i < n$; $\sum_{j=1}^{n-1} x_{i,j} = s_i$ for $1 \leq i < m$; $\sum_{j=1}^{n-1} x_{m,j} = s_m - d_n$; $\sum_{i=1}^{m} x_{i,j} = d_j$ for $1 \leq j < n$; and the coefficients $x_{i,j}$ with $1 \leq i \leq m$ and $1 \leq j < n$ constitute a solution of the relevant transportation problem arising from application of the Minimum Cost Method.

- in *Case II*, the following conditions are satisfied: $s_m \leq d_n$; $x_{m,n} = s_m$; $x_{m,j} = 0$ when $1 \leq j < n$; $\sum_{i=1}^{m-1} x_{i,j} = d_j$ for $1 \leq j < n$; $\sum_{i=1}^{m-1} x_{i,n} = d_n - s_m$; $\sum_{j=1}^{n} x_{i,j} = s_i$ for $1 \leq i < m$; and the coefficients $x_{i,j}$ with $1 \leq i < m$ and $1 \leq j \leq n$ constitute a solution of the relevant transportation problem arising from application of the Minimum Cost Method.

The recursive nature of the Minimum Cost Method therefore enables us to prove that the Minimum Cost Method yields a basic feasible solution by induction on $m+n$, where m is the number of suppliers and n is the number of recipients. The Minimum Cost Method clearly yields a basic feasible solution in the trivial case where $m = n = 1$. We suppose therefore as our inductive hypothesis that the feasible solution determined by application of the Minimum Cost Method is a basic feasible solution in those cases where adding the number of suppliers to the number of recipients results in a number less than $m + n$.

In particular, we may assume that, in applying the Minimum Cost Method to the given problem with m suppliers and n recipients the matrices X' and X'' that result from application of the Minimum Cost Method to a smaller transportation problem as specified in the descriptions of *Case I* and *Case II* above.

Let us now restrict attention to *Case I*. In this case the reduced transportation is a transportation problem with m suppliers and $n-1$ recipients. The inductive hypothesis guarantees that the feasible solution that results from application of the Minimum Cost Method is a basic solution. Therefore there exists a basis B' for this reduced problem with $n+m-2$ elements, Moreover if $1 \leq i \leq m$, $1 \leq j \leq n-1$ and if $x_{i,j} \neq 0$ then $(i,j) \in B'$. The elements of the basis B' take the form of ordered pairs (i,j), where i is some integer between 1 and m and j is some integer between 1 and $n-1$. Let

$$B = B' \cup \{(m,n)\}.$$

We claim that B is a basis for a transportation problem with m suppliers and n recipients.

Let $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ be real numbers, where $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. We must show that there exist unique real numbers $z_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ such that $\sum_{j=1}^{n} z_{i,j} = a_i$ for $i = 1, 2, \ldots, m$, $\sum_{i=1}^{m} z_{i,j} = b_j$ for $j = 1, 2, \ldots, n$, and $z_{i,j} = 0$ unless $(i,j) \in B$.

In particular these equations require that $\sum_{i=1}^{m} z_{i,n} = b_n$. But m is the only value of i for which $(i,n) \in B$. It follows that the coefficients $z_{i,j}$ of any basic solution determined by the basis B must satisfy $z_{i,n} = 0$ for $i < m$ and $z_{m,n} = b_n$.

It then follows that, in *Case I*, if the coefficients $z_{i,j}$ satisfy the equations $\sum_{j=1}^{n} z_{i,j} = a_i$ for $1 \leq i \leq m$ and $\sum_{i=1}^{m} z_{i,j} = b_j$ for $1 \leq j \leq n$, and if $z_{i,j} = 0$ unless $(i,j) \in B$, then these coefficients must satisfy the following conditions:—

(i) $z_{m,n} = b_n$;

(ii) $z_{i,n} = 0$ when $1 \leq i < m$;

(iii) $\sum_{j=1}^{n-1} z_{m,j} = a_m - b_n$

(iv) $\sum_{j=1}^{n-1} z_{i,j} = a_i$ when $1 \leq i < m$;

(v) $\sum_{i=1}^{m} z_{i,j} = b_j$ when $1 \leq j < n$.

(vi) if $j < n$ and $z_{i,j} \neq 0$ then $(i,j) \in B'$.

Now B' is a basis for a transportation problem with m suppliers and $n-1$ recipients. It follows that there exist unique real numbers $z_{i,j}$ for $1 \leq i \leq m$ and $1 \leq j < n$ that satisfy conditions (iii), (iv), (v) and (vi) above. It follows from this that if the numbers $z_{i,n}$ are determined in accordance with conditions (i) and (ii) above then the numbers $z_{i,j}$ are the unique real numbers that solve the equations $\sum_{j=1}^{n} z_{i,j} = a_i$ for $1 \leq i \leq m$ and $\sum_{i=1}^{m} z_{i,j} = b_j$ for $1 \leq j \leq n$, and that also satisfy $z_{i,j} = 0$ whenever $(i,j) \notin B$.

We conclude that, when the Minimum Cost Method proceeds so as to produce a feasible solution to a transportation problem with m suppliers and n recipients that conforms to the conditions specified in *Case I* above, then that feasible solution is a basic feasible solution with associated basis B. A similar argument applies when the feasible solution conforms to the conditions specified in *Case II* above. The feasible solution produced by the Minimum Cost Method conforms to conditions specified in one or other of these two cases. We conclude therefore that the Minimum Cost Method always determines a basic feasible solution to a transportation problem.

3.8 Formal Description of the Minimum Cost Method

We describe the *Minimum Cost Method* for finding an initial basic feasible solution to a transportation problem.

Consider a transportation problem specified by positive integers m and n and non-negative real numbers $s_1, s_2, \ldots, s_m$ and $d_1, d_2, \ldots, d_n$, where $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$. Let $I = \{1, 2, \ldots, m\}$ and let $J = \{1, 2, \ldots, n\}$. A feasible solution consists of an array of non-negative real numbers $x_{i,j}$ for $i \in I$ and $j \in J$ with the property that $\sum_{j \in J} x_{i,j} = s_i$ for all $i \in I$ and $\sum_{i \in I} x_{i,j} = d_j$ for all $j \in J$. The objective of the problem is to find a feasible solution that minimizes cost, where the cost of a feasible solution $(x_{i,j} : i \in I \text{ and} j \in J)$ is $\sum_{i \in I} \sum_{j \in J} c_{i,j} x_{i,j}$.

In applying the Minimum Cost Method to find an initial basic solution to the Transportation we apply an algorithm that corresponds to the determination of elements $(i_1, j_1), (i_2, j_2), \ldots, (i_{m+n-1}, j_{m+n-1})$ of $I \times J$ and of subsets $I_0, I_1, \ldots, I_{m+n-1}$ of I and $J_0, J_1, \ldots, J_{m+n-1}$ of J such that $I_0 = I$,

$J_0 = J$, and such that, for each integer k between 1 and $m+n-1$, exactly one of the following two conditions is satisfied:—

(i) $i_k \notin I_k$, $j_k \in J_k$, $I_{k-1} = I_k \cup \{i_k\}$ and $J_{k-1} = J_k$;

(ii) $i_k \in I_k$, $j_k \notin J_k$, $I_{k-1} = I_k$ and $J_{k-1} = J_k \cup \{j_k\}$;

Indeed let $I_0 = I$, $J_0 = J$ and $B_0 = \{0\}$. The Minimum Cost Method algorithm is accomplished in $m+n-1$ stages.

Let k be an integer satisfying $1 \leq k \leq m+n-1$ and that subsets I_{k-1} of I, J_{k-1} of J and B_{k-1} of $I \times J$ have been determined in accordance with the rules that apply at previous stages of the Minimum Cost algorithm. Suppose also that non-negative real numbers $x_{i,j}$ have been determined for all ordered pairs (i,j) in $I \times J$ that satisfy either $i \notin I_{k-1}$ or $j \notin J_{k-1}$ so as to satisfy the following conditions:—

- $\sum_{j \in J \setminus J_{k-1}} x_{i,j} \leq s_i$ whenever $i \in I_{k-1}$;
- $\sum_{j \in J} x_{i,j} = s_i$ whenever $i \notin I_{k-1}$;
- $\sum_{i \in I \setminus I_{k-1}} x_{i,j} \leq d_j$ whenever $j \in J_{k-1}$;
- $\sum_{i \in I} x_{i,j} = d_j$ whenever $j \notin J_{k-1}$.

The Minimum Cost Method specifies that one should choose $(i_k, j_k) \in I_{k-1} \times J_{k-1}$ so that

$$c_{i_k,j_k} \leq c_{i,j} \quad \text{for all } (i,j) \in I_{k-1} \times J_{k-1},$$

and set $B_k = B_{k-1} \cup \{(i_k, j_k)\}$. Having chosen (i_k, j_k), the non-negative real number x_{i_k,j_k} is then determined so that

$$x_{i_k,j_k} = \min\left(s_{i_k} - \sum_{j \in J \setminus J_{k-1}} x_{i_k,j},\ d_{j_k} - \sum_{i \in I \setminus I_{k-1}} x_{i,j_k} \right).$$

The subsets I_k and J_k of I and J respectively are then determined, along with appropriate values of $x_{i,j}$, according to the following rules:—

(i) if

$$s_{i_k} - \sum_{j \in J \setminus J_{k-1}} x_{i_k,j} < d_{j_k} - \sum_{i \in I \setminus I_{k-1}} x_{i,j_k}$$

then we set $I_k = I_{k-1} \setminus \{i_k\}$ and $J_k = J_{k-1}$, and we also let $x_{i_k,j} = 0$ for all $j \in J_{k-1} \setminus \{j_k\}$;

(ii) if

$$s_{i_k} - \sum_{j \in J \setminus J_{k-1}} x_{i_k,j} > d_{j_k} - \sum_{i \in I \setminus I_{k-1}} x_{i,j_k}$$

then we set $J_k = J_{k-1} \setminus \{j_k\}$ and $I_k = I_{k-1}$, and we also let $x_{i,j_k} = 0$ for all $i \in I_{k-1} \setminus \{i_k\}$;

(iii) if

$$s_{i_k} - \sum_{j \in J \setminus J_{k-1}} x_{i_k,j} = d_{j_k} - \sum_{i \in I \setminus I_{k-1}} x_{i,j_k}$$

then we determine I_k and J_k and the corresponding values of $x_{i,j}$ either in accordance with the specification in rule (i) above or else in accordance with the specification in rule (ii) above.

These rules ensure that the real numbers $x_{i,j}$ determined at this stage are all non-negative, and that the following conditions are satisfied at the conclusion of the kth stage of the Minimum Cost Method algorithm:—

- $\sum_{j \in J \setminus J_k} x_{i,j} \leq s_i$ whenever $i \in I_k$;
- $\sum_{j \in J} x_{i,j} = s_i$ whenever $i \notin I_k$;
- $\sum_{i \in I \setminus I_k} x_{i,j} \leq d_j$ whenever $j \in J_k$;
- $\sum_{i \in I} x_{i,j} = d_j$ whenever $j \notin J_k$.

At the completion of the final stage (for which $k = m + n - 1$) we have determined a subset B of $I \times J$, where $B = B_{m+n-1}$, together with non-negative real numbers $x_{i,j}$ for $i \in I$ and $j \in I$ that constitute a feasible solution to the given transportation problem.

3.9 Formal Description of the Northwest Corner Method

The *Northwest Corner Method* for finding a basic feasible solution proceeds according to the stages of the *Minimum Cost Method* above, differing only from that method in the choice of the ordered pair (i_k, j_k) at the kth stage of the method. In the Minimum Cost Method, the ordered pair (i_k, j_k) is chosen such that $(i_k, j_k) \in I_{k-1} \times J_{k-1}$ and

$$c_{i_k,j_k} \leq c_{i,j} \quad \text{for all } (i,j) \in I_{k-1} \times J_{k-1}$$

(where the sets I_{k-1}, J_{k-1} are determined as in the specification of the Minimum Cost Method). In applying the Northwest Corner Method, costs associated with ordered pairs (i, j) in $I \times J$ are not taken into account. Instead (i_k, j_k) is chosen so that i_k is the minimum of the integers in I_{k-1} and j_k is the minimum of the integers in J_{k-1}. Otherwise the specification of the Northwest Corner Method corresponds to that of the Minimum Cost Method, and results in a basic feasible solution of the given transportation problem.

3.10 A Method for finding Basic Optimal Solutions

We continue with the study of the optimization problem introduced in the discussion of the minimum cost method.

Example We seek to determine non-negative real numbers $x_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ that minimize $\sum_{i=1}^{4} \sum_{j=1}^{3} c_{i,j} x_{i,j}$, where $c_{i,j}$ is the coefficient in the ith row and jth column of the cost matrix C, where

$$C = \begin{pmatrix} 8 & 4 & 16 \\ 3 & 7 & 2 \\ 13 & 8 & 6 \\ 5 & 7 & 8 \end{pmatrix}.$$

subject to the constraints

$$\sum_{j=1}^{3} x_{i,j} = s_i \quad (i = 1, 2, 3, 4)$$

and

$$\sum_{i=1}^{4} x_{i,j} = d_j \quad (j = 1, 2, 3),$$

where

$$s_1 = 13, \quad s_2 = 8, \quad s_3 = 11, \quad s_4 = 13,$$
$$d_1 = 19, \quad d_2 = 12, \quad d_3 = 14.$$

We have found an initial basic feasible solution by the Minimum Cost Method. This solution satisfies $x_{i,j} = (X)_{i,j}$ for all i and j, where

$$X = \begin{pmatrix} 1 & 12 & 0 \\ 0 & 0 & 8 \\ 5 & 0 & 6 \\ 13 & 0 & 0 \end{pmatrix}.$$

We next determine whether this initial basic feasible solution is an optimal solution, and, if not, how to adjust the basis to obtain a solution of lower cost.

We determine u_1, u_2, u_3, u_4 and v_1, v_2, v_3 such that $c_{i,j} = v_j - u_i$ for all $(i, j) \in B$, where B is the initial basis.

We seek a solution with $u_1 = 0$. We then determine $q_{i,j}$ so that $c_{i,j} = v_j - u_i + q_{i,j}$ for all i and j.

We therefore complete the following tableau:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	u_i
1	8 • 0	4 • 0	16 ?	0
2	3 ?	7 ?	2 • 0	?
3	13 • 0	8 ?	6 • 0	?
4	5 • 0	7 ?	8 ?	?
v_j	?	?	?	

Now $u_1 = 0$, $(1, 1) \in B$ and $(1, 2) \in B$ force $v_1 = 8$ and $v_2 = 4$. After entering these values the tableau stands as follows:

$c_{i,j} \searrow q_{i,j}$	1	2	3	u_i
1	8 • 0	4 • 0	16 ?	0
2	3 ?	7 ?	2 • 0	?
3	13 • 0	8 ?	6 • 0	?
4	5 • 0	7 ?	8 ?	?
v_j	8	4	?	

Then $v_1 = 8$, $(3, 1) \in B$ and $(4, 1) \in B$ force $u_3 = -5$ and $u_4 = 3$. After entering these values the tableau stands as follows:

$c_{i,j} \searrow q_{i,j}$	1	2	3	u_i
1	8 $\bullet$	4 $\bullet$	16	0
	0	0	?	
2	3	7	2 $\bullet$	?
	?	?	0	
3	13 $\bullet$	8	6 $\bullet$	-5
	0	?	0	
4	5 $\bullet$	7	8	3
	0	?	?	
v_j	8	4	?	

Then $u_3 = -5$ and $(3, 3) \in B$ force $v_3 = 1$. After entering this value the tableau stands as follows:

$c_{i,j} \searrow q_{i,j}$	1	2	3	u_i
1	8 $\bullet$	4 $\bullet$	16	0
	0	0	?	
2	3	7	2 $\bullet$	?
	?	?	0	
3	13 $\bullet$	8	6 $\bullet$	-5
	0	?	0	
4	5 $\bullet$	7	8	3
	0	?	?	
v_j	8	4	1	

Then $v_3 = 1$ and $(2, 3) \in B$ force $u_2 = -1$.
After entering the numbers u_i and v_j, the tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	u_i
1	8 $\bullet$	4 $\bullet$	16	0
	0	0	?	
2	3	7	2 $\bullet$	-1
	?	?	0	
3	13 $\bullet$	8	6 $\bullet$	-5
	0	?	0	
4	5 $\bullet$	7	8	3
	0	?	?	
v_j	8	4	1	

Computing the numbers $q_{i,j}$ such that $c_{i,j} + u_i = v_j + q_{i,j}$, we find that $q_{1,3} = 15$, $q_{2,1} = -6$, $q_{2,2} = 2$, $q_{3,2} = -1$, $q_{4,2} = 6$ and $q_{4,3} = 10$.

The completed tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1		2		3		u_i
1	8	$\bullet$	4	$\bullet$	16		0
		0		0		15	
2	3		7		2	$\bullet$	-1
		-6		2		0	
3	13	$\bullet$	8		6	$\bullet$	-5
		0		-1		0	
4	5	$\bullet$	7		8		3
		0		6		10	
v_j	8		4		1		

The initial basic feasible solution is not optimal because some of the quantities $q_{i,j}$ are negative. To see this, suppose that the numbers $\overline{x}_{i,j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ constitute a feasible solution to the given problem. Then $\sum_{j=1}^{3} \overline{x}_{i,j} = s_i$ for $i = 1, 2, 3$ and $\sum_{i=1}^{4} \overline{x}_{i,j} = d_j$ for $j = 1, 2, 3, 4$. It follows that

$$\begin{aligned} \sum_{i=1}^{4}\sum_{j=1}^{3} c_{i,j}\overline{x}_{i,j} &= \sum_{i=1}^{4}\sum_{j=1}^{3} (v_j - u_i + q_{i,j})\overline{x}_{i,j} \\ &= \sum_{j=1}^{3} v_j d_j - \sum_{i=1}^{4} u_i s_i + \sum_{i=1}^{4}\sum_{j=1}^{3} q_{i,j}\overline{x}_{i,j}. \end{aligned}$$

Applying this identity to the initial basic feasible solution, we find that $\sum_{j=1}^{3} v_j d_j - \sum_{i=1}^{4} u_i s_i = 238$, given that 238 is the cost of the initial basic feasible solution. Thus the cost $\overline{C}$ of any feasible solution $(\overline{x}_{i,j})$ satisfies

$$\overline{C} = 238 + 15\overline{x}_{1,3} - 6\overline{x}_{2,1} + 2\overline{x}_{2,2} - \overline{x}_{3,2} + 6\overline{x}_{4,2} + 10\overline{x}_{4,3}.$$

One could construct feasible solutions with $\overline{x}_{2,1} < 0$ and $\overline{x}_{i,j} = 0$ for $(i, j) \notin B \cup \{(2, 1)\}$, and the cost of such feasible solutions would be lower than that of the initial basic solution. We therefore seek to bring $(2, 1)$ into the basis, removing some other element of the basis to ensure that the new basis corresponds to a feasible basic solution.

The procedure for achieving this requires us to determine a 4×3 matrix Y satisfying the following conditions:—

- $y_{2,1} = 1$;
- $y_{i,j} = 0$ when $(i, j) \notin B \cup \{(2, 1)\}$;

- all rows and columns of the matrix Y sum to zero.

Accordingly we fill in the following tableau with those coefficients $y_{i,j}$ of the matrix Y that correspond to cells in the current basis (marked with the • symbol), so that all rows sum to zero and all columns sum to zero:—

$y_{i,j}$	1	2	3	
1	? •	? •		0
2	1 ∘		? •	0
3	? •		? •	0
4	? •			0
	0	0	0	0

The constraints that $y_{2,1} = 1$, $y_{i,j} = 0$ when $(i,j) \notin B$ and the constraints requiring the rows and columns to sum to zero determine the values of $y_{i,j}$ for all $y_{i,j} \in B$. These values are recorded in the following tableau:—

$y_{i,j}$	1	2	3	
1	0 •	0 •		0
2	1 ∘		−1 •	0
3	−1 •		1 •	0
4	0 •			0
	0	0	0	0

We now determine those values of λ for which $X+\lambda Y$ is a feasible solution, where

$$X + \lambda Y = \begin{pmatrix} 1 & 12 & 0 \\ \lambda & 0 & 8-\lambda \\ 5-\lambda & 0 & 6+\lambda \\ 13 & 0 & 0 \end{pmatrix}.$$

In order to drive down the cost as far as possible, we should make λ as large as possible, subject to the requirement that all the coefficients of the above matrix should be non-negative numbers.

Accordingly we take $\lambda = 5$. Our new basic feasible solution X is then as follows:—

$$X = \begin{pmatrix} 1 & 12 & 0 \\ 5 & 0 & 3 \\ 0 & 0 & 11 \\ 13 & 0 & 0 \end{pmatrix}.$$

We regard X as the current feasible basic solution.

The cost of the current feasible basic solution X is

$$\begin{aligned} & 8 \times 1 + 4 \times 12 + 3 \times 5 + 2 \times 3 + 6 \times 11 \\ & \qquad + 5 \times 13 \\ = \ & 8 + 48 + 15 + 6 + 66 + 65 \\ = \ & 208. \end{aligned}$$

The cost has gone down by 30, as one would expect (the reduction in the cost being $-\lambda q_{2,1}$ where $\lambda = 5$ and $q_{2,1} = -6$).

The current basic feasible solution X is associated with the basis B where

$$B = \{(1,1),\ (1,2),\ (2,1),\ (2,3),\ (3,3),\ (4,1)\}.$$

We now determine, for the current basis B values u_1, u_2, u_3, u_4 and v_1, v_2, v_3 such that $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$. the initial basis.

We seek a solution with $u_1 = 0$. We then determine $q_{i,j}$ so that $c_{i,j} = v_j - u_i + q_{i,j}$ for all i and j.

We therefore complete the following tableau:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	u_i
1	8 •	4 •	16	0
	0	0	?	
2	3 •	7	2 •	?
	0	?	0	
3	13	8	6 •	?
	?	?	0	
4	5 •	7	8	?
	0	?	?	
v_j	?	?	?	

Now $u_1 = 0$, $(1,1) \in B$ and $(1,2) \in B$ force $v_1 = 8$ and $v_2 = 4$.

Then $v_1 = 8$, $(2,1) \in B$ and $(4,1) \in B$ force $u_2 = 5$ and $u_4 = 3$.

Then $u_2 = 5$ and $(3,3) \in B$ force $v_3 = 7$.

Then $v_3 = 7$ and $(3,3) \in B$ force $u_3 = 1$.

Computing the numbers $q_{i,j}$ such that $c_{i,j} + u_i = v_j + q_{i,j}$, we find that $q_{1,3} = 9$, $q_{2,2} = 8$, $q_{3,1} = 6$, $q_{3,2} = 5$, $q_{4,2} = 6$ and $q_{4,3} = 4$.

The completed tableau is as follows:—

$c_{i,j} \searrow q_{i,j}$	1	2	3	u_i
1	8 •	4 •	16	0
	0	0	9	
2	3 •	7	2 •	5
	0	8	0	
3	13	8	6 •	1
	6	5	0	
4	5 •	7	8	3
	0	6	4	
v_j	8	4	7	

All numbers $q_{i,j}$ are non-negative for the current feasible basic solution. This solution is therefore optimal. Indeed, arguing as before we find that the cost $\overline{C}$ of any feasible solution $(\overline{x}_{i,j})$ satisfies

$$\overline{C} = 208 + 9\overline{x}_{1,3} + 8\overline{x}_{2,2} + 6\overline{x}_{3,1} + 5\overline{x}_{3,2} + 6\overline{x}_{4,2} + 4\overline{x}_{4,3}.$$

We conclude that X is an basic optimal solution, where

$$X = \begin{pmatrix} 1 & 12 & 0 \\ 5 & 0 & 3 \\ 0 & 0 & 11 \\ 13 & 0 & 0 \end{pmatrix}.$$

3.11 Formal Analysis of the Solution of the Transportation Problem

We now describe in general terms the method for solving a transportation problem in which total supply equals total demand.

We suppose that an initial basic feasible solution has been obtained. We apply an iterative method (based on the general Simplex Method for the solution of linear programming problems) that will test a basic feasible solution for optimality and, in the event that the feasible solution is shown not to be optimal, establishes information that (with the exception of certain 'degenerate' cases of the transportation problem) enables one to find a basic feasible solution with lower cost. Iterating this procedure a finite number of times, one should arrive at a basic feasible solution that is optimal for the given transportation problem.

We suppose that the given instance of the Transportation Problem involves m suppliers and n recipients. The required supplies are specified by non-negative real numbers $s_1, s_2, \ldots, s_m$, and the required demands are specified by non-negative real numbers $d_1, d_2, \ldots, d_n$. We further suppose that $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$. A *feasible solution* is represented by non-negative real numbers $x_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, where $\sum_{j=1}^{n} x_{i,j} = s_i$ for $i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} x_{i,j} = d_j$ for $j = 1, 2, \ldots, n$.

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$. A subset B of $I \times J$ is a *basis* for the transportation problem if and only if, given any real numbers $y_1, y_2, \ldots, y_m$ and $z_1, z_2, \ldots, z_n$, where $\sum_{i=1}^{m} y_i = \sum_{j=1}^{n} z_j$, there exist uniquely determined real numbers $\overline{x}_{i,j}$ for $i \in I$ and $j \in J$ such that $\sum_{j=1}^{n} \overline{x}_{i,j} = y_i$ for $i \in I$, $\sum_{i=1}^{m} \overline{x}_{i,j} = z_j$ for $j \in J$, where $\overline{x}_{i,j} = 0$ whenever $(i, j) \notin B$.

A feasible solution $(x_{i,j})$ is said to be a basic feasible solution associated with the basis B if and only if $x_{i,j} = 0$ for all $i \in I$ and $j \in J$ for which $(i, j) \notin B$.

Let $x_{i,j}$ be a non-negative real number for each $i \in I$ and $j \in J$. Suppose that $(x_{i,j})$ is a basic feasible solution to the transportation problem associated with basis B, where $B \subset I \times J$.

The cost associated with a feasible solution $(x_{i,j}$ is given by $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j}$, where the constants $c_{i,j}$ are real numbers for all $i \in I$ and $j \in J$. A feasible solution for a transportation problem is an optimal solution if and only if it minimizes cost amongst all feasible solutions to the problem.

In order to test for optimality of a basic feasible solution $(x_{i,j})$ associated with a basis B, we determine real numbers $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ with the property that $c_{i,j} = v_j - u_i$ for all $(i, j) \in B$. (Proposition 3.10 below guarantees that, given any basis B, it is always possible to find the required quantities u_i and v_j.) Having calculated these quantities u_i and v_j we determine the values of $q_{i,j}$, where $q_{i,j} = c_{i,j} - v_j + u_i$ for all $i \in I$ and $j \in J$. Then $q_{i,j} = 0$ whenever $(i, j) \in B$.

We claim that a basic feasible solution $(x_{i,j})$ associated with the basis B is optimal if and only if $q_{i,j} \geq 0$ for all $i \in I$ and $j \in J$. This is a consequence of the identity established in the following proposition.

Proposition 3.8 *Let $x_{i,j}$, $c_{i,j}$ and $q_{i,j}$ be real numbers defined for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, and let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ be real numbers.*

Suppose that

$$c_{i,j} = v_j - u_i + q_{i,j}$$

for $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$. *Then*

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j} = \sum_{j=1}^{n} v_j d_j - \sum_{i=1}^{m} u_i s_i + \sum_{i=1}^{m} \sum_{j=1}^{n} q_{i,j} x_{i,j},$$

where $s_i = \sum_{j=1}^{n} x_{i,j}$ *for* $i = 1, 2, \ldots, m$ *and* $d_j = \sum_{i=1}^{m} x_{i,j}$ *for* $j = 1, 2, \ldots, n$.

Proof The definitions of the relevant quantities ensure that

$$\begin{aligned}
\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j} &= \sum_{i=1}^{m} \sum_{j=1}^{n} (v_j - u_i + q_{i,j}) x_{i,j} \\
&= \sum_{j=1}^{n} \left(v_j \sum_{i=1}^{m} x_{i,j} \right) - \sum_{i=1}^{m} \left(u_i \sum_{j=1}^{n} x_{i,j} \right) \\
&\qquad + \sum_{i=1}^{m} \sum_{j=1}^{n} q_{i,j} x_{i,j} \\
&= \sum_{j=1}^{n} v_j d_j - \sum_{i=1}^{m} u_i s_i + \sum_{i=1}^{m} \sum_{j=1}^{n} q_{i,j} x_{i,j},
\end{aligned}$$

as required. ∎

Corollary 3.9 *Let* m *and* n *be integers, and let* $I = \{1, 2, \ldots, m\}$ *and* $J = \{1, 2, \ldots, n\}$. *Let* $x_{i,j}$ *and* $c_{i,j}$ *be real numbers defined for all* $i \in I$ *and* $j \in I$, *and let* $u_1, u_2, \ldots, u_m$ *and* $v_1, v_2, \ldots, v_n$ *be real numbers. Suppose that* $c_{i,j} = v_j - u_i$ *for all* $(i, j) \in I \times J$ *for which* $x_{i,j} \neq 0$. *Then*

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j} = \sum_{i=1}^{m} d_j v_j - \sum_{j=1}^{n} s_i u_i,$$

where $s_i = \sum_{j=1}^{n} x_{i,j}$ *for* $i = 1, 2, \ldots, m$ *and* $d_j = \sum_{i=1}^{m} x_{i,j}$ *for* $j = 1, 2, \ldots, n$.

Proof Let $q_{i,j} = c_{i,j} + u_i - v_j$ for all $i \in I$ and $j \in J$. Then $q_{i,j} = 0$ whenever $x_{i,j} \neq 0$. It follows from this that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} q_{i,j} x_{i,j} = 0.$$

It then follows from Proposition 3.8 that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}x_{i,j} = \sum_{i=1}^{m}\sum_{j=1}^{n}(v_j - u_i + q_{i,j})x_{i,j} = \sum_{i=1}^{m} d_j v_j - \sum_{j=1}^{n} s_i u_i,$$

as required. ∎

Let m and n be positive integers, let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let the subset B of $I \times J$ be a basis for a transportation problem with m suppliers and n recipients. Let the cost of a feasible solution $(\overline{x}_{i,j})$ be $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}\overline{x}_{i,j}$. Now $\sum_{j=1}^{n}\overline{x}_{i,j} = s_i$ and $\sum_{i=1}^{m}\overline{x}_{i,j} = d_j$, where the quantities s_i and d_j are determined by the specification of the problem and are the same for all feasible solutions of the problem. Let quantities u_i for $i \in I$ and v_j for $j \in J$ be determined such that $c_{i,j} = v_j - u_i$ for all $(i,j) \in B$, and let $q_{i,j} = c_{i,j} + u_i - v_j$ for all $i \in I$ and $j \in J$. Then $q_{i,j} = 0$ for all $(i,j) \in B$.

It follows from Proposition 3.8 that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}\overline{x}_{i,j} = \sum_{j=1}^{n} v_j d_j - \sum_{i=1}^{m} u_i s_i + \sum_{i=1}^{m}\sum_{j=1}^{n} q_{i,j}\overline{x}_{i,j}.$$

Now if the quantities $x_{i,j}$ for $i \in I$ and $j \in J$ constitute a basic feasible solution associated with the basis B then $x_{i,j} = 0$ whenever $(i,j) \notin B$. It follows that $\sum_{i=1}^{m}\sum_{j=1}^{n} q_{i,j}x_{i,j} = 0$, and therefore

$$\sum_{j=1}^{n} v_j d_j - \sum_{i=1}^{m} u_i s_i = C,$$

where

$$C = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}x_{i,j}.$$

The cost $\overline{C}$ of the feasible solution $(\overline{x}_{i,j})$ then satisfies the equation

$$\overline{C} = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}\overline{x}_{i,j} = C + \sum_{i=1}^{m}\sum_{j=1}^{n} q_{i,j}\overline{x}_{i,j}.$$

If $q_{i,j} \geq 0$ for all $i \in I$ and $j \in J$, then the cost $\overline{C}$ of any feasible solution $(\overline{x}_{i,j})$ is bounded below by the cost of the basic feasible solution $(x_{i,j})$. It follows that, in this case, the basic feasible solution $(x_{i,j})$ is optimal.

Suppose that (i_0, j_0) is an element of $I \times J$ for which $q_{i_0,j_0} < 0$. Then $(i_0, j_0) \notin B$. There is no basis for the transportation problem that includes the set $B \cup \{(i_0, j_0)\}$. A straightforward application of Proposition 3.6 establishes the existence of quantities $y_{i,j}$ for $i \in I$ and $j \in J$ such that $y_{i_0,j_0} = 1$ and $y_{i,j} = 0$ for all $i \in I$ and $j \in J$ for which $(i, j) \notin B \cup \{(i_0, j_0)\}$.

Let the $m \times n$ matrices X and Y be defined so that $(X)_{i,j} = x_{i,j}$ and $(Y)_{i,j} = y_{i,j}$ for all $i \in I$ and $j \in J$. Suppose that $x_{i,j} > 0$ for all $(i, j) \in B$. Then the components of X in the basis positions are strictly positive. It follows that, if λ is positive but sufficiently small, then the components of the matrix $X + \lambda Y$ in the basis positions are also strictly positive, and therefore the components of the matrix $X + \lambda Y$ are non-negative for all sufficiently small non-negative values of λ. There will then exist a maximum value λ_0 that is an upper bound on the values of λ for which all components of the matrix $X+\lambda Y$ are non-negative. It is then a straightforward exercise in linear algebra to verify that $X + \lambda_0 Y$ is another basic feasible solution associated with a basis that includes (i_0, j_0) together with all but one of the elements of the basis B.

Moreover the cost of this new basic feasible solution is $C + \lambda_0 q_{i_0,j_0}$, where C is the cost of the basic feasible solution represented by the matrix X. Thus if $q_{i_0,j_0} < 0$ then the cost of the new basic feasible solution is lower than that of the basic feasible solution X from which it was derived.

Suppose that, for all basic feasible solutions of the given Transportation problem, the coefficients of the matrix specifying the basic feasible solution are strictly positive at the basis positions. Then a finite number of iterations of the procedure discussed above with result in an basic optimal solution of the given transportation problem. Such problems are said to be *non-degenerate*.

However if it turns out that a basic feasible solution $(x_{i,j})$ associated with a basis B satisfies $x_{i,j} = 0$ for some $(i, j) \in B$, then we are in a *degenerate* case of the transportation problem. The theory of degenerate cases of linear programming problems is discussed in detail in textbooks that discuss the details of linear programming algorithms.

We now establish the proposition that guarantees that, given any basis B, there exist quantities $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ such that the costs $c_{i,j}$ associated with the given transportation problem satisfy $c_{i,j} = v_j - u_i$ for all $(i, j) \in B$. This result is an essential component of the method described here for testing basic feasible solutions to determine whether or not they are optimal.

Proposition 3.10 *Let m and n be integers, let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$, and let B be a subset of $I \times J$ that is a basis for the transporta-*

tion problem with m suppliers and n recipients. For each $(i, j) \in B$ let $c_{i,j}$ be a corresponding real number. Then there exist real numbers u_i for $i \in I$ and v_j for $j \in J$ such that $c_{i,j} = v_j - u_i$ for all $(i, j) \in B$. Moreover if $\overline{u}_i$ and $\overline{v}_j$ are real numbers for $i \in I$ and $j \in J$ that satisfy the equations $c_{i,j} = \overline{v}_j - \overline{u}_i$ for all $(i, j) \in B$, then there exists some real number k such that $\overline{u}_i = u_i + k$ for all $i \in I$ and $\overline{v}_j = v_j + k$ for all $j \in J$.

Proof Let

$$M_B = \{X \in M_{m,n}(\mathbb{R}) : (X)_{i,j} = 0 \text{ unless } (i, j) \in B\}.$$

It follows from the definition of bases for transportation problems that there exist unique $m \times n$ matrices $S_1, S_2, \ldots, S_m$ belonging to M_B, where S_1 is the zero matrix, and where, for each integer i satisfying $1 < i \leq m$, the matrix S_k has the properties that

$$\sum_{\ell=1}^{n} (S_i)_{k,\ell} = \begin{cases} 1 & \text{if } k = 1, \\ -1 & \text{if } k = i, \\ 0 & \text{if } k \in I \setminus \{1, i\}, \end{cases}$$

and

$$\sum_{k=1}^{m} (S_i)_{k,\ell} = 0 \text{ for all } \ell \in J.$$

Also there exist unique $m \times n$ matrices $T_1, T_2, \ldots, T_m$ belonging to M_B where, for each integer j satisfying $1 \leq j \leq n$, the matrix T_j has the properties that

$$\sum_{j=1}^{n} (T_j)_{k,l} = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k \in I \setminus \{1\}, \end{cases}$$

and

$$\sum_{i=1}^{m} (T_j)_{k,\ell} = \begin{cases} 1 & \text{if } \ell = j, \\ 0 & \text{if } \ell \in J \setminus \{j\}, \end{cases}$$

Let

$$u_i = \sum_{k=1}^{n} \sum_{\ell=1}^{n} c_{k,\ell} (S_i)_{k,\ell}$$

for $i = 1, 2, \ldots, m$ and

$$v_j = \sum_{k=1}^{m} \sum_{\ell=1}^{n} c_{k,\ell} (T_j)_{k,\ell}.$$

for $j = 1, 2, \ldots, n$. We claim the that numbers $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ have the required properties.

Let X be an $m \times n$ matrix belonging to M_B, and let

$$y_i = \sum_{j=1}^{n} (X)_{i,j} \quad \text{for all } i \in I$$

and

$$z_j = \sum_{i=1}^{m} (X)_{i,j} \quad \text{for all } j \in J,$$

and let

$$\overline{X} = \sum_{\ell=1}^{n} z_\ell T_\ell - \sum_{k=1}^{m} y_k S_k.$$

Then

$$\sum_{i=1}^{m} (\overline{X})_{i,j} = z_j \quad \text{for all } j \in J.$$

and

$$\sum_{j=1}^{n} (\overline{X})_{i,j} = y_i \quad \text{for all } i \in I \setminus \{1\},$$

Moreover

$$\sum_{j=1}^{n} (\overline{X})_{1,j} = \sum_{\ell=1}^{n} z_\ell - \sum_{k=2}^{m} y_k = y_1,$$

because $\sum\limits_{i=1}^{m} y_i = \sum\limits_{j=1}^{n} z_j$.

But the definition of bases for transportation problems ensures that X is the unique $m \times n$ matrix belonging to M_B with the properties that $\sum\limits_{j=1}^{n} (X)_{i,j} = y_i$ for all $i \in I$ and $\sum\limits_{i=1}^{m} (X)_{i,j} = z_j$ for all $j \in J$. It follows that

$$X = \overline{X} = \sum_{j=1}^{n} z_j T_j - \sum_{i=1}^{m} y_i S_i,$$

and therefore

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} c_{k,\ell} (X)_{k,\ell} = \sum_{j=1}^{n} z_j v_j - \sum_{i=1}^{m} y_i u_i.$$

Let $(i,j) \in B$. Then $E^{(i,j)} \in M_B$, where

$$(E^{(i,j)})_{k,\ell} = \begin{cases} 1 & \text{if } k = i \text{ and } j = \ell; \\ 0 & \text{if } k \neq i \text{ or } j \neq \ell. \end{cases}$$

It follows from the result just obtained that

$$c_{i,j} = \sum_{k=1}^{m} \sum_{\ell=1}^{n} c_{k,\ell} (E^{(i,j)})_{k,\ell} = v_j - u_i.$$

We have thus shown that, given any basis B for the transportation problem with m suppliers and n recipients, there exist real numbers $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ with the required property that

$$c_{i,j} = v_j - u_i \quad \text{for all } (i,j) \in B..$$

Now let $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_m$ and $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_n$ be real numbers with the property that

$$c_{i,j} = \overline{v}_j - \overline{u}_i \quad \text{for all } (i,j) \in B..$$

Then $b_j - a_i = 0$ for all $(i,j) \in B$, where $a_i = \overline{u}_i - u_i$ for $i = 1, 2, \ldots, m$ and $b_j = \overline{v}_j - v_j$ for $j = 1, 2, \ldots, n$, and therefore

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} (b_\ell - a_k)(E^{i,j})_{k,\ell} = 0$$

for all $(i,j) \in B$. Now the $m \times n$ matrices $E^{(i,j)}$ for which $(i,j) \in B$ constitute a basis of the vector space M_B. It follows that

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} (b_\ell - a_k)(X)_{k,\ell} = 0$$

for all $X \in M_B$.

In particular

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} (b_\ell - a_k)(S_i)_{k,\ell} = 0$$

for $i = 2, 3, \ldots, m$, and

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} (b_\ell - a_k)(T_j)_{k,\ell} = 0$$

for $j = 1, 2, \ldots, n$.

But it follows from the definitions of the matrices $S_1, S_2, \ldots, S_m$ and $T_1, T_2, \ldots, T_n$ that

$$\begin{aligned}
\sum_{k=1}^{m} \sum_{\ell=1}^{n} b_\ell (S_i)_{k,\ell} &= \sum_{\ell=1}^{n} \left(b_\ell \sum_{k=1}^{m} (S_i)_{k,\ell} \right) = 0, \\
\sum_{k=1}^{m} \sum_{\ell=1}^{n} a_k (S_i)_{k,\ell} &= \sum_{k=1}^{m} \left(a_k \sum_{\ell=1}^{n} (S_i)_{k,\ell} \right) = a_1 - a_i
\end{aligned}$$

for $i = 2, 3, \ldots, m$, and

$$\begin{aligned}
\sum_{k=1}^{m} \sum_{\ell=1}^{n} b_\ell (T_j)_{k,\ell} &= \sum_{\ell=1}^{n} \left(b_\ell \sum_{k=1}^{m} (T_j)_{k,\ell} \right) = b_j, \\
\sum_{k=1}^{m} \sum_{\ell=1}^{n} a_k (S_i)_{k,\ell} &= \sum_{k=1}^{m} \left(a_k \sum_{\ell=1}^{n} (S_i)_{k,\ell} \right) = a_1
\end{aligned}$$

for $j = 1, 2, \ldots, n$.

It follows that $a_i - a_1 = 0$ for $i = 2, \ldots, n$ and $b_j - a_1 = 0$ for $j = 1, 2, \ldots, n$. Thus if $k = a_1$ then $\overline{u}_i = u_i + a_i = u_i + k$ for $i = 1, 2, \ldots, m$ and $\overline{v}_j = v_j + b_j = v_j + k$ for $j = 1, 2, \ldots, n$, as required. ∎