Module MA342R: Annual Examination 2017
Worked solutions

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Module Website

The module website, with online lecture notes, problem sets. etc. are located at

`http://www.maths.tcd.ie/~dwilkins/Courses/MA342R/`

1. (a) Let $\gamma\colon [a,b] \to \mathbb{C}$ be a closed path in the complex plane, and let w be a complex number that does not lie on γ. The *winding number* of γ about w is defined to be the unique integer $n(\gamma, w)$ with the property that $\varphi(b) - \varphi(a) = 2\pi i n(\gamma, w)$ for all paths $\varphi\colon [a,b] \to \mathbb{C}$ in the complex plane that satisfy $\exp(\varphi(t)) = \gamma(t) - w$ for all $t \in [a,b]$.

(b) Note that the inequality satisfied by the functions γ_1 and γ_2 ensures that w does not lie on the path γ_2. Let $\tilde{\gamma}_1\colon [0,1] \to \mathbb{C}$ be a path in the complex plane such that $\exp(\tilde{\gamma}_1(t)) = \gamma_1(t) - w$ for all $t \in [a,b]$, and let

$$\rho(t) = \frac{\gamma_2(t) - w}{\gamma_1(t) - w}$$

for all $t \in [a,b]$ Then

$$|\rho(t) - 1| = \left| \frac{\gamma_2(t) - \gamma_1(t)}{\gamma_1(t) - w} \right| < 1$$

for all $t \in [a,b]$.

There exists a continuous function $F\colon \{z \in \mathbb{C} : |z - 1| < 1\} \to \mathbb{C}$ with the property that $\exp(F(z)) = z$ for all complex numbers z satisfying $|z - 1| < 1$. Let $\tilde{\gamma}_2\colon [0,1] \to \mathbb{C}$ be the path in the complex plane defined such that $\tilde{\gamma}_2(t) = F(\rho(t)) + \tilde{\gamma}_1(t)$ for all $t \in [a,b]$. Then

$$\begin{aligned} \exp(\tilde{\gamma}_2(t)) &= \exp(F(\rho(t))) \exp(\tilde{\gamma}_1(t)) = \rho(t)(\gamma_1(t) - w) \\ &= \gamma_2(t) - w. \end{aligned}$$

Now $\rho(b) = \rho(a)$. It follows that

$$\begin{aligned} 2\pi i n(\gamma_2, w) &= \tilde{\gamma}_2(b) - \tilde{\gamma}_2(a) \\ &= F(\rho(b)) + \tilde{\gamma}_1(b) - F(\rho(a)) - \tilde{\gamma}_1(a) \\ &= \tilde{\gamma}_1(b) - \tilde{\gamma}_1(a) \\ &= 2\pi i n(\gamma_1, w), \end{aligned}$$

as required.

(c) [Seen Similar.] Let $\gamma\colon [0,1] \to \mathbb{C}$ be the closed curve in the complex plane defined such that

$$\gamma(t) = (5 + e^{\cos 2\pi t}) \cos 10\pi t + \sin 8\pi t \sin 12\pi t + i(3 \sin 10\pi t + \cos 4\pi t \sin 12\pi t).$$

for all $t \in [0,1]$, where $i^2 = -1$. Let

$$\gamma_1(t) = (5 + e^{\cos 2\pi t}) \cos 10\pi t + 3i \sin 10\pi t$$

for all $t \in [0,1]$. Then $|\gamma_1(t)| \geq 3$ for all $t \in [0,1]$. Also $|\sin 8\pi t \, \sin 12\pi t| \leq 1$ and $|\cos 4\pi t \, \sin 12\pi t| \leq 1$ and and therefore

$$|\gamma(t) - \gamma_1(t)| = |\sin 8\pi t \, \sin 12\pi t + i \cos 4\pi t \, \sin 12\pi t| < |\gamma_1(t)|$$

for all $t \in [0,1]$. The Dog-Walking Lemma then ensures that $n(\gamma, 0) = n(\gamma_1, 0)$. Another application of the Dog-Walking Lemma then ensures that $n(\gamma_1, 0) = n(\gamma_2, 0)$, where

$$\gamma_2(t) = 3(\cos 10\pi t + i \sin 10\pi t)$$

for all $t \in [0,1]$. Moreover $\gamma_2 = \exp \circ \tilde{\gamma}_2$ where $\tilde{\gamma}_2 \colon [0,1] \to \mathbb{C}$ is the path in $\mathbb{C}$ defined so that $\tilde{\gamma}_2(t) = \log 3 + 10\pi t$ for all $t \in [0,1]$. The definition of winding number ensures that

$$n(\gamma_2, 0) = (2\pi i)^{-1}(\tilde{\gamma}_2(1) - \tilde{\gamma}_2(0)) = 5.$$

Therefore $n(\gamma, 0) = 5$.

2. (a) [Bookwork.] Let X and Y be topological spaces, and let A be a subset of X. Let $f: X \to Y$ and $g: X \to Y$ be continuous maps from X to some topological space Y, where $f|A = g|A$ (i.e., $f(a) = g(a)$ for all $a \in A$). We say that f and g are *homotopic relative to* A (denoted by $f \simeq g$ rel A) if and only if there exists a (continuous) homotopy $H: X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$ and $H(a,t) = f(a) = g(a)$ for all $a \in A$.

(b) [Standard definition, but not stated exactly as below in lecture notes.] Let X be a topological space, let x_0 be some chosen point of X, and let $\pi_1(X, x_0)$ be the set of all based homotopy classes of loops based at the point x_0, where two loops γ_1 and γ_2 are in the same based homotopy class if and only if $\gamma_1 \simeq \gamma_2$ rel $\{0,1\}$. Then $\pi_1(X, x_0)$ is a group, the group multiplication on $\pi_1(X, x_0)$ being defined according to the rule $[\gamma_1][\gamma_2] = [\gamma_1.\gamma_2]$ for all loops γ_1 and γ_2 based at x_0, where $\gamma_1.\gamma_2$ denotes the concatenation of the loops γ_1 and γ_2. This group is the *fundamental group* of X based at the point x_0. The identity element of the fundamental loop is represented by the constant loop at the basepoint x_0. The inverse of a loop $\gamma: [0,1] \to X$ is represented by the loop $\gamma^{-1}: [0,1] \to X$, where $\gamma^{-1}(t) = \gamma(1-t)$ for all $t \in [0,1]$.

(c) [Not bookwork. There are several reasonably obvious approaches to the details. Probably the class will have seen similar problems by the end of teaching.]

Let $f: Z \to W$ be a continuous map from a topological space Z to a topological space W, and let z_0 be a point of Z. Suppose there exists a continuous map $g: W \to Z$ such that $g \circ f \simeq$ identity rel $\{x_0\}$ and $f \circ g \simeq$ identity rel $\{f(x_0)\}$. Then $\pi_1(Z, x_0) \cong \pi_1(W, f(x_0))$.

We can apply this result where $f: X \to S^1$ is defined such that

$$f(x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right),$$

taking $g: S^1 \to X$ to be the inclusion map, and noting that, for all $(x,y,z) \in X$, the points (x,y,z) and $f(x,y,z)$ are the endpoints of a line segment lying wholly in X. It follows that

$$\pi_(X, (1,0,0)) \cong \pi_1(S^1, (1,0)) \cong \mathbb{Z}.$$

For Y, note that radial projection maps Y onto the unit sphere S^2 in $\mathbb{R}^2$. Moreover S^2 is simply-connected. It follows that Y

is simply-connected, and therefore has trivial fundamental group (for any basepoint).

3. (a) [Bookwork.] Let $Z_0 = \{z \in Z : g(z) = h(z)\}$. Note that Z_0 is non-empty, by hypothesis. We show that Z_0 is both open and closed in Z.

Let z be a point of Z. There exists an open set U in X containing the point $p(g(z))$ which is evenly covered by the covering map p. Then $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the covering map p. One of these open sets contains $g(z)$; let this set be denoted by $\tilde{U}$. Also one of these open sets contains $h(z)$; let this open set be denoted by $\tilde{V}$. Let $N_z = g^{-1}(\tilde{U}) \cap h^{-1}(\tilde{V})$. Then N_z is an open set in Z containing z.

Consider the case when $z \in Z_0$. Then $g(z) = h(z)$, and therefore $\tilde{V} = \tilde{U}$. It follows from this that both g and h map the open set N_z into $\tilde{U}$. But $p \circ g = p \circ h$, and $p|\tilde{U} : \tilde{U} \to U$ is a homeomorphism. Therefore $g|N_z = h|N_z$, and thus $N_z \subset Z_0$. We have thus shown that, for each $z \in Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z_0$. We conclude that Z_0 is open.

Next consider the case when $z \in Z \setminus Z_0$. In this case $\tilde{U} \cap \tilde{V} = \emptyset$, since $g(z) \neq h(z)$. But $g(N_z) \subset \tilde{U}$ and $h(N_z) \subset \tilde{V}$. Therefore $g(z') \neq h(z')$ for all $z' \in N_z$, and thus $N_z \subset Z \setminus Z_0$. We have thus shown that, for each $z \in Z \setminus Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z \setminus Z_0$. We conclude that $Z \setminus Z_0$ is open.

The subset Z_0 of Z is therefore both open and closed. Also Z_0 is non-empty by hypothesis. We deduce that $Z_0 = Z$, since Z is connected. Thus $g = h$, as required.

(b) [Bookwork.]

Let S be the subset of $[a, b]$ defined as follows: an element c of $[a, b]$ belongs to S if and only if there exists a continuous map $\eta_c : [a, c] \to \tilde{X}$ such that $\eta_c(a) = w$ and $p(\eta_c(t)) = \gamma(t)$ for all $t \in [a, c]$. Note that S is non-empty, since a belongs to S. Let $s = \sup S$.

There exists an open neighbourhood U of $\gamma(s)$ which is evenly covered by the map p, since $p: \tilde{X} \to X$ is a covering map. It then follows from the continuity of the path γ that there exists some $\delta > 0$ such that $\gamma(J(s, \delta)) \subset U$, where

$$J(s, \delta) = \{t \in [a, b] : |t - s| < \delta\}.$$

Now $S \cap J(s, \delta)$ is non-empty, because s is the supremum of the set S. Choose some element c of $S \cap J(s, \delta)$. Then there exists a continuous map $\eta_c: [a, c] \to \tilde{X}$ such that $\eta_c(a) = w$ and $p(\eta_c(t)) = \gamma(t)$ for all $t \in [a, c]$. Now the open set U is evenly covered by the map p. Therefore $p^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically onto U by the covering map p. One of these open sets contains the point $\eta_c(c)$; let this open set be denoted by $\tilde{U}$. There then exists a unique continuous map $\sigma: U \to \tilde{U}$ defined such that, for all $x \in U$, $\sigma(x)$ is the unique element of $\tilde{U}$ for which $p(\sigma(x)) = x$. Then $\sigma(\gamma(c)) = \eta_c(c)$.

Then, given any $d \in J(s, \delta)$, let $\eta_d: [a, d] \to \tilde{X}$ be the function from $[a, d]$ to $\tilde{X}$ defined so that

$$\eta_d(t) = \begin{cases} \eta_c(t) & \text{if } a \leq t \leq c; \\ \sigma(\gamma(t)) & \text{if } c \leq t \leq d. \end{cases}$$

Then $\eta_d(a) = w$ and $p(\eta_d(t)) = \gamma(t)$ for all $t \in [a, d]$. The restrictions of the function $\eta_d: [a, d] \to \tilde{X}$ to the intervals $[a, c]$ and $[c, d]$ are continuous. It follows from the Pasting Lemma that η_d is continuous on $[a, d]$. Thus $d \in S$. We conclude from this that $J(s, \delta) \subset S$. However s is defined to be the supremum of the set S. Therefore $s = b$, and b belongs to S. It follows that that there exists a continuous map $\tilde{\gamma}: [a, b] \to \tilde{X}$ for which $\tilde{\gamma}(a) = w$ and $p \circ \tilde{\gamma} = \gamma$, as required.

4. (a) [Definition] Let X and Y be topological spaces and let $q: X \to Y$ be a function from X to Y. The function q is said to be an *identification map* if and only if the following conditions are satisfied:
   - the function $q: X \to Y$ is surjective,
   - a subset U of Y is open in Y if and only if $q^{-1}(U)$ is open in X.

(b) [Definition.] Let G be a group with identity element e, and let X be a topological space. The group G is said to act *freely and properly discontinuously* on X if each element g of G determines a corresponding continuous map $\theta_g: X \to X$, where the following conditions are satisfied:

   (i) $\theta_{gh} = \theta_g \circ \theta_h$ for all $g, h \in G$;

   (ii) the continuous map θ_e determined by the identity element e of G is the identity map of X;

   (iii) given any point x of X, there exists an open set U in X such that $x \in U$ and $\theta_g(U) \cap U = \emptyset$ for all $g \in G$ satisfying $g \neq e$.

(c) [Bookwork.] The quotient map $q: X \to X/G$ is surjective. Let V be an open set in X. Then $q^{-1}(q(V))$ is the union $\bigcup_{g \in G} \theta_g(V)$ of the open sets $\theta_g(V)$ as g ranges over the group G, since $q^{-1}(q(V))$ is the subset of X consisting of all elements of X that belong to the orbit of some element of V. But any union of open sets in a topological space is an open set. We conclude therefore that if V is an open set in X then $q(V)$ is an open set in X/G.

Let x be a point of X. Then there exists an open set U in X such that $x \in U$ and $\theta_g(U) \cap U = \emptyset$ for all $g \in G$ satisfying $g \neq e$. Now $q^{-1}(q(U)) = \bigcup_{g \in G} \theta_g(U)$. We claim that the sets $\theta_g(U)$ are disjoint. Let g and h be elements of G. Suppose that $\theta_g(U) \cap \theta_h(U) \neq \emptyset$. Then $\theta_{h^{-1}}(\theta_g(U) \cap \theta_h(U)) \neq \emptyset$. But $\theta_{h^{-1}}: X \to X$ is a bijection, and therefore

$$\theta_{h^{-1}}(\theta_g(U) \cap \theta_h(U)) = \theta_{h^{-1}}(\theta_g(U)) \cap \theta_{h^{-1}}(\theta_h(U)) = \theta_{h^{-1}g}(U) \cap U,$$

and therefore $\theta_{h^{-1}g}(U) \cap U \neq \emptyset$. It follows that $h^{-1}g = e$, where e denotes the identity element of G, and therefore $g = h$. Thus if g and h are elements of g, and if $g \neq h$, then $\theta_g(U) \cap \theta_h(U) = \emptyset$. We conclude therefore that the preimage $q^{-1}(q(U))$ of $q(U)$ is the disjoint union of the sets $\theta_g(U)$ as g ranges over the group G. Moreover each these sets $\theta_g(U)$ is an open set in X.

Now $U \cap [u]_G = \{u\}$ for all $u \in U$, since $[u]_G = \{\theta_g(u) : g \in G\}$ and $U \cap \theta_g(U) = \emptyset$ when $g \neq e$. Thus if u and v are elements of U, and if $q(u) = q(v)$ then $[u]_G = [v]_G$ and therefore $u = v$. It follows that the restriction $q|U\colon U \to X/G$ of the quotient map q to U is injective, and therefore q maps U bijectively onto $q(U)$. But q maps open sets onto open sets, and any continuous bijection that maps open sets onto open sets is a homeomorphism. We conclude therefore that the restriction of $q\colon X \to X/G$ to the open set U maps U homeomorphically onto $q(U)$. Moreover, given any element g of G, the quotient map q satisfies $q = q \circ \theta_{g^{-1}}$, and the homeomorphism $\theta_{g^{-1}}$ maps $\theta_g(U)$ homeomorphically onto U. It follows that the quotient map q maps $\theta_g(U)$ homeomorphically onto $q(U)$ for all $g \in U$. We conclude therefore that $q(U)$ is an evenly covered open set in X/G whose preimage $q^{-1}(q(U))$ is the disjoint union of the open sets $\theta_g(U)$ as g ranges over the group G. It follows that the quotient map $q\colon X \to X/G$ is a covering map, as required.