Partial Worked Solutions to Sample Paper - MA3429

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May 23, 2011

1(c)

Use formular from 1(b). Now $x = F(u,v) = u^2 + v^2$ so

$$\frac{\partial F}{\partial u} = 2u, \quad \frac{\partial F}{\partial v} = 2v.$$

So

$$\begin{aligned} \varphi_* \left(u^3 \left. \frac{\partial}{\partial u} \right|_p + v^3 \left. \frac{\partial}{\partial v} \right|_p \right) &= \left(u(p)^3 \frac{\partial F}{\partial u} + v(p)^3 \frac{\partial F}{\partial v} \right) \frac{d}{dx} \\ &= 2(u(p)^4 + v(p)^4) \left. \frac{d}{dx} \right|_{\varphi(p)} \end{aligned}$$

for all points p of $\mathbb{R}^2$. Now we evaluate at $u = 1$, $v = 0$. to obtain

$$\varphi_* \left(u^3 \left. \frac{\partial}{\partial u} \right|_{(1,0)} + v^3 \left. \frac{\partial}{\partial v} \right|_{(1,0)} \right) = 2 \left. \frac{d}{dx} \right|_1 .$$

(Maybe those were not particularly interesting values of of u and v to choose.)

Alternatively, to obtain the result more directly, let $f: \mathbb{R} \to \mathbb{R}$ be a smooth real-valued function on R. Then $(\varphi_* X_p)[f] = X_p[f \circ \varphi]$. Now $(f \circ \varphi)(u,v) = f(u^2 + v^2)$, so if

$$X_p = u^3 \left. \frac{\partial}{\partial u} \right|_p + v^3 \left. \frac{\partial}{\partial v} \right|_p$$

at a point p of $\mathbb{R}^2$, where $p = (u,v)$, then

$$(\varphi_* X_p)[f] = u^3 \left. \frac{\partial f(u^2+v^2)}{\partial u} \right|_p + v^3 \left. \frac{\partial f(u^2+v^2)}{\partial v} \right|_p$$

$$
\begin{aligned}
&= u^3 \left.\frac{\partial f(x)}{\partial x}\right|_{x=u^2+v^2} \frac{\partial u^2+v^2}{\partial u} + v^3 \left.\frac{\partial f(x)}{\partial x}\right|_{x=u^2+v^2} \frac{\partial u^2+v^2}{\partial v} \\
&= 2u^4 \left.\frac{\partial f(x)}{\partial x}\right|_{x=u^2+v^2} + 2v^4 \left.\frac{\partial f(x)}{\partial x}\right|_{x=u^2+v^2},
\end{aligned}
$$

and thus

$$\varphi_* X_p = 2(u^4+v^4) \left.\frac{\partial}{\partial x}\right|_{\varphi(p)}.$$

3

Note that the formula for the Riemann curvature tensor should have read

$$R(W,Z,X,Y) = g(W, R(X,Y)Z).$$

3(b)

$$
\begin{aligned}
[E_t, E_z] &= \left[\sqrt{1+r^2}\,\frac{\partial}{\partial t}, \frac{1}{\sin r}\frac{\partial}{\partial z}\right] \\
&= \sqrt{1+r^2}\frac{\partial}{\partial t}\left(\frac{1}{\sin r}\right)\frac{\partial}{\partial z} - \frac{1}{\sin r}\frac{\partial}{\partial z}\sqrt{1+r^2}\,\frac{\partial}{\partial t} \\
&= 0, \\
[E_t, E_r] &= \left[\sqrt{1+r^2}\,\frac{\partial}{\partial t}, \frac{\partial}{\partial r}\right] \\
&= -\frac{\partial}{\partial r}\left(\sqrt{1+r^2}\right)\frac{\partial}{\partial t} \\
&= -\frac{r}{\sqrt{1+r^2}}\frac{\partial}{\partial t} \\
&= -\frac{r}{1+r^2}E_t \\
&\ \text{etc.}
\end{aligned}
$$

3(c)

$g(E_t, E_t) = -1$ everywhere, and is thus a constant function on M.

$$0 = E_r[g(E_t,E_t)] = g(\nabla_{E_r}E_t, E_t) + g(E_t, \nabla_{E_r}E_t) = 2g(\nabla_{E_r}E_t, E_t).$$

Thus $g(\nabla_{E_r}E_t, E_t) = 0$. Similarly $g(\nabla_{E_z}E_t, E_t) = 0$ and $g(\nabla_{E_\theta}E_t, E_t) = 0$.

Now

$$\begin{aligned} g(\nabla_{E_t} E_r, E_t) &= g(\nabla_{E_t} E_r - \nabla_{E_r} E_t, E_t) + g(\nabla_{E_r} E_t, E_t) \\ &= g([E_t, E_r], E_t) + 0 \\ &= -\frac{r}{1+r^2} g(E_t, E_t) = \frac{r}{1+r^2}. \end{aligned}$$

Similarly for other cases.

4(c)

Curve γ is a geodesic if and only if

$$\frac{D}{dt}\gamma'(t) = 0.$$

$$\begin{aligned} 2|\gamma'(t)|\frac{d}{dt}|\gamma'(t)| &= \frac{d}{dt}\left(|\gamma'(t)|^2\right) = \frac{d}{dt} g(\gamma'(t), \gamma'(t)) \\ &= 2g\left(\gamma'(t), \frac{D}{dt}\gamma'(t)\right) = 0, \end{aligned}$$

hence result.

4(d)

$$\begin{aligned} \frac{d}{dt}\left(g(V(t), \gamma'(t)\right) &= g\left(\frac{DV(t)}{dt}, \gamma'(t)\right) + g\left(V(t), \frac{D}{dt}\gamma'(t)\right) \\ &= f(t) g\left(V(t)\, \gamma'(t)\right) \end{aligned}$$

Thus if $h(t) = g(V(t), \gamma'(t)$ for all $t \in \mathbb{R}$ then the function h satisfies the differential equation

$$\frac{dh(t)}{dt} = f(t)h(t),$$

with initial condition $h(0) = 0$. The solution to such a differential equation is uniquely determined by the initial condition. In this case it must be the zero solution. Thus $g(V(t), \gamma'(t)) = h(t) = 0$ for all $t \in \mathbb{R}$.