# MA3427—Algebraic Topology I
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# Section 2: Winding Numbers of Closed Paths in the Complex Plane

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# 2. Winding Numbers of Closed Curves in the Plane

## 2.1. Paths in the Complex Plane

Let $D$ be a subset of the complex plane $\mathbb{C}$. We define a *path* in $D$ to be a continuous complex-valued function $\gamma\colon [a, b] \to D$ defined over some closed interval $[a, b]$. We shall denote the range $\gamma([a, b])$ of the function $\gamma$ defining the path by $[\gamma]$. Now it follows from the Heine-Borel Theorem (Theorem 1.37) that the closed bounded interval $[a, b]$ is compact. Moreover continuous functions map compact sets to compact sets (see Lemma 1.39). It follows that $[\gamma]$ is a closed bounded subset of the complex plane.

**Lemma 2.1**

*Let $\gamma\colon [a,b] \to \mathbb{C}$ be a path in the complex plane, and let $w$ be a complex number that does not lie on the path $\gamma$. Then there exists some positive real number $\varepsilon_0$ such that $|\gamma(t) - w| \geq \varepsilon_0 > 0$ for all $t \in [a,b]$.*

**Proof**

The closed unit interval $[a,b]$ is a closed bounded subset of $\mathbb{R}$. Now any continuous real-valued function on a compact set is bounded above and below on that set (Lemma 1.40). Therefore there exists some positive real number $M$ such that $|\gamma(t) - w|^{-1} \leq M$ for all $t \in [a,b]$. Let $\varepsilon_0 = M^{-1}$. Then the positive real number $\varepsilon_0$ has the required property. ∎

**Definition**

A path $\gamma\colon [a, b] \to \mathbb{C}$ in the complex plane is said to be *closed* if $\gamma(a) = \gamma(b)$.

**Remark**

The use of the technical term *closed* as in the above definition has no relation to the notions of open and closed sets.) Thus a *closed path* is a path that returns to its starting point.

Let $\gamma\colon [a, b] \to \mathbb{C}$ be a path in the complex plane. We say that a complex number $w$ *lies on* the path $\gamma$ if $w \in [\gamma]$, where $[\gamma] = \gamma([a, b])$.

## 2.2. The Exponential Map

The exponential map $\exp\colon \mathbb{C} \to \mathbb{C}$ is defined on the complex plane so that

$$\exp(x + iy) = e^x \cos y + i\, e^x \sin y$$

for all real numbers $x$ and $y$, where $i^2 = -1$. Then

$$\exp(x + iy) = u(x, y) + i\, v(x, y)$$

where

$$u(x, y) = e^x \cos y, \quad v(x, y) = e^x \sin y$$

for all real numbers $x$ and $y$. The functions $u\colon \mathbb{R}^2 \to \mathbb{R}$ and $v\colon \mathbb{R}^2 \to \mathbb{R}$ satisfy the partial differential equations

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

These partial differential equations are the *Cauchy-Riemann equations* that are satisfied by the real and imaginary parts of a function of a complex variable if and only if that function is holomorphic.

**Lemma 2.2**

*The exponential map* $\exp\colon \mathbb{C} \to \mathbb{C}$ *satisfies the identities* $\exp(z + w) = \exp(z)\,\exp(w)$ *and* $\exp(-z) = \exp(z)^{-1}$ *for all complex numbers* $z$ *and* $w$.

**Proof**

Let $z = x + iy$ and $w = u + iv$, where $x$, $y$, $u$ and $v$ are real numbers and $i^2 = -1$. Then

$$\begin{aligned}
\exp(z + w) &= e^{x+u}(\cos(y + v) + i \sin(y + v)) \\
&= e^x e^u\big(\cos y\, \cos v - \sin y\, \sin v \\
&\qquad + i \sin y\, \cos v + i \cos y\, \sin v\big) \\
&= e^x e^u(\cos y + i \sin y)(\cos v + i \sin v) \\
&= \exp(z)\, \exp(w).
\end{aligned}$$

Applying this result with $w = -z$, we see that $\exp(z) \exp(-z) = \exp(0) = 1$, and therefore $\exp(-z) = \exp(z)^{-1}$, as required. ∎

### Lemma 2.3

*Let $z$ and $w$ be complex numbers. Then $\exp(z) = \exp(w)$ if and only if $w = z + 2\pi i n$ for some integer $n$.*

**Proof**

Suppose that $w = z + 2\pi i n$ for some integer $n$. Then

$$\begin{aligned} \exp(w) &= \exp(z)\exp(2\pi i n) = \exp(z)(\cos 2\pi n + i \sin 2\pi n) \\ &= \exp(z). \end{aligned}$$

Conversely suppose that $\exp(w) = \exp(z)$. Let $w - z = u + iv$, where $u$ and $v$ are real numbers. Then

$$e^u(\cos v + i \sin v) = \exp(w - z) = \exp(w)\exp(z)^{-1} = 1.$$

Taking the modulus of both sides, we see that $e^u = 1$, and thus $u = 0$. Also $\cos v = 1$ and $\sin v = 0$, and therefore $v = 2\pi n$ for some integer $n$. The result follows. ∎

**Remark**

The infinite series $\displaystyle\sum_{n=0}^{+\infty} \frac{z^n}{n!}$ converges absolutely for all complex numbers $z$. Standard theorems concerning power series then ensure that the infinite series converges uniformly in $z$ over any closed disk of positive radius about zero in the complex plane. A standard theorem of analysis regarding Cauchy products of absolutely convergent infinite series then ensures that

$$\left(\sum_{n=0}^{+\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{+\infty} \frac{w^n}{n!}\right) = \left(\sum_{n=0}^{+\infty} \frac{(z+w)^n}{n!}\right)$$

for all complex numbers $z$.

It follows that if $z = x + iy$, where $x$ and $y$ are real numbers and $i^2 = -1$, then

$$\begin{aligned}\sum_{n=0}^{+\infty} \frac{z^n}{n!} &= \left(\sum_{n=0}^{+\infty} \frac{x^n}{n!}\right)\left(\sum_{k=0}^{+\infty} \frac{(-1)^k y^{2k}}{(2k)!} + i \sum_{k=0}^{+\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!}\right) \\ &= e^x(\cos y + i \sin y)\end{aligned}$$

for all real numbers $x$ and $y$. Thus

$$\exp z = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$$

for all complex numbers $z$.

**Lemma 2.4**

*Let $w$ be a non-zero complex number, and let*

$$D_{w,|w|} = \{z \in \mathbb{C} : |z - w| < |w|\}.$$

*Then there exists a continuous function $F_w \colon D_{w,|w|} \to \mathbb{C}$ with the property that $\exp(F_w(z)) = z$ for all $z \in D_{w,|w|}$.*

**Proof**

Let $U = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$, and let $\log\colon U \to \mathbb{C}$ be the "principal branch" of the logarithm function, defined so that $\log(re^{i\theta}) = \log r + i\theta$ for all real numbers $r$ and $\theta$ satisfying $r > 0$ and $-\pi < \theta < \pi$. Then the function $\log\colon U \to \mathbb{C}$ is continuous, and $\exp(\log z) = z$ for all $z \in U$. Let $\zeta$ be a complex number satisfying $\exp \zeta = w$. Then $z/w \in U$ for all $z \in D_{w,|w|}$. Let $F_w\colon D_{w,|w|} \to \mathbb{C}$ be defined so that $F_w(z) = \zeta + \log(z/w)$ for all $z \in D_{w,|w|}$. Then

$$\exp(F_w(z)) = \exp(\zeta)\exp(\log(z/w)) = w(z/w) = z$$

for all $z \in D(w,|w|)$, as required. ∎

## 2.3. Path-Lifting with respect to the Exponential Map

**Theorem 2.5**

*Let $\gamma\colon [a,b] \to \mathbb{C} \setminus \{0\}$ be a path in the set $\mathbb{C} \setminus \{0\}$ of non-zero complex numbers. Then there exists a path $\tilde{\gamma}\colon [a,b] \to \mathbb{C}$ in the complex plane which satisfies $\exp(\tilde{\gamma}(t)) = \gamma(t)$ for all $t \in [a,b]$.*

**Proof**

The complex number $\gamma(t)$ is non-zero for all $t \in [a,b]$, and therefore there exists some positive number $\varepsilon_0$ such that $|\gamma(t)| \geq \varepsilon_0$ for all $t \in [a,b]$. (Lemma 2.1). Now any continuous complex-valued function on a closed bounded interval is uniformly continuous. (This follows, for example, from Theorem 1.48.) Therefore there exists some positive real number $\delta$ such that $|\gamma(t) - \gamma(s)| < \varepsilon_0$ for all $s, t \in [a,b]$ satisfying $|t - s| < \delta$.

Let $m$ be a positive integer satisfying $m > |b-a|/\delta$, and let $t_j = a + j(b-a)/m$ for $j = 0, 1, 2, \ldots, m$. Then $|t_j - t_{j-1}| < \delta$ for $j = 1, 2, \ldots, m$. It follows from this that

$$|\gamma(t) - \gamma(t_j)| < \varepsilon_0 \leq |\gamma(t_j)|$$

for all $t \in [t_{j-1}, t_j]$, and thus

$$\gamma([t_{j-1}, t_j]) \subset D_{\gamma(t_j), |\gamma(t_j)|}$$

for $j = 1, 2, \ldots, n$, where

$$D_{w,|w|} = \{z \in \mathbb{C} : |z - w| < |w|\}$$

for all $w \in \mathbb{C}$.

Now there exist continuous functions $F_j\colon D_{\gamma(t_j),|\gamma(t_j)|} \to \mathbb{C}$ with the property that $\exp(F_j(z)) = z$ for all $z \in D_{\gamma(t_j),|\gamma(t_j)|}$ (see Lemma 2.4). Let $\tilde{\gamma}_j(t) = F_j(\gamma(t))$ for all $t \in [t_{j-1}, t_j]$. Then, for each integer $j$ between 1 and $m$, the function $\tilde{\gamma}_j\colon [t_{j-1}, t_j] \to \mathbb{C}$ is continuous, and is thus a path in the complex plane with the property that $\exp(\tilde{\gamma}_j(t)) = \gamma(t)$ for all $t \in [t_{j-1}, t_j]$. Now

$$\exp(\tilde{\gamma}_j(t_j)) = \gamma(t_j) = \exp(\tilde{\gamma}_{j+1}(t_j))$$

for each integer $j$ between 1 and $m-1$. The periodicity properties of the exponential function (Lemma 2.3) therefore ensure that there exist integers $k_1, k_2, \ldots, k_{m-1}$ such that $\tilde{\gamma}_{j+1}(t_j) = \tilde{\gamma}_j(t_j) + 2\pi i k_j$ for $j = 1, 2, \ldots, m-1$. Then

$$\tilde{\gamma}_{j+1}(t_j) - 2\pi i \sum_{r=1}^{j} k_r = \tilde{\gamma}_j(t_j) - 2\pi i \sum_{r=1}^{j-1} k_r$$

for $j = 1, 2, \ldots, m-1$. The Pasting Lemma (Lemma 1.24) then ensures the existence of a continuous function $\tilde{\gamma}\colon [a, b] \to \mathbb{C}$ defined so that $\tilde{\gamma}(t) = \tilde{\gamma}_1(t)$ whenever $t \in [a, t_1]$, and

$$\tilde{\gamma}(t) = \tilde{\gamma}_j(t) - 2\pi i \sum_{r=1}^{j-1} k_r$$

whenever $t \in [t_{j-1}, t_j]$ for some integer $j$ between 2 and $m$. Moreover $\exp(\tilde{\gamma}(t)) = \gamma(t)$ for all $t \in [a, b]$. We have thus proved the existence of a path $\tilde{\gamma}$ in the complex plane with the required properties. $\blacksquare$

## 2.4. Winding Numbers

Let $\gamma\colon [a, b] \to \mathbb{C}$ be a closed path in the complex plane, and let $w$ be a complex number that does not lie on $\gamma$. Then there exists a path $\tilde{\gamma}_w\colon [a, b] \to \mathbb{C}$ in the complex plane such that $\exp(\tilde{\gamma}_w(t)) = \gamma(t) - w$ for all $t \in [a, b]$ (Theorem 2.5). Now the definition of closed paths ensures that $\gamma(b) = \gamma(a)$. Also two complex numbers $z_1$ and $z_2$ satisfy $\exp z_1 = \exp z_2$ if and only if $(2\pi i)^{-1}(z_2 - z_1)$ is an integer (Lemma 2.3). It follows that there exists some integer $n(\gamma, w)$ such that $\tilde{\gamma}_w(b) = \tilde{\gamma}_w(a) + 2\pi i n(\gamma, w)$.

Now let $\varphi\colon [a, b] \to \mathbb{C}$ be any path with the property that $\exp(\varphi(t)) = \gamma(t) - w$ for all $t \in [a, b]$. Then the function sending $t \in [a, b]$ to $(2\pi i)^{-1}(\varphi(t) - \tilde{\gamma}_w(t))$ is a continuous integer-valued function on the interval $[a, b]$, and is therefore constant on this interval (Corollary 1.58). It follows that

$$\varphi(b) - \varphi(a) = \tilde{\gamma}_w(b) - \tilde{\gamma}_w(a) = 2\pi i n(\gamma, w).$$

It follows from this that the value of the integer $n(\gamma, w)$ depends only on the choice of $\gamma$ and $w$, and is independent of the choice of path $\tilde{\gamma}_w$ satisfying $\exp(\tilde{\gamma}_w(t)) = \gamma(t) - w$ for all $t \in [a, b]$.

## Definition

Let $\gamma\colon [a,b] \to \mathbb{C}$ be a closed path in the complex plane, and let $w$ be a complex number that does not lie on $\gamma$. The *winding number* of $\gamma$ about $w$ is defined to be the unique integer $n(\gamma, w)$ with the property that $\varphi(b) - \varphi(a) = 2\pi i n(\gamma, w)$ for all paths $\varphi\colon [a,b] \to \mathbb{C}$ in the complex plane that satisfy $\exp(\varphi(t)) = \gamma(t) - w$ for all $t \in [a,b]$.

**Example**

Let $n$ be an integer, and let $\gamma_n\colon [0,1] \to \mathbb{C}$ be the closed path in the complex plane defined by $\gamma_n(t) = \exp(2\pi i n t)$. Then $\gamma_n(t) = \exp(\varphi_n(t))$ for all $t \in [0,1]$ where $\varphi_n\colon [0,1] \to \mathbb{C}$ is the path in the complex plane defined such that $\varphi_n(t) = 2\pi i n t$ for all $t \in [0,1]$. It follows that $n(\gamma_n, 0) = (2\pi i)^{-1}(\varphi_n(1) - \varphi_n(0)) = n$.

Given a closed path $\gamma$, and given a complex number $w$ that does not lie on $\gamma$, the winding number $n(\gamma, w)$ measures the number of times that the path $\gamma$ winds around the point $w$ of the complex plane in the anticlockwise direction.

**Lemma 2.6 (Dog-Walking Lemma)**

*Let $\gamma_1 \colon [a, b] \to \mathbb{C}$ and $\gamma_2 \colon [a, b] \to \mathbb{C}$ be closed paths in the complex plane, and let $w$ be a complex number that does not lie on $\gamma_1$. Suppose that $|\gamma_2(t) - \gamma_1(t)| < |\gamma_1(t) - w|$ for all $t \in [a, b]$. Then $n(\gamma_2, w) = n(\gamma_1, w)$.*

**Proof**

Note that the inequality satisfied by the functions $\gamma_1$ and $\gamma_2$ ensures that $w$ does not lie on the path $\gamma_2$. Let $\tilde{\gamma}_1 \colon [0,1] \to \mathbb{C}$ be a path in the complex plane such that $\exp(\tilde{\gamma}_1(t)) = \gamma_1(t) - w$ for all $t \in [a,b]$, and let

$$\rho(t) = \frac{\gamma_2(t) - w}{\gamma_1(t) - w}$$

for all $t \in [a,b]$ Then

$$|\rho(t) - 1| = \left| \frac{\gamma_2(t) - \gamma_1(t)}{\gamma_1(t) - w} \right| < 1$$

for all $t \in [a,b]$.

Now it follows from Lemma 2.4 that there exists a continuous function $F\colon \{z \in \mathbb{C} : |z-1| < 1\} \to \mathbb{C}$ with the property that $\exp(F(z)) = z$ for all complex numbers $z$ satisfying $|z-1| < 1$. Let $\tilde{\gamma}_2\colon [0,1] \to \mathbb{C}$ be the path in the complex plane defined such that $\tilde{\gamma}_2(t) = F(\rho(t)) + \tilde{\gamma}_1(t)$ for all $t \in [a,b]$. Then

$$\begin{aligned} \exp(\tilde{\gamma}_2(t)) &= \exp(F(\rho(t)))\exp(\tilde{\gamma}_1(t)) = \rho(t)(\gamma_1(t) - w) \\ &= \gamma_2(t) - w. \end{aligned}$$

Now $\rho(b) = \rho(a)$. It follows that

$$\begin{aligned} 2\pi i n(\gamma_2, w) &= \tilde{\gamma}_2(b) - \tilde{\gamma}_2(a) \\ &= F(\rho(b)) + \tilde{\gamma}_1(b) - F(\rho(a)) - \tilde{\gamma}_1(a) \\ &= \tilde{\gamma}_1(b) - \tilde{\gamma}_1(a) \\ &= 2\pi i n(\gamma_1, w), \end{aligned}$$

as required. ∎

**Remark**

Imagine that you are exercising a dog in a park. You walk along a path close to the perimeter of the park that remains at all times at at least 200 metres from an oak tree in the centre of the park. Your dog runs around in your vicinity, but remains at all times within 100 metres of you. In order to leave the park you and your dog return to the point at which you entered the park. The Dog-Walking Lemma then ensures that the number of times that your dog went around the oak tree in the centre of the park is equal to the number of times that you yourself went around that tree.

**Example**

Let $\gamma\colon [0,1] \to \mathbb{C}$ be the closed curve in the complex plane defined such that

$$\begin{aligned}\gamma(t) &= 3\cos 6\pi t + 4i\sin 6\pi t + (\sin 16\pi t)(\sin 8\pi t) \\ &\qquad - 2ie^{\cos 8\pi t - 1}\cos 8\pi t\end{aligned}$$

for all $t \in [0,1]$, where $i^2 = -1$. Let

$$\gamma_1(t) = 3\cos 6\pi t + 4i\sin 6\pi t$$

for all $t \in [0,1]$. Then $|\gamma_1(t)| \geq 3$ for all $t \in [0,1]$. Also $|\sin 16\pi t| \leq 1$ and $0 \leq e^{\cos 8\pi t - 1} \leq 1$ for all $t \in [0,1]$, and therefore

$$\begin{aligned}&\left|(\sin 16\pi t)(\sin 8\pi t) - 2ie^{\cos 8\pi t - 1}\cos 8\pi t\right|^2 \\ &\qquad \leq \quad \sin^2 8\pi t + 4\cos^2 8\pi t \leq 4\end{aligned}$$

for all $t \in [0,1]$. It follows that

$$\begin{aligned}|\gamma(t) - \gamma_1(t)| &= \left|(\sin 16\pi t)(\sin 8\pi t) - 2ie^{\cos 8\pi t - 1}\cos 8\pi t\right| \\ &\leq 2 < |\gamma_1(t)|\end{aligned}$$

for all $t \in [0,1]$. The Dog-Walking Lemma (Lemma 2.6) then ensures that $n(\gamma, 0) = n(\gamma_1, 0)$. Another application of the Dog-Walking Lemma then ensures that $n(\gamma_1, 0) = n(\gamma_2, 0)$, where

$$\gamma_2(t) = 3(\cos 6\pi t + i\sin 6\pi t)$$

for all $t \in [0,1]$. Moreover $\gamma_2 = \exp\circ\tilde{\gamma}_2$ where $\tilde{\gamma}_2\colon [0,1] \to \mathbb{C}$ is the path in $\mathbb{C}$ defined so that $\tilde{\gamma}_2(t) = \log 3 + 6\pi t$ for all $t \in [0,1]$.

The definition of winding number ensures that

$$n(\gamma_2, 0) = (2\pi i)^{-1}(\tilde{\gamma}_2(1) - \tilde{\gamma}_2(0)) = 3.$$

Therefore $n(\gamma, 0) = 3$.

### Lemma 2.7

*Let $\gamma \colon [a, b] \to \mathbb{C}$ be a closed path in the complex plane and let $W$ be the set $\mathbb{C} \setminus [\gamma]$ of all points of the complex plane that do not lie on the curve $\gamma$. Then the function that sends $w \in W$ to the winding number $n(\gamma, w)$ of $\gamma$ about $w$ is a continuous function on $W$.*

**Proof**

Let $w \in W$. It then follows from Lemma 2.1 that there exists some positive real number $\varepsilon_0$ such that $|\gamma(t) - w| \geq \varepsilon_0 > 0$ for all $t \in [a, b]$. Let $w_1$ be a complex number satisfying $|w_1 - w| < \varepsilon_0$, and let $\gamma_1 \colon [a, b] \to \mathbb{C}$ be the closed path in the complex plane defined such that $\gamma_1(t) = \gamma(t) + w - w_1$ for all $t \in [a, b]$. Then $\gamma(t) - w_1 = \gamma_1(t) - w$ for all $t \in [a, b]$, and therefore $n(\gamma, w_1) = n(\gamma_1, w)$. Also $|\gamma_1(t) - \gamma(t)| < |\gamma(t) - w|$ for all $t \in [a, b]$. It follows from the Dog-Walking Lemma (Lemma 2.6) that $n(\gamma, w_1) = n(\gamma_1, w) = n(\gamma, w)$. This shows that the function sending $w \in W$ to $n(\gamma, w)$ is continuous on $W$, as required. ∎

## Lemma 2.8

*Let $\gamma\colon [a,b] \to \mathbb{C}$ be a closed path in the complex plane, and let $R$ be a positive real number with the property that $|\gamma(t)| < R$ for all $t \in [a,b]$. Then $n(\gamma, w) = 0$ for all complex numbers $w$ satisfying $|w| \geq R$.*

**Proof**

Let $\gamma_0\colon [a,b] \to \mathbb{C}$ be the constant path defined by $\gamma_0(t) = 0$ for all $[a,b]$. If $|w| > R$ then $|\gamma(t) - \gamma_0(t)| = |\gamma(t)| < |w| = |\gamma_0(t) - w|$. It follows from the Dog-Walking Lemma (Lemma 2.6) that $n(\gamma, w) = n(\gamma_0, w) = 0$, as required. ∎

**Proposition 2.9**

*Let $[a, b]$ and $[c, d]$ be closed bounded intervals, and, for each $s \in [c, d]$, let $\gamma_s \colon [a, b] \to \mathbb{C}$ be a closed path in the complex plane. Let $w$ be a complex number that does not lie on any of the paths $\gamma_s$. Suppose that the function $H \colon [a, b] \times [c, d] \to \mathbb{C}$ is continuous, where $H(t, s) = \gamma_s(t)$ for all $t \in [a, b]$ and $s \in [c, d]$. Then $n(\gamma_c, w) = n(\gamma_d, w)$.*

**Proof**

The rectangle $[a, b] \times [c, d]$ is a closed bounded subset of $\mathbb{R}^2$, and is therefore compact. It follows that the continuous function on the closed rectangle $[a, b] \times [c, d]$ that sends a point $(t, s)$ of the rectangle to $|H(t, s) - w|^{-1}$ is a bounded function on $[a, b] \times [c, d]$ (see, for example, Lemma 1.40). Therefore there exists some positive number $\varepsilon_0$ such that $|H(t, s) - w| \geq \varepsilon_0 > 0$ for all $t \in [a, b]$ and $s \in [c, d]$.

Now any continuous complex-valued function on a closed bounded subset of a Euclidean space is uniformly continuous. (This follows, for example, on combining the results of Theorem 1.50 and Theorem 1.48.) Therefore there exists some positive real number $\delta$ such that $|H(t,s) - H(t,u)| < \varepsilon_0$ for all $t \in [a,b]$ and for all $s, u \in [c,d]$ satisfying $|s-u| < \delta$. Let $s_0, s_1, \ldots, s_m$ be real numbers chosen such that $c = s_0 < s_1 < \ldots < s_m = d$ and $|s_j - s_{j-1}| < \delta$ for $j = 1, 2, \ldots, m$. Then

$$\begin{aligned} |\gamma_{s_j}(t) - \gamma_{s_{j-1}}(t)| &= |H(t,s_j) - H(t,s_{j-1})| \\ &< \varepsilon_0 \leq |H(t,s_{j-1}) - w| = |\gamma_{s_{j-1}}(t) - w| \end{aligned}$$

for all $t \in [a,b]$, and for each integer $j$ between 1 and $m$. It therefore follows from the Dog-Walking Lemma (Lemma 2.6) that $n(\gamma_{s_{j-1}}, w) = n(\gamma_{s_j}, w)$ for each integer $j$ between 1 and $m$. But then $n(\gamma_c, w) = n(\gamma_d, w)$, as required. $\blacksquare$

**Definition**

Let $D$ be a subset of the complex plane, and let $\gamma\colon [a, b] \to D$ be a closed path in $D$. The closed path $\gamma$ is said to be *contractible* in $D$ if and only if there exists a continuous function $H\colon [a, b] \times [0, 1] \to D$ and an element $z_0$ of $D$ such that $H(t, 0) = \gamma(t)$ and $H(t, 1) = z_0$ for all $t \in [a, b]$, and $H(a, s) = H(b, s)$ for all $s \in [0, 1]$.

**Corollary 2.10**

*Let $D$ be a subset of the complex plane, and let $\gamma\colon [a, b] \to D$ be a closed path in $D$. Suppose that $\gamma$ is contractible in $D$. Then $n(\gamma, w) = 0$ for all $w \in \mathbb{C} \setminus D$, where $n(\gamma, w)$ denotes the winding number of $\gamma$ about $w$.*

**Proof**

The closed curve $\gamma$ is contractible, and therefore there exists an element $z_0$ of $D$ and a continuous function $H\colon [a, b] \times [0, 1] \to D$ such that $H(t, 0) = \gamma(t)$ and $H(t, 1) = z_0$ for all $t \in [a, b]$, and $H(a, s) = H(b, s)$ for all $s \in [0, 1]$. For each $s \in [0, 1]$ let $\gamma_s\colon [a, b] \to D$ be the closed path in $D$ defined such that $\gamma_s(t) = H(t, s)$ for all $t \in [a, b]$. Then $\gamma_1$ is a constant path, and therefore $n(\gamma_1, w) = 0$ for all points $w$ that do not lie on $\gamma_1$. Let $w$ be an element of $w \in \mathbb{C} \setminus D$. Then $w$ does not lie on any of the paths $\gamma_s$. It follows from Proposition 2.9 that

$$n(\gamma, w) = n(\gamma_0, w) = n(\gamma_1, w) = 0,$$

as required. ∎

## 2.5. Simply-Connected Subsets of the Complex Plane

**Definition**

A subset $D$ of the complex plane is said to be *path-connected* if, given any elements $z_1$ and $z_2$, there exists a path in $D$ from $z_1$ and $z_2$.

**Definition**

A path-connected subset $D$ of the complex plane is said to be *simply-connected* if every closed loop in $D$ is contractible.

**Definition**

An subset $D$ of the complex plane is said to be a *star-shaped* if there exists some complex number $z_0$ in $D$ with the property that

$$\{(1-t)z + tz_0 : t \in [0,1]\} \subset D$$

for all $z \in D$. (Thus an open set in the complex plane is a star-shaped if and only if the line segment joining any point of $D$ to $z_0$ is contained in $D$.)

## Lemma 2.11

*Star-shaped subsets of the complex plane are simply-connected.*

**Proof**

Let $D$ be a star-shaped subset of the complex plane. Then there exists some element $z_0$ of $D$ such that the line segment joining $z_0$ to $z$ is contained in $D$ for all $z \in D$. The star-shaped set $D$ is obviously path-connected. Let $\gamma\colon [a,b] \to D$ be a closed path in $D$, and let $H(t,s) = (1-s)\gamma(t) + sz_0$ for all $t \in [a,b]$ and $s \in [0,1]$. Then $H(t,s) \in D$ for all $t \in [a,b]$ and $s \in [0,1]$, $H(t,0) = \gamma(t)$ and $H(t,1) = z_0$ for all $t \in [a,b]$. Also $\gamma(a) = \gamma(b)$, and therefore $H(a,s) = H(b,s)$ for all $s \in [0,1]$. It follows that the closed path $\gamma$ is contractible. Thus $D$ is simply-connected. ∎

The following result is an immediate consequence of Corollary 2.10

**Proposition 2.12**

*Let $D$ be a simply-connected subset of the complex plane, and let $\gamma$ be a closed path in $D$. Then $n(\gamma, w) = 0$ for all $w \in \mathbb{C} \setminus D$.*

## 2.6. The Fundamental Theorem of Algebra

**Theorem 2.13**

*(The Fundamental Theorem of Algebra) Let $P\colon \mathbb{C} \to \mathbb{C}$ be a non-constant polynomial with complex coefficients. Then there exists some complex number $z_0$ such that $P(z_0) = 0$.*

**Proof**

We shall prove that any polynomial that is everywhere non-zero must be a constant polynomial.

Let $P(z) = a_0 + a_1 z + \cdots + a_m z^m$, where $a_1, a_2, \ldots, a_m$ are complex numbers and $a_m \neq 0$. We write $P(z) = P_m(z) + Q(z)$, where $P_m(z) = a_m z^m$ and $Q(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}$. Let

$$R = \frac{|a_0| + |a_1| + \cdots + |a_m|}{|a_m|}.$$

If $|z| > R$ then $|z| \geq 1$, and therefore

$$\begin{aligned}
\left|\frac{Q(z)}{P_m(z)}\right| &= \frac{1}{|a_m z|}\left|\frac{a_0}{z^{m-1}} + \frac{a_1}{z^{m-2}} + \cdots + a_{m-1}\right| \\
&\leq \frac{1}{|a_m|\,|z|}\left(\left|\frac{a_0}{z^{m-1}}\right| + \left|\frac{a_1}{z^{m-2}}\right| + \cdots + |a_{m-1}|\right) \\
&\leq \frac{1}{|a_m|\,|z|}(|a_0| + |a_1| + \cdots + |a_{m-1}|) \leq \frac{R}{|z|} < 1.
\end{aligned}$$

It follows that $|P(z) - P_m(z)| < |P_m(z)|$ for all complex numbers $z$ satisfying $|z| > R$.

For each non-negative real number $r$, let $\gamma_r \colon [0,1] \to \mathbb{C}$ and $\varphi_r \colon [0,1] \to \mathbb{C}$ be the closed paths defined such that $\gamma_r(t) = P(r \exp(2\pi i t))$ and $\varphi_r(t) = P_m(r \exp(2\pi i t)) = a_m r^m \exp(2\pi i m t)$ for all $t \in [0,1]$. If $r > R$ then $|\gamma_r(t) - \varphi_r(t)| < |\varphi_r(t)|$ for all $t \in [0,1]$. It then follows from the Dog-Walking Lemma (Lemma 2.6) that $n(\gamma_r, 0) = n(\varphi_r, 0) = m$ whenever $r > R$.

Now if the polynomial $P$ is everywhere non-zero then it follows on applying Proposition 2.9 that the function sending each non-negative real number $r$ to the winding number $n(\gamma_r, 0)$ of the closed path $\gamma_r$ about zero is a continuous function on the set of non-negative real numbers. But any continuous integer-valued function on an interval is necessarily constant (see Corollary 1.58). It follows that $n(\gamma_r, 0) = n(\gamma_0, 0)$ for all positive real-numbers $r$. But $\gamma_0$ is the constant path defined by $\gamma_0(t) = P(0)$ for all $t \in [0, 1]$, and therefore $n(\gamma_0, 0) = 0$. It follows that is the polynomial $P$ is everywhere non-zero then $n(\gamma_r, 0) = 0$ for all non-negative real numbers $r$. But we have shown that $n(\gamma_r, 0) = m$ for sufficiently large values of $r$, where $m$ is the degree of the polynomial $P$. It follows that if the polynomial $P$ is everywhere non-zero, then it must be a constant polynomial. The result follows. ∎

## 2.7. The Kronecker Principle

The proof of the Fundamental Theorem of Algebra given above depends on the continuity of the polynomial $P$, together with the fact that the winding number $n(P \circ \sigma_r, 0)$ is non-zero for sufficiently large $r$, where $\sigma_r$ denotes the circle of radius $r$ about zero, described once in the anticlockwise direction. We can therefore generalize the proof of the Fundamental Theorem of Algebra in order to obtain the following result (sometimes referred to as the *Kronecker Principle*).

**Proposition 2.14**

*Let $f\colon D \to \mathbb{C}$ be a continuous map defined on the closed unit disk $D$ in $\mathbb{C}$, and let $w \in \mathbb{C} \setminus f(D)$. Then $n(f \circ \sigma, w) = 0$, where $\sigma\colon [0,1] \to \mathbb{C}$ is the parameterization of unit circle defined by $\sigma(t) = \exp(2\pi i t)$, and $n(f \circ \sigma, w)$ is the winding number of $f \circ \sigma$ about $w$.*

**Proof**

Define $\gamma_s(t) = f(s \exp(2\pi i t))$ for all $t \in [0,1]$ and $s \in [0,1]$. Then none of the closed curves $\gamma_s$ passes through $w$, and $\gamma_0$ is the constant curve with value $f(0)$. It follows from Proposition 2.9 that

$$n(f \circ \sigma, w) = n(\gamma_1, w) = n(\gamma_0, w) = 0,$$

as required. ∎

## 2.8. The Brouwer Fixed Point Theorem

We now use Proposition 2.14 to show that there is no continuous 'retraction' mapping the closed unit disk onto its boundary circle.

### Corollary 2.15

*There does not exist a continuous map $r\colon D \to \partial D$ with the property that $r(z) = z$ for all $z \in \partial D$, where $\partial D$ denotes the boundary circle of the closed unit disk $D$.*

**Proof**

Let $\sigma\colon [0,1] \to \mathbb{C}$ be defined by $\sigma(t) = \exp(2\pi i t)$. If a continuous map $r\colon D \to \partial D$ with the required property were to exist, then $r(z) \neq 0$ for all $z \in D$ (since $r(D) \subset \partial D$), and therefore $n(\sigma, 0) = n(r \circ \sigma, 0) = 0$, by Proposition 2.14. But $\sigma = \exp \circ \tilde{\sigma}$, where $\tilde{\sigma}(t) = 2\pi i t$ for all $t \in [0,1]$, and thus

$$n(\sigma, 0) = \frac{\tilde{\sigma}(1) - \tilde{\sigma}(0)}{2\pi i} = 1.$$

This shows that there cannot exist any continuous map $r$ with the required property. ∎

**Theorem 2.16**

*(The Brouwer Fixed Point Theorem in Two Dimensions) Let* $f: D \to D$ *be a continuous map which maps the closed unit disk* $D$ *into itself. Then there exists some* $z_0 \in D$ *such that* $f(z_0) = z_0$*.*

**Proof**

Suppose that there did not exist any fixed point $z_0$ of $f: D \to D$. Then one could define a continuous map $r: D \to \partial D$ as follows: for each $z \in D$, let $r(z)$ be the point on the boundary $\partial D$ of $D$ obtained by continuing the line segment joining $f(z)$ to $z$ beyond $z$ until it intersects $\partial D$ at the point $r(z)$. Then $r: D \to \partial D$ would be a continuous map, and moreover $r(z) = z$ for all $z \in \partial D$. But Corollary 2.15 shows that there does not exist any continuous map $r: D \to \partial D$ with this property. We conclude that $f: D \to D$ must have at least one fixed point. ∎

**Remark**

The Brouwer Fixed Point Theorem is also valid in higher dimensions. This theorem states that any continuous map from the closed $n$-dimensional ball into itself must have at least one fixed point. The proof of the theorem for $n > 2$ is analogous to the proof for $n = 2$, once one has shown that there is no continuous map from the closed $n$-dimensional ball to its boundary which is the identity map on the boundary. However winding numbers cannot be used to prove this result, and thus more powerful topological techniques need to be employed.

## 2.9. The Borsuk-Ulam Theorem

**Lemma 2.17**

*Let $f: S^1 \to \mathbb{C} \setminus \{0\}$ be a continuous function defined on $S^1$, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Suppose that $f(-z) = -f(z)$ for all $z \in S^1$. Then the winding number $n(f \circ \sigma, 0)$ of $f \circ \sigma$ about 0 is odd, where $\sigma: [0,1] \to S^1$ is given by $\sigma(t) = \exp(2\pi i t)$.*

**Proof**

It follows from the Path Lifting Theorem (Theorem 2.5) that there exists a continuous path $\tilde{\gamma}\colon [0,1] \to \mathbb{C}$ in $\mathbb{C}$ such that $\exp(\tilde{\gamma}(t)) = f(\sigma(t))$ for all $t \in [0,1]$. Now $f(\sigma(t+\frac{1}{2})) = -f(\sigma(t))$ for all $t \in [0, \frac{1}{2}]$, since $\sigma(t+\frac{1}{2}) = -\sigma(t)$ and $f(-z) = -f(z)$ for all $z \in \mathbb{C}$. Thus $\exp(\tilde{\gamma}(t+\frac{1}{2})) = \exp(\tilde{\gamma}(t) + \pi i)$ for all $t \in [0, \frac{1}{2}]$. It follows that $\tilde{\gamma}(t+\frac{1}{2}) = \tilde{\gamma}(t) + (2m+1)\pi i$ for some integer $m$. (The value of $m$ for which this identity is valid does not depend on $t$, since every continuous function from $[0, \frac{1}{2}]$ to the set of integers is necessarily constant.) Hence

$$n(f\circ\sigma, 0) = \frac{\tilde{\gamma}(1) - \tilde{\gamma}(0)}{2\pi i} = \frac{\tilde{\gamma}(1) - \tilde{\gamma}(\frac{1}{2})}{2\pi i} + \frac{\tilde{\gamma}(\frac{1}{2}) - \tilde{\gamma}(0)}{2\pi i} = 2m+1.$$

Thus $n(f\circ\sigma, 0)$ is an odd integer, as required. $\blacksquare$

We shall identify the space $\mathbb{R}^2$ with $\mathbb{C}$, identifying $(x, y) \in \mathbb{R}^2$ with the complex number $x + iy \in \mathbb{C}$ for all $x, y \in \mathbb{R}$. This is permissible, since we are interested in purely topological results concerning continuous functions defined on appropriate subsets of these spaces. Under this identification the closed unit disk $D$ is given by

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

As usual, we define

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

**Lemma 2.18**

*Let $f\colon S^2 \to \mathbb{R}^2$ be a continuous map with the property that $f(-\mathbf{n}) = -f(\mathbf{n})$ for all $\mathbf{n} \in S^2$. Then there exists some point $\mathbf{n}_0$ of $S^2$ with the property that $f(\mathbf{n}_0) = 0$.*

**Proof**

Let $\varphi\colon D \to S^2$ be the map defined by

$$\varphi(x, y) = (x, y, +\sqrt{1 - x^2 - y^2}).$$

(Thus the map $\varphi$ maps the closed disk $D$ homeomorphically onto the upper hemisphere in $\mathbb{R}^3$.) Let $\sigma\colon [0, 1] \to S^2$ be the parameterization of the equator in $S^2$ defined by

$$\sigma(t) = (\cos 2\pi t, \sin 2\pi t, 0)$$

for all $t \in [0, 1]$. Let $f\colon S^2 \to \mathbb{R}^2$ be a continuous map with the property that $f(-\mathbf{n}) = -f(\mathbf{n})$ for all $\mathbf{n} \in S^2$. If $f(\sigma(t_0)) = 0$ for some $t_0 \in [0, 1]$ then the function $f$ has a zero at $\sigma(t_0)$. It remains to consider the case in which $f(\sigma(t)) \neq 0$ for all $t \in [0, 1]$. In that case the winding number $n(f \circ \sigma, 0)$ is an odd integer, by Lemma 2.17, and is thus non-zero.

It follows from Proposition 2.14, applied to $f \circ \varphi \colon D \to \mathbb{R}^2$, that $0 \in f(\varphi(D))$, (since otherwise the winding number $n(f \circ \sigma, 0)$ would be zero). Thus $f(\mathbf{n}_0) = 0$ for some $\mathbf{n}_0 = \varphi(D)$, as required. $\blacksquare$

**Theorem 2.19**

*(Borsuk-Ulam) Let $f\colon S^2 \to \mathbb{R}^2$ be a continuous map. Then there exists some point $\mathbf{n}$ of $S^2$ with the property that $f(-\mathbf{n}) = f(\mathbf{n})$.*

**Proof**

This result follows immediately on applying Lemma 2.18 to the continuous function $g\colon S^2 \to \mathbb{R}^2$ defined by $g(\mathbf{n}) = f(\mathbf{n}) - f(-\mathbf{n})$. ∎

**Remark**

It is possible to generalize the Borsuk-Ulam Theorem to $n$ dimensions. Let $S^n$ be the unit $n$-sphere centered on the origin in $\mathbb{R}^n$. The Borsuk-Ulam Theorem in $n$-dimensions states that if $f\colon S^n \to \mathbb{R}^n$ is a continuous map then there exists some point $\mathbf{x}$ of $S^n$ with the property that $f(\mathbf{x}) = f(-\mathbf{x})$.