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Section 4: Covering Maps

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4. Covering Maps

4.1. Evenly-Covered Open Sets and Covering Maps

Definition

Let X and $\tilde{X}$ be topological spaces and let $p\colon \tilde{X} \to X$ be a continuous map. An open subset U of X is said to be *evenly covered* by the map p if and only if $p^{-1}(U)$ is a disjoint union of open sets of $\tilde{X}$ each of which is mapped homeomorphically onto U by p. The map $p\colon \tilde{X} \to X$ is said to be a *covering map* if $p\colon \tilde{X} \to X$ is surjective and in addition every point of X is contained in some open set that is evenly covered by the map p.

If $p\colon \tilde{X} \to X$ is a covering map, then we say that $\tilde{X}$ is a *covering space* of X.

Example

Let S^1 be the unit circle in $\mathbb{R}^2$. Then the map $p\colon \mathbb{R} \to S^1$ defined by

$$p(t) = (\cos 2\pi t, \sin 2\pi t)$$

is a covering map. Indeed let $\mathbf{n}$ be a point of S^1. Consider the open set U in S^1 containing $\mathbf{n}$ defined by $U = S^1 \setminus \{-\mathbf{n}\}$. Now $\mathbf{n} = (\cos 2\pi t_0, \sin 2\pi t_0)$ for some $t_0 \in \mathbb{R}$. Then $p^{-1}(U)$ is the union of the disjoint open sets J_n for all integers n, where

$$J_n = \{t \in \mathbb{R} : t_0 + n - \tfrac{1}{2} < t < t_0 + n + \tfrac{1}{2}\}.$$

Each of the open sets J_n is mapped homeomorphically onto U by the map p. This shows that $p\colon \mathbb{R} \to S^1$ is a covering map.

Example

Let $p_{\exp}\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ be the map from the complex plane $\mathbb{C}$ to the open subset $\mathbb{C} \setminus \{0\}$ of $\mathbb{C}$ defined such that $p_{\exp}(z) = \exp(z)$ for all complex numbers z. We show that $p_{\exp}(z)$ is a covering map.

Given any real number s, let

$$L_s = \{-re^{is} : r \in \mathbb{R} \text{ and } r \geq 0\}.$$

Then L_s is a ray in the complex plane starting at zero and passing through $-\cos s - i \sin s$. Moreover every complex number belonging to the complement $\mathbb{C} \setminus L_s$ of the ray L_s in $\mathbb{C}$ can be expressed uniquely in the form re^{it}, where r and t are real numbers satisfying $r > 0$ and $s - \pi < t < s + \pi$.

Let

$$W_s = \{w \in \mathbb{C} : s - \pi < \mathrm{Im}[w] < s + \pi\},$$

where $\mathrm{Im}[w]$ denotes the imaginary part of w for all complex numbers w, and let $F_s\colon \mathbb{C} \setminus L_s \to W_s$ be the complex-valued function on the open subset $\mathbb{C} \setminus L_s$ of the complex plane defined such that

$$F_s(re^{it}) = \log r + it$$

for all real numbers r and t satisfying $r > 0$ and $s - \pi < t < s + \pi$. Then $F_s\colon \mathbb{C} \setminus L_s \to W_s$ is a continuous map, $\exp(F_s(z)) = z$ for all $z \in \mathbb{C} \setminus L_s$ and $F_s(\exp(w)) = w$ for all $w \in W_s$. It follows that $F_s\colon \mathbb{C} \setminus L_s \to W_s$ is a homeomorphism between $\mathbb{C} \setminus L_s$ and W_s.

Let w be a complex number for which $\exp(w) \in \mathbb{C} \setminus L_s$. Then there exists a unique integer m such that $s + 2\pi m - \pi < \mathrm{Im}[w] < s + 2\pi m + \pi$. Then $w \in F_{s+m}(\exp w)$. It follows from this that, for each real number s, the preimage $p_{\exp}^{-1}(\mathbb{C} \setminus L_s)$ is the disjoint union of the sets $W_{s+2\pi m}$ as m ranges over the set $\mathbb{Z}$ of integers. Also $W_{s+2\pi m} \cap W_{s+2\pi n} = \emptyset$ when m and n are integers and $m \neq n$, and $p_{\exp}\colon \mathbb{C} \setminus \mathbb{C} \setminus \{0\}$ maps the open set $W_{s+2\pi m}$ homeomorphically onto $\mathbb{C} \setminus L_s$ for all integers m, where $p_{\exp}(w) = \exp(w)$ for all $w \in \mathbb{C}$. Thus $p_{\exp}\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is a covering map.

Example

Let

$$\begin{aligned} X &= \{(x,y)\in\mathbb{R}^2 : (x,y)\neq(0,0)\},\\ \tilde{X} &= \{(x,y,z)\in\mathbb{R}^3 : (x,y)\neq(0,0),\\ &\qquad x=\sqrt{x^2+y^2}\cos 2\pi z \text{ and } y=\sqrt{x^2+y^2}\sin 2\pi z\}, \end{aligned}$$

and let $p\colon \tilde{X}\to X$ be defined so that $p(x,y,z)=(x,y)$ for all $(x,y,z)\in\tilde{X}$. Now $\exp(w)=T(p(h(w)))$ for all $w\in\mathbb{C}$, where

$$h(u+iv)=\left(e^u\cos v, e^u\sin v, \frac{v}{2\pi}\right)$$

for all real numbers u and v and $T(x,y)=x+iy$ for all $(x,y)\in X$.

Moreover $h\colon \mathbb{C} \to \tilde{X}$ is a homeomorphism whose inverse h^{-1} satisfies

$$h^{-1}(z) = \tfrac{1}{2}\log(x^2 + y^2) + 2\pi i z$$

for all $(x, y, z) \in \tilde{X}$.

The map $p\colon \tilde{X} \to X$ is a covering map. Indeed let

$$W_{s,m} = \{(x, y, z) \in \tilde{X} : s + m - \tfrac{1}{2} < z < s + m + \tfrac{1}{2}\}$$

and let $V_{s,m} = p(W_{s,0})$ for all real numbers s and integers m. Then $V_{s,0}$ is an open set in X, $p^{-1}(V_{s,0}) = \bigcup_{m\in\mathbb{Z}} W_{s,m}$ and p maps $W_{s,m}$ homeomorphically onto $V_{s,0}$ for all $s \in \mathbb{R}$ and $m \in \mathbb{Z}$.

The surface $\tilde{X}$ is a *helicoid* in $\mathbb{R}^3$.

Example

Consider the map $\alpha\colon (-2,2) \to S^1$, where $\alpha(t) = (\cos 2\pi t, \sin 2\pi t)$ for all $t \in (-2,2)$. It can easily be shown that there is no open set U containing the point $(1,0)$ that is evenly covered by the map α. Indeed suppose that there were to exist such an open set U. Then there would exist some δ satisfying $0 < \delta < \frac{1}{2}$ such that $U_\delta \subset U$, where

$$U_\delta = \{(\cos 2\pi t, \sin 2\pi t) : -\delta < t < \delta\}.$$

The open set U_δ would then be evenly covered by the map α. However the connected components of $\alpha^{-1}(U_\delta)$ are $(-2,-2+\delta)$, $(-1-\delta,-1+\delta)$, $(-\delta,\delta)$, $(1-\delta,1+\delta)$ and $(2-\delta,2)$, and neither $(-2,-2+\delta)$ nor $(2-\delta,2)$ is mapped homeomorphically onto U_δ by α.

Example

Let $Z = \mathbb{C} \setminus \{1, -1\}$, let

$$\tilde{Z} = \{(z, w) \in \mathbb{C}^2 : w \neq 0 \text{ and } w^2 = z^2 - 1\},$$

and let $p\colon \tilde{Z} \to Z$ be defined such that $p(z, w) = z$ for all $(z, w) \in \tilde{Z}$. Let $(z_0, w_0) \in \tilde{Z}$, and let $z = z_0 + \zeta$. Then

$$\begin{aligned} z^2 - 1 &= z_0^2 - 1 + 2z_0\zeta + \zeta^2 = w_0^2 + 2z_0\zeta + \zeta^2 \\ &= w_0^2 \left(1 + \frac{2z_0\zeta + \zeta^2}{w_0^2}\right). \end{aligned}$$

Now the continuity at zero of the function sending each complex number ζ to $(2z_0\zeta + \zeta^2)/w_0^2$ ensures that there exists some positive real number δ such that

$$\left| \frac{2z_0\zeta + \zeta^2}{w_0^2} \right| < 1$$

whenever $|\zeta| < \delta$. Let $D(z_0, \delta)$ be the open disk of radius δ about z_0 in the complex plane, and let

$$F(z) = \frac{1}{2} \log \left(1 + \frac{2z_0(z - z_0) + (z - z_0)^2}{w_0^2} \right)$$

for all $z \in D(z_0, \delta)$, where $\log(re^{i\theta}) = \log r + i\theta$ for all real numbers r and θ satisfying $r > 0$ and $-\pi < \theta < \pi$. Then $F(z)$ is a continuous function of z on $D(z_0, \delta)$, and

$$\exp(F(z))^2 = 1 + \frac{2z_0(z - z_0) + (z - z_0)^2}{w_0^2} = \frac{z^2 - 1}{w_0^2}$$

for all $z \in D(z_0, \delta)$.

Let $(z, w) \in p^{-1}(D(z_0, \delta))$. Then $z \in D(z_0, \delta)$ and

$$w^2 = z^2 - 1 = \Big(w_0 \exp(F(z))\Big)^2,$$

and therefore $w = \pm w_0 \exp(F(z))$. It follows that $p^{-1}(D(z_0, \delta)) = W_+ \cup W_-$ where

$$\begin{aligned} W_+ &= \{(z, w) \in \mathbb{C}^2 : z \in D(z_0, \delta) \text{ and } w = w_0 \exp(F(z))\}, \\ W_- &= \{(z, w) \in \mathbb{C}^2 : z \in D(z_0, \delta) \text{ and } w = -w_0 \exp(F(z))\}, \end{aligned}$$

Now

$$\mathrm{Re}\left[1 + \frac{2z_0(z - z_0) + (z - z_0)^2}{w_0^2}\right] > 0$$

for all $z \in D(z_0, \delta)$. It follows from the definition of $F(z)$ that

$$-\tfrac{1}{4}\pi < \mathrm{Im}[F(z)] < \tfrac{1}{4}\pi$$

for all $z \in D(z_0, \delta)$, and therefore

$$\mathrm{Re}[\exp(F(z))] = \exp(\mathrm{Re}[F(z)]) \cos(\mathrm{Im}[F(z)]) > 0$$

for all $z \in D(z_0, \delta)$. It follows that

$$\begin{aligned} W_+ &= \left\{ (z, w) \in \tilde{Z} : z \in D(z_0, \delta) \text{ and } \mathrm{Re}\left[\frac{w}{w_0}\right] > 0 \right\}, \\ &= \left\{ (z, w) \in p^{-1}(D(z_0, \delta)) : \mathrm{Re}\left[\frac{w}{w_0}\right] > 0 \right\}, \\ W_- &= \left\{ (z, w) \in \tilde{Z} : z \in D(z_0, \delta) \text{ and } \mathrm{Re}\left[\frac{w}{w_0}\right] < 0 \right\}, \\ &= \left\{ (z, w) \in p^{-1}(D(z_0, \delta)) : \mathrm{Re}\left[\frac{w}{w_0}\right] < 0 \right\}. \end{aligned}$$

Now $p^{-1}(D(z_0, \delta)$ is open in $\tilde{Z}$, because the it is the preimage of the open subset $D(z_0, \delta)$ of Z under the continuous map $p\colon \tilde{Z} \to Z$. Moreover the function mapping (z, w) to the real part of w/w_0 is continuous on $p^{-1}(D(z_0, \delta)$. It follows that W_+ and W_- are open in $\tilde{Z}$. Also $W_+ \cap W_- = \emptyset$, and the map $p\colon \tilde{Z} \to Z$ maps each of the sets W_+ and W_- homeomorphically onto Z, where $Z = \mathbb{C} \setminus \{1, -1\}$. It follows that the open disk $D(z_0, \delta)$ is evenly covered by the map $p\colon \tilde{Z} \to Z$. We have therefore shown that this map is a covering map.

Let $\tilde{f}(z, w) = w$ for all $(z, w) \in \tilde{Z}$. Then

$$\tilde{f}(\tilde{z})^2 = p(\tilde{z})^2 - 1$$

for all $\tilde{z} \in \tilde{Z}$. It follows that the function $\tilde{f} \colon \tilde{Z} \to \mathbb{C}$ represents in some sense the many-valued 'function' $\sqrt{z^2 - 1}$. However this function $\tilde{z}$ is not defined on the open subset Z of the complex plane, but is instead defined over a covering space $\tilde{Z}$ of this open set. This covering space is the *Riemann surface* for the 'function' $\sqrt{z^2 - 1}$. This method of representing many-valued 'functions' of a complex variable using single-valued functions defined over a covering space was initiated and extensively developed by Bernhard Riemann (1826–1866) in his doctoral thesis.

Proposition 4.1

Let $p\colon \tilde{X} \to X$ be a covering map. Then $p(V)$ is open in X for every open set V in $\tilde{X}$.

Proof

Let V be open in $\tilde{X}$, and let $x \in p(V)$. Then $x = p(v)$ for some $v \in V$. Now there exists an open set U containing the point x which is evenly covered by the covering map p. Then $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the covering map p. One of these open sets contains v; let $\tilde{U}$ be this open set, and let $N_x = p(V \cap \tilde{U})$. Now N_x is open in X, since $V \cap \tilde{U}$ is open in $\tilde{U}$ and $p|\tilde{U}$ is a homeomorphism from $\tilde{U}$ to U. Also $x \in N_x$ and $N_x \subset p(V)$. It follows that $p(V)$ is the union of the open sets N_x as x ranges over all points of $p(V)$, and thus $p(V)$ is itself an open set, as required. ∎

Corollary 4.2

A bijective covering map is a homeomophism.

Proof

This result follows directly from Proposition 4.1 the fact that a continuous bijection is a homeomorphism if and only if it maps open sets to open sets. ∎

4.2. Uniqueness of Lifts into Covering Spaces

Definition

Let $p\colon \tilde{X} \to X$ be a covering map, let Z be a topological space, and let $f\colon Z \to X$ be a continuous map from Z to X. A continuous map $\tilde{f}\colon Z \to \tilde{X}$ is said to be a *lift* of $f\colon Z \to X$ to the covering space $\tilde{X}$ if $p \circ \tilde{f} = f$.

Much of the general theory of covering maps is concerned with the development of necessary and sufficient conditions to determine whether or not maps into the base space of a covering map can be lifted to the covering space.

We prove that any lift of a given map from a connected topological topological space into the base space of a covering map is determined by its value at a single point of its domain.

Proposition 4.3

Let $p\colon \tilde{X} \to X$ be a covering map, let Z be a connected topological space, and let $g\colon Z \to \tilde{X}$ and $h\colon Z \to \tilde{X}$ be continuous maps. Suppose that $p \circ g = p \circ h$ and that $g(z) = h(z)$ for at least one point z of Z. Then $g = h$.

Proof

Let $Z_0 = \{z \in Z : g(z) = h(z)\}$. Note that Z_0 is non-empty, by hypothesis. We show that Z_0 is both open and closed in Z.

Let z be a point of Z. There exists an open set U in X containing the point $p(g(z))$ which is evenly covered by the covering map p. Then $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the covering map p. One of these open sets contains $g(z)$; let this set be denoted by $\tilde{U}$. Also one of these open sets contains $h(z)$; let this open set be denoted by $\tilde{V}$. Let $N_z = g^{-1}(\tilde{U}) \cap h^{-1}(\tilde{V})$. Then N_z is an open set in Z containing z.

Consider the case when $z \in Z_0$. Then $g(z) = h(z)$, and therefore $\tilde{V} = \tilde{U}$. It follows from this that both g and h map the open set N_z into $\tilde{U}$. But $p \circ g = p \circ h$, and $p|\tilde{U}\colon \tilde{U} \to U$ is a homeomorphism. Therefore $g|N_z = h|N_z$, and thus $N_z \subset Z_0$. We have thus shown that, for each $z \in Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z_0$. We conclude that Z_0 is open.

Next consider the case when $z \in Z \setminus Z_0$. In this case $\tilde{U} \cap \tilde{V} = \emptyset$, since $g(z) \neq h(z)$. But $g(N_z) \subset \tilde{U}$ and $h(N_z) \subset \tilde{V}$. Therefore $g(z') \neq h(z')$ for all $z' \in N_z$, and thus $N_z \subset Z \setminus Z_0$. We have thus shown that, for each $z \in Z \setminus Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z \setminus Z_0$. We conclude that $Z \setminus Z_0$ is open.

The subset Z_0 of Z is therefore both open and closed. Also Z_0 is non-empty by hypothesis. We deduce that $Z_0 = Z$, since Z is connected. Thus $g = h$, as required. ∎

Corollary 4.4

Let $p\colon \tilde{X} \to X$ be a covering map over a topological space X. Let Z be a connected topological space, and let $f : Z \to \tilde{X}$ be a continuous map. Suppose that $p(f(z)) = x_0$ for all $z \in Z$, where x_0 is some point of X. Then $f(z) = \tilde{x}_0$ for all $z \in Z$, where $\tilde{x}_0$ is some point of $\tilde{X}$ which satisfies $p(\tilde{x}_0) = x_0$.

Proof

Let z_0 be some point of Z. Let $\tilde{x}_0 = f(z_0)$, and let $c\colon Z \to \tilde{X}$ be the constant map defined by $c(z) = \tilde{x}_0$ for all $z \in Z$. Then $c(z_0) = f(z_0)$ and $p \circ c = p \circ f$. It follows from Theorem 4.3 that $f = c$, as required. ∎

4.3. The Path-Lifting Theorem

Theorem 4.5 (Path-Lifting Theorem)

Let $p\colon \tilde{X} \to X$ be a covering map over a topological space X. Let $\gamma\colon [a, b] \to X$ be a continuous map from the closed interval $[a, b]$ to X, and let w be a point of $\tilde{X}$ for which $p(w) = \gamma(a)$. Then there exists a unique continuous map $\tilde{\gamma}\colon [a, b] \to \tilde{X}$ for which $\tilde{\gamma}(a) = w$ and $p \circ \tilde{\gamma} = \gamma$.

Proof

Let S be the subset of $[a, b]$ defined as follows: an element c of $[a, b]$ belongs to S if and only if there exists a continuous map $\eta_c\colon [a, c] \to \tilde{X}$ such that $\eta_c(a) = w$ and $p(\eta_c(t)) = \gamma(t)$ for all $t \in [a, c]$. Note that S is non-empty, since a belongs to S. Let $s = \sup S$.

There exists an open neighbourhood U of $\gamma(s)$ which is evenly covered by the map p, since $p\colon \tilde{X} \to X$ is a covering map. It then follows from the continuity of the path γ that there exists some $\delta > 0$ such that $\gamma(J(s,\delta)) \subset U$, where

$$J(s,\delta) = \{t \in [a,b] : |t - s| < \delta\}.$$

Now $S \cap J(s,\delta)$ is non-empty, because s is the supremum of the set S. Choose some element c of $S \cap J(s,\delta)$. Then there exists a continuous map $\eta_c\colon [a,c] \to \tilde{X}$ such that $\eta_c(a) = w$ and $p(\eta_c(t)) = \gamma(t)$ for all $t \in [a,c]$. Now the open set U is evenly covered by the map p. Therefore $p^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically onto U by the covering map p. One of these open sets contains the point $\eta_c(c)$; let this open set be denoted by $\tilde{U}$.

There then exists a unique continuous map $\sigma\colon U \to \tilde{U}$ defined such that, for all $x \in U$, $\sigma(x)$ is the unique element of $\tilde{U}$ for which $p(\sigma(x)) = x$. Then $\sigma(\gamma(c)) = \eta_c(c)$.

Then, given any $d \in J(s, \delta)$, let $\eta_d\colon [a, d] \to \tilde{X}$ be the function from $[a, d]$ to $\tilde{X}$ defined so that

$$\eta_d(t) = \begin{cases} \eta_c(t) & \text{if } a \leq t \leq c; \\ \sigma(\gamma(t)) & \text{if } c \leq t \leq d. \end{cases}$$

Then $\eta_d(a) = w$ and $p(\eta_d(t)) = \gamma(t)$ for all $t \in [a, d]$. The restrictions of the function $\eta_d\colon [a, d] \to \tilde{X}$ to the intervals $[a, c]$ and $[c, d]$ are continuous. It follows from the Pasting Lemma (Lemma 1.24) that η_d is continuous on $[a, d]$. Thus $d \in S$. We conclude from this that $J(s, \delta) \subset S$. However s is defined to be the supremum of the set S. Therefore $s = b$, and b belongs to S. It follows that that there exists a continuous map $\tilde{\gamma}\colon [a, b] \to \tilde{X}$ for which $\tilde{\gamma}(a) = w$ and $p \circ \tilde{\gamma} = \gamma$, as required. ∎

4.4. The Homotopy-Lifting Theorem

Theorem 4.6 (Homotopy-Lifting Theorem)

Let $p\colon \tilde{X} \to X$ be a covering map over a topological space X. Let Z be a topological space, and let $F\colon Z \times [0,1] \to X$ and $g\colon Z \to \tilde{X}$ be continuous maps with the property that $p(g(z)) = F(z,0)$ for all $z \in Z$. Then there exists a unique continuous map $G\colon Z \times [0,1] \to \tilde{X}$ such that $G(z,0) = g(z)$ for all $z \in Z$ and $p \circ G = F$.

Proof

For each $z \in Z$, consider the path $\gamma_z \colon [0,1] \to Z$ defined by $\gamma_z(t) = F(z,t)$ for all $t \in [0,1]$. Note that $p(g(z)) = \gamma_z(0)$. It follows from the Path-Lifting Theorem (Theorem 4.5) that there exists a unique continuous path $\tilde{\gamma}_z \colon [0,1] \to \tilde{X}$ such that $\tilde{\gamma}_z(0) = g(z)$ for all $z \in Z$ and $p \circ \tilde{\gamma}_z = \gamma_z$. Let the function $G \colon Z \times [0,1] \to \tilde{X}$ be defined by $G(z,t) = \tilde{\gamma}_z(t)$ for all $z \in Z$ and $t \in [0,1]$. Then $G(z,0) = g(z)$ for all $z \in Z$ and

$$p(G(z,t)) = p(\tilde{\gamma}_z(t)) = \gamma_z(t) = F(z,t)$$

for all $z \in Z$ and $t \in [0,1]$. It remains to show that the function $G \colon Z \times [0,1] \to \tilde{X}$ is continuous and that it is unique.

Given any $z \in Z$, let S_z denote the set of all real numbers c belonging to the closed interval $[0, 1]$ which have the following property:

> *there exists an open set N in Z such that $z \in N$ and the function G is continuous on $N \times [0, c]$.*

Let s_z be the supremum $\sup S_z$ (i.e., the least upper bound) of the set S_z. We prove that s_z belongs to the set S_z and that $s_z = 1$.

Choose some $z \in Z$, and let $w \in \tilde{X}$ be given by $w = G(z, s_z)$. There exists an open neighbourhood U of $p(w)$ in X which is evenly covered by the map p. Thus $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the covering map p. One of these open sets contains the point w; let this open set be denoted by $\tilde{U}$. Then there exists a unique continuous map $\sigma\colon U \to \tilde{U}$ defined such that, for all $x \in U$, $\sigma(x)$ is the unique element of $\tilde{U}$ for which $p(\sigma(x)) = x$. Then $\sigma(F(z, s_z)) = w$.

Now $F(z, s_z) = p(w)$. It follows from the continuity of the map F that there exists some positive real number δ and some open set M_1 in Z such that $z \in M_1$ and $F(M_1 \times J(s_z, \delta)) \subset U$, where

$$J(s_z, \delta) = \{t \in [0, 1] : s_z - \delta < t < s_z + \delta\}.$$

Now we can choose some c belonging to S_z which satisfies $s_z - \delta < c \leq s_z$, because s_z is the least upper bound of the set S_z. It then follows from the definition of the set S_z that there exists an open set M_2 in Z such that $z \in M_2$ and the function G is continuous on $M_2 \times [0, c]$. Let

$$N = \{z' \in M_1 \cap M_2 : G(z', c) \in \tilde{U}\}.$$

Then $z \in N$, and the continuity of the function G on $M_2 \times [0, c]$ ensures that N is open in Z. Moreover the function G is continuous on $N \times [0, c]$ and $F(N \times J(s_z, \delta)) \subset U$.

Let $z' \in N$. Then $G(z', c) \in \tilde{U}$ and $p(G(z', c)) = F(z', c)$. It follows from the definition of the map $\sigma\colon U \to \tilde{X}$ that $G(z', c) = \sigma(F(z', c))$. Also the interval $J(s_z, \delta)$ is connected, and

$$p(G(z', t)) = F(z', t) = p(\sigma(F(z', t))$$

for all $t \in J(s_z, \delta)$. It follows from Theorem 4.3 that $G(z', t) = \sigma(F(z', t)$ for all $t \in J(s_z, \delta)$.

We have thus shown that the function G is equal to the continuous function $\sigma \circ F$ on $N \times J(s_z, \delta)$. The function G is therefore continuous on both $N \times [0, c]$ and $N \times [c, t]$ for all $t \in J(s_z, \delta)$ satisfying $t \geq c$. It then follows from the Pasting Lemma (Lemma 1.24) that the function G is continuous on $N \times [0, t]$ for all $t \in J(s_z, \delta)$, and thus $J(s_z, \delta) \subset S_z$. This however contradicts the definition of S_z unless $s_z \in S_z$ and $s_z = 1$. We conclude therefore that $1 \in S_z$, and thus there exists an open set N in Z such that $z \in N$ and $G|N \times [0, 1]$ is continuous.

We conclude from this that every point of $Z \times [0, 1]$ is contained in some open subset of $Z \times [0, 1]$ on which that function G is continuous. It follows that $G\colon Z \times [0, 1] \to \tilde{X}$ is continuous (see Proposition 1.23).

The uniqueness of the map $G\colon Z \times [0,1] \to \tilde{X}$ follows directly from the fact that for any $z \in Z$ there is a unique continuous path $\tilde{\gamma}_z\colon [0,1] \to \tilde{X}$ such that $\tilde{\gamma}_z(0) = g(z)$ and $p(\tilde{\gamma}_z(t)) = F(z,t)$ for all $t \in [0,1]$. $\blacksquare$

4.5. Path-Lifting and the Fundamental Group

Let $p\colon \tilde{X} \to X$ be a covering map and let $\alpha\colon [0,1] \to X$ and $\beta\colon [0,1] \to X$ be paths in the base space X which both start at some point x_0 of X and finish at some point x_1 of X, so that

$$\alpha(0) = \beta(0) = x_0 \quad \text{and} \quad \alpha(1) = \beta(1) = x_1.$$

Let $\tilde{x}_0$ be some point of the covering space $\tilde{X}$ that projects down to x_0, so that $p(\tilde{x}_0) = x_0$. It follows from the Path-Lifting Theorem (Theorem 4.5) that there exist paths $\tilde{\alpha}\colon [0,1] \to \tilde{X}$ and $\tilde{\beta}\colon [0,1] \to \tilde{X}$ in the covering space $\tilde{X}$ that both start at $\tilde{x}_0$ and that are lifts of the paths α and β respectively.

Thus

$$\tilde{\alpha}(0) = \tilde{\beta}(0) = \tilde{x}_0,$$

$$p(\tilde{\alpha}(t)) = \alpha(t) \quad \text{and} \quad p(\tilde{\beta}(t)) = \beta(t) \quad \text{for all } t \in [0, 1].$$

These lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of the paths α and β are uniquely determined by their starting point $\tilde{x}_0$ (see Proposition 4.3).

Now, though the lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of the paths α and β have been chosen such that they start at the same point $\tilde{x}_0$ of the covering space $\tilde{X}$, they need not in general end at the same point of $\tilde{X}$. However we shall prove that if $\alpha \simeq \beta \operatorname{rel} \{0, 1\}$, then the lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β respectively that both start at some point $\tilde{x}_0$ of $\tilde{X}$ will both finish at some point $\tilde{x}_1$ of $\tilde{x}$, so that $\tilde{\alpha}(1) = \tilde{\beta}(1) = \tilde{x}_1$. This result is established in Proposition 4.7 below.

Proposition 4.7

Let $p\colon \tilde{X} \to X$ be a covering map over a topological space X, let $\alpha\colon [0,1] \to X$ and $\beta\colon [0,1] \to X$ be paths in X, where $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$, and let $\tilde{\alpha}\colon [0,1] \to \tilde{X}$ and $\tilde{\beta}\colon [0,1] \to \tilde{X}$ be paths in $\tilde{X}$ such that $p \circ \tilde{\alpha} = \alpha$ and $p \circ \tilde{\beta} = \beta$. Suppose that $\tilde{\alpha}(0) = \tilde{\beta}(0)$ and that $\alpha \simeq \beta \operatorname{rel} \{0,1\}$. Then $\tilde{\alpha}(1) = \tilde{\beta}(1)$ and $\tilde{\alpha} \simeq \tilde{\beta} \operatorname{rel} \{0,1\}$.

Proof

Let x_0 and x_1 be the points of X given by

$$x_0 = \alpha(0) = \beta(0), \qquad x_1 = \alpha(1) = \beta(1).$$

Now $\alpha \simeq \beta \text{ rel } \{0, 1\}$, and therefore there exists a homotopy $F\colon [0, 1] \times [0, 1] \to X$ such that

$$F(t, 0) = \alpha(t) \quad \text{and} \quad F(t, 1) = \beta(t) \quad \text{for all } t \in [0, 1],$$

and

$$F(0, \tau) = x_0 \quad \text{and} \quad F(1, \tau) = x_1 \quad \text{for all } \tau \in [0, 1].$$

It then follows from the Homotopy-Lifting Theorem (Theorem 4.6) that there exists a continuous map $G\colon [0,1] \times [0,1] \to \tilde{X}$ such that $p \circ G = F$ and $G(0,0) = \tilde{\alpha}(0)$. Then $p(G(0,\tau)) = x_0$ and $p(G(1,\tau)) = x_1$ for all $\tau \in [0,1]$. A straightforward application of Proposition 4.3 shows that any continuous lift of a constant path must itself be a constant path. Therefore $G(0,\tau) = \tilde{x}_0$ and $G(1,\tau) = \tilde{x}_1$ for all $\tau \in [0,1]$, where

$$\tilde{x}_0 = G(0,0) = \tilde{\alpha}(0), \qquad \tilde{x}_1 = G(1,0).$$

However

$$G(0,0) = G(0,1) = \tilde{x}_0 = \tilde{\alpha}(0) = \tilde{\beta}(0),$$

$$p(G(t,0)) = F(t,0) = \alpha(t) = p(\tilde{\alpha}(t))$$

and

$$p(G(t,1)) = F(t,1) = \beta(t) = p(\tilde{\beta}(t))$$

for all $t \in [0,1]$. It follows that the map that sends $t \in [0,1]$ to $G(t,0)$ is a lift of the path α that starts at $\tilde{x}_0$, and the map that sends $t \in [0,1]$ to $G(t,1)$ is a lift of the path β that also starts at $\tilde{x}_0$.

However Proposition 4.3 ensures that the lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of the paths α and β are uniquely determined by their starting points. It follows that $G(t,0) = \tilde{\alpha}(t)$ and $G(t,1) = \tilde{\beta}(t)$ for all $t \in [0,1]$. In particular,

$$\tilde{\alpha}(1) = G(1,0) = \tilde{x}_1 = G(1,1) = \tilde{\beta}(1).$$

Moreover the map $G\colon [0,1] \times [0,1] \to \tilde{X}$ is a homotopy between the paths $\tilde{\alpha}$ and $\tilde{\beta}$ which satisfies $G(0,\tau) = \tilde{x}_0$ and $G(1,\tau) = \tilde{x}_1$ for all $\tau \in [0,1]$. It follows that $\tilde{\alpha} \simeq \tilde{\beta} \operatorname{rel} \{0,1\}$, as required. ∎

Proposition 4.8

Let $p \colon \tilde{X} \to X$ be a covering map, and let $\tilde{x}_0$ be a point of the covering space $\tilde{X}$. Then the homomorphism

$$p_\# \colon \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, p(\tilde{x}_0))$$

of fundamental groups induced by the covering map p is injective.

Proof

Let σ_0 and σ_1 be loops in $\tilde{X}$ based at the point $\tilde{x}_0$, representing elements $[\sigma_0]$ and $[\sigma_1]$ of $\pi_1(\tilde{X}, \tilde{x}_0)$. Suppose that $p_\#[\sigma_0] = p_\#[\sigma_1]$. Then $p \circ \sigma_0 \simeq p \circ \sigma_1 \text{ rel } \{0, 1\}$. Also $\sigma_0(0) = \tilde{x}_0 = \sigma_1(0)$. Therefore $\sigma_0 \simeq \sigma_1 \text{ rel } \{0, 1\}$, by Proposition 4.7, and thus $[\sigma_0] = [\sigma_1]$. We conclude that the homomorphism $p_\# \colon \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, p(\tilde{x}_0))$ is injective. ∎

Proposition 4.9

Let $p\colon \tilde{X} \to X$ be a covering map, let $\tilde{x}_0$ be a point of the covering space $\tilde{X}$, and let γ be a loop in X based at $p(\tilde{x}_0)$. Then $[\gamma] \in p_{\#}\bigl(\pi_1(\tilde{X}, \tilde{x}_0)\bigr)$ if and only if there exists a loop $\tilde{\gamma}$ in $\tilde{X}$, based at the point $\tilde{x}_0$, such that $p \circ \tilde{\gamma} = \gamma$.

Proof

If $\gamma = p \circ \tilde{\gamma}$ for some loop $\tilde{\gamma}$ in $\tilde{X}$ based at $\tilde{x}_0$ then $[\gamma] = p_\#[\tilde{\gamma}]$, and therefore $[\gamma] \in p_\#\bigl(\pi_1(\tilde{X}, \tilde{x}_0)\bigr)$.

Conversely suppose that $[\gamma] \in p_\#\bigl(\pi_1(\tilde{X}, \tilde{x}_0)\bigr)$. We must show that there exists some loop $\tilde{\gamma}$ in $\tilde{X}$ based at $\tilde{x}_0$ such that $\gamma = p \circ \tilde{\gamma}$. Now there exists a loop σ in $\tilde{X}$ based at the point $\tilde{x}_0$ such that $[\gamma] = p_\#([\sigma])$ in $\pi_1(X, p(\tilde{x}_0))$. Then $\gamma \simeq p \circ \sigma \text{ rel } \{0, 1\}$. It follows from the Path-Lifting Theorem for covering maps (Theorem 4.5) that there exists a unique path $\tilde{\gamma} \colon [0, 1] \to \tilde{X}$ in $\tilde{X}$ for which $\tilde{\gamma}(0) = \tilde{x}_0$ and $p \circ \tilde{\gamma} = \gamma$. It then follows from Proposition 4.7 that $\tilde{\gamma}(1) = \sigma(1)$ and $\tilde{\gamma} \simeq \sigma \text{ rel } \{0, 1\}$. But $\sigma(1) = \tilde{x}_0$. Therefore the path $\tilde{\gamma}$ is the required loop in $\tilde{X}$ based the point $\tilde{x}_0$ which satisfies $p \circ \tilde{\gamma} = \gamma$. ∎

Corollary 4.10

Let $p\colon \tilde{X} \to X$ be a covering map over a topological space X, let w_0 and w_1 be points of $\tilde{X}$ satisfying $p(w_0) = p(w_1)$, and let $\alpha\colon [0,1] \to \tilde{X}$ be a path in $\tilde{X}$ from w_0 to w_1. Suppose that $[p \circ \alpha] \in p_\#\bigl(\pi_1(\tilde{X}, w_0)\bigr)$. Then the path α is a loop in $\tilde{X}$, and thus $w_0 = w_1$.

Proof

It follows from Proposition 4.9 that there exists a loop β based at w_0 satisfying $p \circ \beta = p \circ \alpha$. Then $\alpha(0) = \beta(0)$. Now Proposition 4.3 ensures that the lift to $\tilde{X}$ of any path in X is uniquely determined by its starting point. It follows that $\alpha = \beta$. But then the path α must be a loop in $\tilde{X}$, and therefore $w_0 = w_1$, as required. ∎

Theorem 4.11

Let $p\colon \tilde{X} \to X$ be a covering map over a topological space X. Suppose that $\tilde{X}$ is path-connected and that X is simply-connected. Then the covering map $p\colon \tilde{X} \to X$ is a homeomorphism.

Proof

We show that the map $p\colon \tilde{X} \to X$ is a bijection. This map is surjective (since covering maps are by definition surjective). We must show that it is injective. Let w_0 and w_1 be points of $\tilde{X}$ with the property that $p(w_0) = p(w_1)$. Then there exists a path $\alpha\colon [0,1] \to \tilde{X}$ with $\alpha(0) = w_0$ and $\alpha(1) = w_1$, since $\tilde{X}$ is path-connected. Then $p \circ \alpha$ is a loop in X based at the point x_0, where $x_0 = p(w_0)$. However $\pi_1(X, p(w_0))$ is the trivial group, since X is simply-connected. It follows from Corollary 4.10 that the path α is a loop in $\tilde{X}$ based at w_0, and therefore $w_0 = w_1$. This shows that the the covering map $p\colon \tilde{X} \to X$ is injective.

Thus the map $p\colon \tilde{X} \to X$ is a bijection, and thus has a well-defined inverse $p^{-1}\colon X \to \tilde{X}$. But any bijective covering map is a homeomorphism (Corollary 4.2). The result follows. $\blacksquare$