Module MA3427: Annual Examination 2015
Worked solutions

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Module Website

The module website, with online lecture notes, problem sets. etc. are located at

`http://www.maths.tcd.ie/~dwilkins/Courses/MA3427/`

Module Contents

Students will be informed that they will be examined on the work from Section 3 (*Product and Quotient Topologies*) onwards.

1. (a) [Bookwork.] Let $f\colon Z \to X$ be a function with the property that $p_i \circ f$ is continuous for all i. Let U be an open set in X. We must show that $f^{-1}(U)$ is open in Z.

Let z be a point of $f^{-1}(U)$, and let $f(z) = (u_1, u_2, \ldots, u_n)$. Now U is open in X, and therefore there exist open sets $V_1, V_2, \ldots, V_n$ in $X_1, X_2, \ldots, X_n$ respectively such that $u_i \in V_i$ for all i and $V_1 \times V_2 \times \cdots \times V_n \subset U$. Let

$$N_z = f_1^{-1}(V_1) \cap f_2^{-1}(V_2) \cap \cdots \cap f_n^{-1}(V_n),$$

where $f_i = p_i \circ f$ for $i = 1, 2, \ldots, n$. Now $f_i^{-1}(V_i)$ is an open subset of Z for $i = 1, 2, \ldots, n$, since V_i is open in X_i and $f_i\colon Z \to X_i$ is continuous. Thus N_z, being a finite intersection of open sets, is itself open in Z. Moreover

$$f(N_z) \subset V_1 \times V_2 \times \cdots \times V_n \subset U,$$

so that $N_z \subset f^{-1}(U)$. It follows that $f^{-1}(U)$ is the union of the open sets N_z as z ranges over all points of $f^{-1}(U)$. Therefore $f^{-1}(U)$ is open in Z. This shows that $f\colon Z \to X$ is continuous, as required.

(b) [Definition. From printed lecture notes.] Let X and Y be topological spaces and let $q\colon X \to Y$ be a function from X to Y. The function q is said to be an *identification map* if and only if the following conditions are satisfied:

- the function $q\colon X \to Y$ is surjective,
- a subset U of Y is open in Y if and only if $q^{-1}(U)$ is open in X.

(c) [Bookwork adapted from printed lecture notes] Let τ be the collection consisting of all subsets U of Y for which $q^{-1}(U)$ is open in X. Now $q^{-1}(\emptyset) = \emptyset$, and $q^{-1}(Y) = X$, so that $\emptyset \in \tau$ and $Y \in \tau$. It follows directly from (b) that, given any collection of subsets of Y, the union of the preimages of the sets is the preimage of the union of those sets, and the intersection of the preimages of the sets is the preimage of the intersection of those sets. Therefore unions and finite intersections of sets belonging to τ must themselves belong to τ. Thus τ is a topology on Y, and the function $q\colon X \to Y$ is an identification map with respect to the topology τ. Moreover the definition of identification maps ensures that the open subsets of Y must be the subsets belonging to τ, and thus τ is the unique topology on Y for which the function $q\colon X \to Y$ is an identification map.

(d) [From printed lecture notes.] Suppose that f is continuous. Then the composition function $f \circ q$ is a composition of continuous functions and hence is itself continuous.

Conversely suppose that $f \circ q$ is continuous. Let U be an open set in Z. Then $q^{-1}(f^{-1}(U))$ is open in X (since $f \circ q$ is continuous), and hence $f^{-1}(U)$ is open in Y (since the function q is an identification map). Therefore the function f is continuous, as required.

2. (a) [Definition.] Let X and $\tilde{X}$ be topological spaces and let $p\colon \tilde{X} \to X$ be a continuous map. An open subset U of X is said to be *evenly covered* by the map p if and only if $p^{-1}(U)$ is a disjoint union of open sets of $\tilde{X}$ each of which is mapped homeomorphically onto U by p. The map $p\colon \tilde{X} \to X$ is said to be a *covering map* if $p\colon \tilde{X} \to X$ is surjective and in addition every point of X is contained in some open set that is evenly covered by the map p.

(b) [Mostly bookwork. The covering map from the helicoid to the punctured plane is discussed extensively in notes, but in various places as free-form text to introduce the ideas of covering maps and path lifting rather than as a labelled proposition or example.]

The map p is a surjective map from the helicoid $\tilde{X}$ to the punctured plane X.

Let (x, y) be a point of the punctured plane X, and let $\rho = \sqrt{x^2 + y^2}$. Then there exists some real number z such that $x = \rho \cos 2\pi z$ and $y = \rho \sin 2\pi z$. Then $(x, y) = p(x, y, z)$. Thus the map $p\colon \tilde{X} \to X$ is surjective.

[Material from this point on is quoted verbatim from the lecture notes, up to the final paragraph of the worked solution.]

Given any real number θ, let

$$\tilde{U}_\theta = \left\{ (x, y, z) \in \tilde{X} : \left| z - \frac{\theta}{2\pi} \right| < \frac{1}{2} \right\},$$

and let $U_\theta = p(\tilde{U}_\theta)$. Then U_θ is the sector of the punctured plane consisting all all half-lines starting at the origin that make an angle of less than π with the half-line in the direction of the vector $(\cos\theta, \sin\theta)$. It follows that $U_\theta = X \setminus L_\theta$, where L_θ is the half-line from the origin in the direction of the vector $(-\cos\theta, -\sin\theta)$, defined so that

$$L_\theta = \{(-t\cos\theta, -t\sin\theta) : t \in \mathbb{R} \text{ and } t > 0\}.$$

Then the preimage $p^{-1}(U_\theta)$ of U_θ is the disjoint union $\bigcup\limits_{n \in \mathbb{Z}} V_n$ of the open subsets V_n of $\tilde{X}$ for all integers n, where

$$\begin{aligned} V_n &= \{(x, y, z) \in \tilde{X} : (x, y, z - n) \in \tilde{U}_\theta\} \\ &= \left\{ (x, y, z) \in \tilde{X} : \frac{\theta}{2\pi} + n - \frac{1}{2} < z < \frac{\theta}{2\pi} + n + \frac{1}{2} \right\}. \end{aligned}$$

Each of these open set V_n is mapped homeomorphically onto U_θ by the map $p: \tilde{X} \to X$. Indeed let $s_n: U_\theta \to V_n$ be defined such that

$$s_n(\rho\cos(\theta+\varphi), \rho\sin(\theta+\varphi)) = \left(\rho\cos(\theta+\varphi), \rho\sin(\theta+\varphi), \frac{\theta+\varphi}{2\pi} + n\right)$$

for all angles φ satisfying $-\pi < \varphi < \pi$. Then $s_n: U_\theta \to V_n$ is a continuous map, and this map is the inverse of the restriction of the map $p: \tilde{X} \to X$ to V_n. It follows that the preimage $p^{-1}(U_\theta)$ of the open subset U_θ of X is a disjoint union of open sets, each of which is mapped homeomorphically onto U_θ by the map $p: \tilde{X} \to X$. We say that the open set U_θ is *evenly covered* by the continuous map $p: \tilde{X} \to X$.

[The direct quote from the lecture notes ends here.]

The continuous map $p: \tilde{X} \to X$ is surjective, and we have verified that, given any point of X, there exists an open neighbourhood of that point that is evenly covered by the map $p: \tilde{X} \to X$. It follows that $p: \tilde{X} \to X$ is a covering map, as required.

3. [Based on lecture notes.] Let X be a topological space, and let x_0 and x_1 be points of X. A *path* in X from x_0 to x_1 is defined to be a continuous map $\gamma\colon [0,1] \to X$ for which $\gamma(0) = x_0$ and $\gamma(1) = x_1$. A *loop* in X based at x_0 is defined to be a continuous map $\gamma\colon [0,1] \to X$ for which $\gamma(0) = \gamma(1) = x_0$.

   We can concatenate paths. Let $\gamma_1\colon [0,1] \to X$ and $\gamma_2\colon [0,1] \to X$ be paths in some topological space X. Suppose that $\gamma_1(1) = \gamma_2(0)$. We define the *product path* $\gamma_1.\gamma_2\colon [0,1] \to X$ by

$$(\gamma_1.\gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

   If $\gamma\colon [0,1] \to X$ is a path in X then we define the *inverse path* $\gamma^{-1}\colon [0,1] \to X$ by $\gamma^{-1}(t) = \gamma(1-t)$.

   Let X be a topological space, and let $x_0 \in X$ be some chosen point of X. We define an equivalence relation on the set of all (continuous) loops based at the basepoint x_0 of X, where two such loops γ_0 and γ_1 are equivalent if and only if $\gamma_0 \simeq \gamma_1$ rel $\{0,1\}$. We denote the equivalence class of a loop $\gamma\colon [0,1] \to X$ based at x_0 by $[\gamma]$. This equivalence class is referred to as the *based homotopy class* of the loop γ. The set of equivalence classes of loops based at x_0 is denoted by $\pi_1(X, x_0)$.

   Let X be a topological space, let x_0 be some chosen point of X, and let $\pi_1(X, x_0)$ be the set of all based homotopy classes of loops based at the point x_0. We show $\pi_1(X, x_0)$ is a group, the group multiplication on $\pi_1(X, x_0)$ being defined according to the rule $[\gamma_1][\gamma_2] = [\gamma_1.\gamma_2]$ for all loops γ_1 and γ_2 based at x_0. This group is the *fundamental group* of the topological space X based at x_0.

   First we show that the group operation on $\pi_1(X, x_0)$ is well-defined. Let γ_1, γ_1', γ_2 and γ_2' be loops in X based at the point x_0. Suppose that $[\gamma_1] = [\gamma_1']$ and $[\gamma_2] = [\gamma_2']$. Let the map $F\colon [0,1] \times [0,1] \to X$ be defined by

$$F(t,\tau) = \begin{cases} F_1(2t,\tau) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ F_2(2t-1,\tau) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

   where $F_1\colon [0,1] \times [0,1] \to X$ is a homotopy between γ_1 and γ_1', $F_2\colon [0,1] \times [0,1] \to X$ is a homotopy between γ_2 and γ_2', and where the homotopies F_1 and F_2 map $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. Then F is itself a homotopy from $\gamma_1.\gamma_2$ to $\gamma_1'.\gamma_2'$, and maps $(0,\tau)$ and $(1,\tau)$ to x_0 for all

$\tau \in [0,1]$. Thus $[\gamma_1.\gamma_2] = [\gamma_1'.\gamma_2']$, showing that the group operation on $\pi_1(X, x_0)$ is well-defined.

Next we show that the group operation on $\pi_1(X, x_0)$ is associative. Let γ_1, γ_2 and γ_3 be loops based at x_0, and let $\alpha = (\gamma_1.\gamma_2).\gamma_3$. Then $\gamma_1.(\gamma_2.\gamma_3) = \alpha \circ \theta$, where

$$\theta(t) = \begin{cases} \frac{1}{2}t & \text{if } 0 \leq t \leq \frac{1}{2}; \\ t - \frac{1}{4} & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4}; \\ 2t - 1 & \text{if } \frac{3}{4} \leq t \leq 1. \end{cases}$$

Thus the map $G\colon [0,1] \times [0,1] \to X$ defined by $G(t,\tau) = \alpha((1-\tau)t + \tau\theta(t))$ is a homotopy between $(\gamma_1.\gamma_2).\gamma_3$ and $\gamma_1.(\gamma_2.\gamma_3)$, and moreover this homotopy maps $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. It follows that $(\gamma_1.\gamma_2).\gamma_3 \simeq \gamma_1.(\gamma_2.\gamma_3)$ rel $\{0,1\}$ and hence $([\gamma_1][\gamma_2])[\gamma_3] = [\gamma_1]([\gamma_2][\gamma_3])$. This shows that the group operation on $\pi_1(X, x_0)$ is associative.

Let $\varepsilon\colon [0,1] \to X$ denote the constant loop at x_0, defined by $\varepsilon(t) = x_0$ for all $t \in [0,1]$. Then $\varepsilon.\gamma = \gamma \circ \theta_0$ and $\gamma.\varepsilon = \gamma \circ \theta_1$ for any loop γ based at x_0, where

$$\theta_0(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 2t - 1 & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases} \qquad \theta_1(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

for all $t \in [0,1]$. But the continuous map $(t,\tau) \mapsto \gamma((1-\tau)t + \tau\theta_j(t))$ is a homotopy between γ and $\gamma \circ \theta_j$ for $j = 0, 1$ which sends $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. Therefore $\varepsilon.\gamma \simeq \gamma \simeq \gamma.\varepsilon$ rel $\{0,1\}$, and hence $[\varepsilon][\gamma] = [\gamma] = [\gamma][\varepsilon]$. We conclude that $[\varepsilon]$ represents the identity element of $\pi_1(X, x_0)$.

It only remains to verify the existence of inverses. Now the map $K\colon [0,1] \times [0,1] \to X$ defined by

$$K(t,\tau) = \begin{cases} \gamma(2\tau t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \gamma(2\tau(1-t)) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

is a homotopy between the loops $\gamma.\gamma^{-1}$ and ε, and moreover this homotopy sends $(0,\tau)$ and $(1,\tau)$ to x_0 for all $\tau \in [0,1]$. Therefore $\gamma.\gamma^{-1} \simeq \varepsilon \operatorname{rel}\{0,1\}$, and thus $[\gamma][\gamma^{-1}] = [\gamma.\gamma^{-1}] = [\varepsilon]$. On replacing γ by γ^{-1}, we see also that $[\gamma^{-1}][\gamma] = [\varepsilon]$, and thus $[\gamma^{-1}] = [\gamma]^{-1}$, as required.

4. (a) [Bookwork.] Let $\gamma\colon [0,1] \to X/G$ be a loop in the orbit space with $\gamma(0) = \gamma(1) = q(x_0)$. It follows from the Path Lifting Theorem for covering maps that there exists a unique path $\tilde{\gamma}\colon [0,1] \to X$ for which $\tilde{\gamma}(0) = x_0$ and $q \circ \tilde{\gamma} = \gamma$. Now $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$ must belong to the same orbit, since $q(\tilde{\gamma}(0)) = \gamma(0) = \gamma(1) = q(\tilde{\gamma}(1))$. Therefore there exists some element g of G such that $\tilde{\gamma}(1) = \theta_g(x_0)$. This element g is uniquely determined, since the group G acts freely on X. Moreover the value of g is determined by the based homotopy class $[\gamma]$ of γ in $\pi_1(X/G, q(x_0))$. Indeed it follows from a basic result (stated on the examination paper) that if σ is a loop in X/G based at $q(x_0)$, if $\tilde{\sigma}$ is the lift of σ starting at x_0 (so that $q \circ \tilde{\sigma} = \sigma$ and $\tilde{\sigma}(0) = x_0$), and if $[\gamma] = [\sigma]$ in $\pi_1(X/G, q(x_0))$ (so that $\gamma \simeq \sigma$ rel $\{0,1\}$), then $\tilde{\gamma}(1) = \tilde{\sigma}(1)$. We conclude therefore that there exists a well-defined function

$$\lambda\colon \pi_1(X/G, q(x_0)) \to G,$$

which is characterized by the property that $\tilde{\gamma}(1) = \theta_{\lambda([\gamma])}(x_0)$ for any loop γ in X/G based at $q(x_0)$, where $\tilde{\gamma}$ denotes the unique path in X for which $\tilde{\gamma}(0) = x_0$ and $q \circ \tilde{\gamma} = \gamma$.

Now let $\alpha\colon [0,1] \to X/G$ and $\beta\colon [0,1] \to X/G$ be loops in X/G based at $q(x_0)$, and let $\tilde{\alpha}\colon [0,1] \to X$ and $\tilde{\beta}\colon [0,1] \to X$ be the lifts of α and β respectively starting at x_0, so that $q \circ \tilde{\alpha} = \alpha$, $q \circ \tilde{\beta} = \beta$ and $\tilde{\alpha}(0) = \tilde{\beta}(0) = x_0$. Then $\tilde{\alpha}(1) = \theta_{\lambda([\alpha])}(x_0)$ and $\tilde{\beta}(1) = \theta_{\lambda([\beta])}(x_0)$. Then the path $\theta_{\lambda([\alpha])} \circ \tilde{\beta}$ is also a lift of the loop β, and is the unique lift of β starting at $\tilde{\alpha}(1)$. Let $\alpha.\beta$ be the concatenation of the loops α and β, where

$$(\alpha.\beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \beta(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Then the unique lift of $\alpha.\beta$ to X starting at x_0 is the path $\sigma\colon [0,1] \to X$, where

$$\sigma(t) = \begin{cases} \tilde{\alpha}(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \theta_{\lambda([\alpha])}(\tilde{\beta}(2t-1)) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

It follows that

$$\begin{aligned} \theta_{\lambda([\alpha][\beta])}(x_0) &= \theta_{\lambda([\alpha.\beta])}(x_0) = \sigma(1) = \theta_{\lambda([\alpha])}(\tilde{\beta}(1)) \\ &= \theta_{\lambda([\alpha])}(\theta_{\lambda([\beta])}(x_0)) = \theta_{\lambda([\alpha])\lambda([\beta])}(x_0) \end{aligned}$$

and therefore $\lambda([\alpha][\beta]) = \lambda([\alpha])\lambda([\beta])$. Therefore the function

$$\lambda\colon \pi_1(X/G, q(x_0)) \to G$$

is a homomorphism.

Let $g \in G$. Then there exists a path α in X from x_0 to $\theta_g(x_0)$, since the space X is path-connected. Then $q \circ \alpha$ is a loop in X/G based at $q(x_0)$, and $g = \lambda([q \circ \alpha])$. This shows that the homomorphism λ is surjective.

(b) [Bookwork.] Let $\gamma\colon [0, 1] \to X/G$ be a loop in X/G based at $q(x_0)$. Suppose that $[\gamma] \in \ker \lambda$. Then $\tilde{\gamma}(1) = \theta_e(x_0) = x_0$, and therefore $\tilde{\gamma}$ is a loop in X based at x_0. Moreover $[\gamma] = q_\#[\tilde{\gamma}]$, and therefore $[\gamma] \in q_\#(\pi_1(X, x_0))$. On the other hand, if $[\gamma] \in q_\#(\pi_1(X, x_0))$ then $\gamma = q \circ \tilde{\gamma}$ for some loop $\tilde{\gamma}$ in X based at x_0. But then $x_0 = \tilde{\gamma}(1) = \theta_{\lambda([\gamma])}(x_0)$, and therefore $\lambda([\gamma]) = e$, where e is the identity element of G. Thus $\ker \lambda = q_\#(\pi_1(X, x_0))$, as required.