# Module MA3427: Michaelmas Term 2014 Problems

1. A function $f\colon X \to Y$ between topological spaces $X$ and $Y$ is continuous if and only if, for every open set $V$ in $Y$, the preimage $f^{-1}(V)$ of $V$ is an open set in $X$.

   (a) Explain why the identity map $1_X\colon X \to X$ of any topological space $X$ is continuous.

   (b) Let $X$ and $Y$ be topological spaces, let $q$ be a point of $Y$, and let $c_q\colon X \to Y$ be the constant map defined such that $c_q(x) = q$ for all $x \in X$. Explain why $c_q\colon X \to Y$ is continuous.

2. Prove that

$$\left\{ (x, y, z) \in \mathbb{R}^3 : x > 0 \text{ and } y^2 + z^2 \geq \frac{1}{x^2} \right\}$$

   is a closed set in $\mathbb{R}^3$.

3. An infinite sequence $x_1, x_2, x_3, \ldots$ of points in a topological space $X$ is said to *converge* to some point $p$ of $X$ if and only if, given any open set $V$ in $Y$ with $p \in V$, there exists some positive integer $N$ such that $x_j \in V$ whenever $j \geq N$. Prove that an infinite sequence of points of a Hausdorff space can converge to at most one point of that Hausdorff space.

4. (a) Let $f\colon X \to Y$ be a function from a topological space $X$ to a topological space $Y$, and let $A$ and $B$ be subsets of $X$ for which $X = A \cup B$. Suppose that the restrictions $f|A$ and $f|B$ of $f$ to the sets $A$ and $B$ are continuous. Is $f\colon X \to Y$ necessarily continuous on $X$? [Give proof or counterexample.]

   (b) Let $f\colon X \to Y$ be a function from a topological space $X$ to a topological space $Y$, and let $\mathcal{F}$ be a (not necessarily finite) collection of closed subsets of $X$ whose union is the whole of $X$. Suppose that the restriction $f|A$ of $f$ to $A$ is continuous for all closed sets $A$ in the collection $\mathcal{F}$. Is $f\colon X \to Y$ necessarily continuous on $X$? [Give proof or counterexample.]

5. Let $f\colon X \to Y$ be a function from a topological space $X$ to a topological space $Y$, and let $\mathcal{U}$ be a collection of open subsets of $X$ whose union is the whole of $X$. Suppose that the restriction $f|W$ of $f$ to $W$ is continuous for all open sets $W$ in the collection $\mathcal{U}$. Prove that $f\colon X \to Y$ is continuous on $X$.

6. Let $X$ be a topological space, let $A$ be a subset of $X$, and let $B$ be the complement $X \setminus A$ of $A$ in $X$. Prove that the interior of $B$ is the complement of the closure of $A$.

7. Determine which of the following subsets of $\mathbb{R}^3$ are compact.

   (i) The x-axis $\{(x, y, z) \in \mathbb{R}^3 : y = z = 0\}$

   (ii) The surface of a tetrahedron in $\mathbb{R}^3$.

   (iii) $\{(x, y, z) \in \mathbb{R}^3 : x > 0 \text{ and } x^2 + y^2 - z^2 \leq 1\}$.

8. Let $X$ be a topological space. Suppose that $X = A \cup B$, where $A$ and $B$ are path-connected subsets of $X$ and $A \cap B$ is non-empty. Show that $X$ is path-connected.

9. Let $f\colon X \to Y$ be a continuous map between topological spaces $X$ and $Y$. Suppose that $X$ is path-connected. Prove that the image $f(X)$ of the map $f$ is also path-connected.

10. Determine the connected components of the following subsets of $\mathbb{R}^2$:

    (i) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$,

    (ii) $\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$,

    (iii) $\{(x, y) \in \mathbb{R}^2 : y^2 = x(x^2 - 1)\}$,

    (iv) $\{(x, y) \in \mathbb{R}^2 : (x - n)^2 + y^2 > \frac{1}{4} \text{ for all } n \in \mathbb{Z}\}$.

    [Fully justify your answers.]

11. A topological space $X$ is said to be *locally path-connected* if, given any point $x$ of $X$ there exists a path-connected open set $U$ in $X$ such that $x \in U$.

(a) Let $X$ be a locally path-connected topological space, and let $p$ be a point of $X$. Let $A$ be the set of all points $x$ of $X$ for which there exists a path from $p$ to $x$, and let $B$ be the complement of $A$ in $X$. Prove that $A$ and $B$ are open in $X$.

(b) Use the result of (a) to show that any connected and locally path-connected topological space is path-connected.

12. Let $X$ and $Y$ be path-connected topological spaces, let $q$ be a point of $Y$, and let $i_q\colon X \to X \times Y$ be the function from $X$ to the Cartesian product $X \times Y$ of the topological spaces $X$ and $Y$ defined such that $i_q(x) = (x, q)$ for all $x \in X$. Explain why $i_q\colon X \to X \times Y$ is guaranteed to be continuous.

13. Let $X$ and $Y$ be path-connected topological spaces. Explain why the Cartesian product $X \times Y$ of $X$ and $Y$ is path-connected.

14. Let $n$ an integer satisfying $n \geq 2$, let $S^n$ be the unit sphere in $\mathbb{R}^{n+1}$, and let $E_n$ denote the closed unit ball in $\mathbb{R}^n$, so that

$$\begin{aligned} S^n &= \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \cdots + x_n^2 + x_{n+1}^2 = 1\}, \\ E^n &= \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1\}. \end{aligned}$$

Real projective $n$-dimensional space $RP^n$ can be identified with the set of lines passing through the origin in $\mathbb{R}^{n+1}$, and there is a two-to-one function $q\colon S^n \to RP^n$ that maps each point $\mathbf{x}$ to the element of $RP^n$ represented by the line through the origin that contains the point $\mathbf{x}$. Points $\mathbf{u}$ and $\mathbf{v}$ of $S^n$ satisfy $q(\mathbf{u}) = q(\mathbf{v})$ if and only if $\mathbf{v} = \pm\mathbf{u}$. The topology on $RP^n$ is the quotient topology with respect to which the function $q\colon S^n \to RP^n$ is an identification map.

(a) Let $h\colon E^n \to S^n$ be the function from $E^n$ to the 'upper hemisphere' of $S^n$ defined such that

$$h(x_1, x_2, \ldots, x_n) = \left(x_1, x_2, \ldots, x_n, \sqrt{1 - x_1^2 - \cdots - x_n^2}\right)$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, and let $r\colon E^n \to RP^n$ be the composition $q \circ h$ of the continuous map $h\colon E^n \to S^n$ and the identification map $q\colon S^n \to RP^n$. Prove that $r\colon E^n \to RP^n$ is also an identification map.

(b) Let $h\colon E^n \to RP^n$ be the continuous map defined in (a). Describe necessary and sufficient conditions to be satisfied by points $\mathbf{y}$ and $\mathbf{z}$ of $E^n$ to ensure that $r(\mathbf{y}) = r(\mathbf{z})$.

15. Let $X$ be a convex subset of $\mathbb{R}^n$. (A subset $X$ of $\mathbb{R}^n$ is said to be *convex* if $(1-t)\mathbf{x} + t\mathbf{y} \in X$ for all $\mathbf{x}, \mathbf{y} \in X$ and real numbers $t$ satisfying $0 \leq t \leq 1$.)

    (a) Prove that any two continuous functions mapping some topological space into $X$ are homotopic.

    (b) Prove that any two continuous functions mapping $X$ into some path-connected topological space $Y$ are homotopic.

16. Determine which of the following maps are covering maps:—

    (i) the map from $\mathbb{R}$ to $[-1, 1]$ sending $\theta$ to $\sin\theta$,

    (ii) the map from $S^1$ to $S^1$ sending $(\cos\theta, \sin\theta)$ to $(\cos n\theta, \sin n\theta)$, where $n$ is some non-zero integer,

    (iii) the map from $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ to $\{z \in \mathbb{C} : 0 < |z| < 1\}$ sending $z$ to $\exp(z)$,

    (iv) the map from $\{z \in \mathbb{C} : -4\pi < \operatorname{Im} z < 4\pi\}$ to $\{z \in \mathbb{C} : |z| > 0\}$ sending $z$ to $\exp(z)$.

    [Briefly justify your answers.]

17. A continuous function $f\colon X \to Y$ between topological spaces $X$ and $Y$ is said to be a *local homeomorphism* if, given any point $x$ of $X$ there exists an open set $V$ in $X$ containing the point $x$ and an open set $W$ in $Y$ containing the point $f(x)$ such that the function $f$ maps $V$ homeomorphically onto $W$. Explain why any covering map is a local homeomorphism.

18. Determine which of the maps described in question 16 are local homeomorphisms.

19. (a) Let $W$, $X$, $Y$ and $Z$ be topological spaces, and let $A$ be a subset of $X$. Let $f\colon X \to Y$ and $g\colon X \to Y$ be continuous maps. Suppose that $f \simeq g \operatorname{rel} A$. Show that $h \circ f \simeq h \circ g \operatorname{rel} A$ for all continuous maps $h\colon Y \to Z$, and that $f \circ e \simeq g \circ e \operatorname{rel} e^{-1}(A)$ for all continuous maps $e\colon W \to X$.

(b) Using $(a)$, explain why, given any continuous map $f: X \to Y$ between topological spaces $X$ and $Y$, there is a well-defined homomorphism $f_\#: \pi_1(X, x) \to \pi_1(Y, f(x))$ of fundamental groups for any $x \in X$ which sends $[\gamma]$ to $[f \circ \gamma]$ for any loop $\gamma$ based at the point $x$.

(c) Let $f: X \to Y$ and $g: X \to Y$ be continuous maps satisfying $f(x) = g(x)$ and $f \simeq g \operatorname{rel}\{x\}$. Show that the homomorphisms $f_\#$ and $g_\#$ from $\pi_1(X, x)$ to $\pi_1(Y, f(x))$ induced by the maps $f$ and $g$ are equal.

20. (a) Let $X$ and $Y$ be topological spaces, let $f: X \to Y$ and $h: Y \to X$ continuous maps, and let $x$ be a point of $X$. Suppose that $h(f(x)) = x$ and that $h \circ f \simeq 1_X \operatorname{rel}\{x\}$ and $f \circ h \simeq 1_Y \operatorname{rel}\{f(x)\}$, where $1_X$ and $1_Y$ denote the identity maps of the spaces $X$ and $Y$. Explain why the fundamental groups $\pi_1(X, x)$ and $\pi_1(Y, f(x))$ are isomorphic.

    (b) Using $(a)$, explain why the fundamental groups $\pi_1(\mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbf{p})$ and $\pi^1(S^{n-1}, \mathbf{p})$ of $\mathbb{R}^n \setminus \{\mathbf{0}\}$ and the $(n-1)$-dimensional sphere $S^{n-1}$ are isomorphic for all $n > 1$, where $\mathbf{p} \in S^{n-1}$.

21. Let $X$ be a topological space, and let $\alpha: [0, 1] \to X$ and $\beta: [0, 1] \to X$ be paths in $X$. We say that the path $\beta$ is a *reparameterization* of the path $\alpha$ if there exists a strictly increasing continuous function $\sigma: [0, 1] \to [0, 1]$ such that $\sigma(0) = 0$, $\sigma(1) = 1$ and $\beta = \alpha \circ \sigma$. (Note that if $\beta$ is a reparameterization of $\alpha$ then $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$, and the paths $\alpha$ and $\beta$ have the same image in $X$.)

    (a) Show that there is a well-defined equivalence relation on the set of all paths in $X$, where a path $\alpha$ is related to a path $\beta$ if and only if $\beta$ is a reparameterization of a path $\alpha$. [Hint: use the basic result of analysis which states that a strictly increasing continuous function mapping one interval onto another has a continuous inverse.]

    (b) Show that if the path $\beta$ is a reparameterization of the path $\alpha$, then $\beta \simeq \alpha \operatorname{rel}\{0, 1\}$.

    Given paths $\gamma_1, \gamma_2, \ldots, \gamma_n$ in a topological space $X$, where $\gamma_i(1) = \gamma_{i+1}(0)$ for $i = 1, 2, \ldots, n-1$, we define the *concatenation* $\gamma_1.\gamma_2.\cdots.\gamma_n$ of the paths by the formula $(\gamma_1.\gamma_2.\ \ldots\ .\gamma_n)(t) = \gamma_i(nt - i + 1)$ for all $t$ satisfying $(i-1)/n \leq t \leq i/n$.

    (c) Show that the path $(\gamma_1.\ \ldots\ .\gamma_r).(\gamma_{r+1}.\ \ldots\ .\gamma_n)$ is a reparameterization of $\gamma_1.\gamma_2.\cdots.\gamma_n$ for any $r$ between 1 and $n-1$.

(d) By making repeated applications of $(c)$, or otherwise, show that $(\gamma_1.\gamma_2).\gamma_3.(\gamma_4.\gamma_5)$ is a reparameterization of $\gamma_1.(\gamma_2.\gamma_3.\gamma_4).\gamma_5$ for all paths $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$ in $X$ satisfying $\gamma_i(1) = \gamma_{i+1}(0)$ for $i = 1, 2, 3, 4$.

22. Let $X$ be a topological space.

(a) Show that, given any path $\alpha\colon [0,1] \to X$ in $X$, there is a well-defined homomorphism $\Theta_\alpha\colon \pi_1(X, \alpha(1)) \to \pi_1(X, \alpha(0))$ of fundamental groups which sends the homotopy class $[\gamma]$ of any loop $\gamma$ based at $\alpha(1)$ to the homotopy class $[\alpha.\gamma.\alpha^{-1}]$ of the loop $\alpha.\gamma.\alpha^{-1}$, where

$$(\alpha.\gamma.\alpha^{-1})(t) = \begin{cases} \alpha(3t) & \text{if } 0 \le t \le \frac{1}{3}, \\ \gamma(3t-1) & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ \alpha(3-3t) & \text{if } \frac{2}{3} \le t \le 1 \end{cases}$$

(i.e., $\alpha.\gamma.\alpha^{-1}$ represents '$\alpha$ followed by $\gamma$ followed by $\alpha$ reversed').

(b) Show that $\Theta_{\alpha.\beta} = \Theta_\alpha \circ \Theta_\beta$ for all paths $\alpha$ and $\beta$ in $X$ satisfying $\beta(0) = \alpha(1)$.

(c) Show that $\Theta_\alpha$ is the identity homomorphism whenever $\alpha$ is a constant path.

(d) Let $\alpha$ and $\hat{\alpha}$ be paths in $X$ satisfying $\alpha(0) = \hat{\alpha}(0)$ and $\alpha(1) = \hat{\alpha}(1)$. Suppose that $\alpha(0) \simeq \hat{\alpha}(0) \operatorname{rel}\{0,1\}$. Show that $\Theta_\alpha = \Theta_{\hat{\alpha}}$.

(e) Explain why the homomorphism $\Theta_\alpha\colon \pi_1(X, \alpha(1)) \to \pi_1(X, \alpha(0))$ is an isomorphism for all paths $\alpha$ in $X$. (This shows that, up to isomorphism, the fundamental group of a path-connected topological space does not depend on the choice of basepoint.)

23. Let $X$ and $Y$ be topological spaces, and let $x_0$ and $y_0$ be points of $X$ and $Y$. Prove that $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$. [Hint: you should make use of the result that a function mapping a topological space into $X \times Y$ is continuous if and only if its components are continuous.]