Module MA3427: Michaelmas Term 2012
Problems

1. A topological space X is said to be *locally path-connected* if, given any point x of X there exists a path-connected open set U in X such that $x \in U$.

   (a) Let X be a locally path-connected topological space, and let p be a point of X. Let A be the set of all points x of X for which there exists a path from p to x, and let B be the complement of A in X. Prove that A and B are open in X.

   (b) Use the result of (a) to show that any connected and locally path-connected topological space is path-connected.

2. Let X be a convex subset of $\mathbb{R}^n$. (A subset X of $\mathbb{R}^n$ is said to be *convex* if $(1-t)\mathbf{x} + t\mathbf{y} \in X$ for all $\mathbf{x}, \mathbf{y} \in X$ and real numbers t satisfying $0 \leq t \leq 1$.)

   (a) Prove that any two continuous functions mapping some topological space into X are homotopic.

   (b) Prove that any two continuous functions mapping X into some path-connected topological space Y are homotopic.

3. Determine which of the following maps are covering maps:—

   (i) the map from $\mathbb{R}$ to $[-1, 1]$ sending θ to $\sin\theta$,

   (ii) the map from S^1 to S^1 sending $(\cos\theta, \sin\theta)$ to $(\cos n\theta, \sin n\theta)$, where n is some non-zero integer,

   (iii) the map from $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ to $\{z \in \mathbb{C} : 0 < |z| < 1\}$ sending z to $\exp(z)$,

   (iv) the map from $\{z \in \mathbb{C} : -4\pi < \operatorname{Im} z < 4\pi\}$ to $\{z \in \mathbb{C} : |z| > 0\}$ sending z to $\exp(z)$.

   [Briefly justify your answers.]

4. A continuous function $f\colon X \to Y$ between topological spaces X and Y is said to be a *local homeomorphism* if, given any point x of X there exists an open set V in X containing the point x and an open set W in Y containing the point $f(x)$ such that the function f maps V homeomorphically onto W. Explain why any covering map is a local homeomorphism.

5. Determine which of the maps described in question 3 are local homeomorphisms.

6. (a) Let W, X, Y and Z be topological spaces, and let A be a subset of X. Let $f\colon X \to Y$ and $g\colon X \to Y$ be continuous maps. Suppose that $f \simeq g \operatorname{rel} A$. Show that $h \circ f \simeq h \circ g \operatorname{rel} A$ for all continuous maps $h\colon Y \to Z$, and that $f \circ e \simeq g \circ e \operatorname{rel} e^{-1}(A)$ for all continuous maps $e\colon W \to X$.

   (b) Using (a), explain why, given any continuous map $f\colon X \to Y$ between topological spaces X and Y, there is a well-defined homomorphism $f_\#\colon \pi_1(X,x) \to \pi_1(Y, f(x))$ of fundamental groups for any $x \in X$ which sends $[\gamma]$ to $[f \circ \gamma]$ for any loop γ based at the point x.

   (c) Let $f\colon X \to Y$ and $g\colon X \to Y$ be continuous maps satisfying $f(x) = g(x)$ and $f \simeq g \operatorname{rel}\{x\}$. Show that the homomorphisms $f_\#$ and $g_\#$ from $\pi_1(X,x)$ to $\pi_1(Y, f(x))$ induced by the maps f and g are equal.

7. (a) Let X and Y be topological spaces, let $f\colon X \to Y$ and $h\colon Y \to X$ continuous maps, and let x be a point of X. Suppose that $h(f(x)) = x$ and that $h \circ f \simeq 1_X \operatorname{rel}\{x\}$ and $f \circ h \simeq 1_Y \operatorname{rel}\{f(x)\}$, where 1_X and 1_Y denote the identity maps of the spaces X and Y. Explain why the fundamental groups $\pi_1(X,x)$ and $\pi_1(Y, f(x))$ are isomorphic.

   (b) Using (a), explain why the fundamental groups $\pi_1(\mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbf{p})$ and $\pi^1(S^{n-1}, \mathbf{p})$ of $\mathbb{R}^n \setminus \{\mathbf{0}\}$ and the $(n-1)$-dimensional sphere S^{n-1} are isomorphic for all $n > 1$, where $\mathbf{p} \in S^{n-1}$.

8. Let X be a topological space, and let $\alpha\colon [0,1] \to X$ and $\beta\colon [0,1] \to X$ be paths in X. We say that the path β is a *reparameterization* of the path α if there exists a strictly increasing continuous function $\sigma\colon [0,1] \to [0,1]$ such that $\sigma(0) = 0$, $\sigma(1) = 1$ and $\beta = \alpha \circ \sigma$. (Note that if β is a reparameterization of α then $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$, and the paths α and β have the same image in X.)

(a) Show that there is a well-defined equivalence relation on the set of all paths in X, where a path α is related to a path β if and only if β is a reparameterization of a path α. [Hint: use the basic result of analysis which states that a strictly increasing continuous function mapping one interval onto another has a continuous inverse.]

(b) Show that if the path β is a reparameterization of the path α, then $\beta \simeq \alpha \operatorname{rel}\{0,1\}$.

Given paths $\gamma_1, \gamma_2, \ldots, \gamma_n$ in a topological space X, where $\gamma_i(1) = \gamma_{i+1}(0)$ for $i = 1, 2, \ldots, n-1$, we define the *concatenation* $\gamma_1.\gamma_2.\,\cdots\,.\gamma_n$ of the paths by the formula $(\gamma_1.\gamma_2.\,\ldots\,.\gamma_n)(t) = \gamma_i(nt - i + 1)$ for all t satisfying $(i-1)/n \leq t \leq i/n$.

(c) Show that the path $(\gamma_1.\,\ldots\,.\gamma_r).(\gamma_{r+1}.\,\ldots\,.\gamma_n)$ is a reparameterization of $\gamma_1.\gamma_2.\,\cdots\,.\gamma_n$ for any r between 1 and $n-1$.

(d) By making repeated applications of (c), or otherwise, show that $(\gamma_1.\gamma_2).\gamma_3.(\gamma_4.\gamma_5)$ is a reparameterization of $\gamma_1.(\gamma_2.\gamma_3.\gamma_4).\gamma_5$ for all paths $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$ in X satisfying $\gamma_i(1) = \gamma_{i+1}(0)$ for $i = 1, 2, 3, 4$.

9. Let X be a topological space.

(a) Show that, given any path $\alpha: [0,1] \to X$ in X, there is a well-defined homomorphism $\Theta_\alpha: \pi_1(X, \alpha(1)) \to \pi_1(X, \alpha(0))$ of fundamental groups which sends the homotopy class $[\gamma]$ of any loop γ based at $\alpha(1)$ to the homotopy class $[\alpha.\gamma.\alpha^{-1}]$ of the loop $\alpha.\gamma.\alpha^{-1}$, where

$$(\alpha.\gamma.\alpha^{-1})(t) = \begin{cases} \alpha(3t) & \text{if } 0 \leq t \leq \frac{1}{3}, \\ \gamma(3t-1) & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ \alpha(3-3t) & \text{if } \frac{2}{3} \leq t \leq 1 \end{cases}$$

(i.e., $\alpha.\gamma.\alpha^{-1}$ represents 'α followed by γ followed by α reversed').

(b) Show that $\Theta_{\alpha.\beta} = \Theta_\alpha \circ \Theta_\beta$ for all paths α and β in X satisfying $\beta(0) = \alpha(1)$.

(c) Show that Θ_α is the identity homomorphism whenever α is a constant path.

(d) Let α and $\hat{\alpha}$ be paths in X satisfying $\alpha(0) = \hat{\alpha}(0)$ and $\alpha(1) = \hat{\alpha}(1)$. Suppose that $\alpha(0) \simeq \hat{\alpha}(0) \operatorname{rel}\{0,1\}$. Show that $\Theta_\alpha = \Theta_{\hat{\alpha}}$.

(e) Explain why the homomorphism $\Theta_\alpha \colon \pi_1(X, \alpha(1)) \to \pi_1(X, \alpha(0))$ is an isomorphism for all paths α in X. (This shows that, up to isomorphism, the fundamental group of a path-connected topological space does not depend on the choice of basepoint.)

10. Let X and Y be topological spaces, and let x_0 and y_0 be points of X and Y. Prove that $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$. [Hint: you should make use of the result that a function mapping a topological space into $X \times Y$ is continuous if and only if its components are continuous.]