## Module MA3427: Michaelmas Term 2010 Problems

1. (a) Let  $f: X \to Y$  be a function from a topological space X to a topological space Y, and let A and B be subsets of X for which  $X = A \cup B$ . Suppose that the restrictions f|A and f|B of f to the sets A and B are continuous. Is  $f: X \to Y$  necessarily continuous on X? [Give proof or counterexample.]

(b) Let  $f: X \to Y$  be a function from a topological space X to a topological space Y, and let  $\mathcal{F}$  be a (not necessarily finite) collection of closed subsets of X whose union is the whole of X. Suppose that the restriction f|A of f to A is continuous for all closed sets A in the collection  $\mathcal{F}$ . Is  $f: X \to Y$  necessarily continuous on X? [Give proof or counterexample.]

- 2. Let  $f: X \to Y$  be a function from a topological space X to a topological space Y, and let  $\mathcal{U}$  be a collection of open subsets of X whose union is the whole of X. Suppose that the restriction f|W of f to W is continuous for all open sets W in the collection  $\mathcal{U}$ . Prove that  $f: X \to Y$  is continuous on X.
- 3. Let X be a topological space, let A be a subset of X, and let B be the complement  $X \setminus A$  of A in X. Prove that the interior of B is the complement of the closure of A.
- 4. Determine which of the following subsets of  $\mathbb{R}^3$  are compact.
  - (i) The x-axis  $\{(x, y, z) \in \mathbb{R}^3 : y = z = 0\}$
  - (ii) The surface of a tetrahedron in  $\mathbb{R}^3$ .
  - (iii)  $\{(x, y, z) \in \mathbb{R}^3 : x > 0 \text{ and } x^2 + y^2 z^2 \le 1\}.$
- 5. Let X be a topological space. Suppose that  $X = A \cup B$ , where A and B are path-connected subsets of X and  $A \cap B$  is non-empty. Show that X is path-connected.
- 6. Let  $f: X \to Y$  be a continuous map between topological spaces X and Y. Suppose that X is path-connected. Prove that the image f(X) of the map f is also path-connected.

- 7. Let X and Y be path-connected topological spaces. Explain why the Cartesian product  $X \times Y$  of X and Y is path-connected.
- 8. Determine the connected components of the following subsets of  $\mathbb{R}^2$ :
  - (i)  $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},\$
  - (ii)  $\{(x,y) \in \mathbb{R}^2 : x^2 y^2 = 1\},\$
  - (iii)  $\{(x,y) \in \mathbb{R}^2 : y^2 = x(x^2 1)\},\$
  - (iv)  $\{(x,y) \in \mathbb{R}^2 : (x-n)^2 + y^2 > \frac{1}{4} \text{ for all } n \in \mathbb{Z}\}.$

[Fully justify your answers.]

9. A topological space X is said to be *locally path-connected* if, given any point x of X there exists a path-connected open set U in X such that  $x \in U$ .

(a) Let X be a locally path-connected topological space, and let p be a point of X. Let A be the set of all points x of X for which there exists a path from p to x, and let B be the complement of A in X. Prove that A and B are open in X.

(b) Use the result of (a) to show that any connected and locally pathconnected topological space is path-connected.

10. Let X be a convex subset of  $\mathbb{R}^n$ . (A subset X of  $\mathbb{R}^n$  is said to be *convex* if  $(1 - t)\mathbf{x} + t\mathbf{y} \in X$  for all  $\mathbf{x}, \mathbf{y} \in X$  and real numbers t satisfying  $0 \le t \le 1$ .)

(a) Prove that any two continuous functions mapping some topological space into X are homotopic.

(b) Prove that any two continuous functions mapping X into some path-connected topological space Y are homotopic.

11. Determine which of the following maps are covering maps:—

(i) the map from  $\mathbb{R}$  to [-1, 1] sending  $\theta$  to  $\sin \theta$ ,

(ii) the map from  $S^1$  to  $S^1$  sending  $(\cos \theta, \sin \theta)$  to  $(\cos n\theta, \sin n\theta)$ , where n is some non-zero integer,

(iii) the map from  $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$  to  $\{z \in \mathbb{C} : 0 < |z| < 1\}$  sending z to  $\exp(z)$ ,

(iv) the map from  $\{z \in \mathbb{C} : -4\pi < \text{Im } z < 4\pi\}$  to  $\{z \in \mathbb{C} : |z| > 0\}$  sending z to  $\exp(z)$ .

[Briefly justify your answers.]

- 12. A continuous function  $f: X \to Y$  between topological spaces X and Y is said to be a *local homeomorphism* if, given any point x of X there exists an open set V in X containing the point x and an open set W in Y containing the point f(x) such that the function f maps V homeomorphically onto W. Explain why any covering map is a local homeomorphism.
- 13. Determine which of the maps described in question 11 are local homeomorphisms.
- 14. (a) Let W, X, Y and Z be topological spaces, and let A be a subset of X. Let  $f: X \to Y$  and  $g: X \to Y$  be continuous maps. Suppose that  $f \simeq g \operatorname{rel} A$ . Show that  $h \circ f \simeq h \circ g \operatorname{rel} A$  for all continuous maps  $h: Y \to Z$ , and that  $f \circ e \simeq g \circ e \operatorname{rel} e^{-1}(A)$  for all continuous maps  $e: W \to X$ .

(b) Using (a), explain why, given any continuous map  $f: X \to Y$  between topological spaces X and Y, there is a well-defined homomorphism  $f_{\#}: \pi_1(X, x) \to \pi_1(Y, f(x))$  of fundamental groups for any  $x \in X$ which sends  $[\gamma]$  to  $[f \circ \gamma]$  for any loop  $\gamma$  based at the point x.

(c) Let  $f: X \to Y$  and  $g: X \to Y$  be continuous maps satisfying f(x) = g(x) and  $f \simeq g \operatorname{rel}\{x\}$ . Show that the homomorphisms  $f_{\#}$  and  $g_{\#}$  from  $\pi_1(X, x)$  to  $\pi_1(Y, f(x))$  induced by the maps f and g are equal.

15. (a) Let X and Y be topological spaces, let  $f: X \to Y$  and  $h: Y \to X$ continuous maps, and let x be a point of X. Suppose that h(f(x)) = xand that  $h \circ f \simeq 1_X \operatorname{rel}\{x\}$  and  $f \circ h \simeq 1_Y \operatorname{rel}\{f(x)\}$ , where  $1_X$  and  $1_Y$  denote the identity maps of the spaces X and Y. Explain why the fundamental groups  $\pi_1(X, x)$  and  $\pi_1(Y, f(x))$  are isomorphic.

(b) Using (a), explain why the fundamental groups  $\pi_1(\mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbf{p})$  and  $\pi^1(S^{n-1}, \mathbf{p})$  of  $\mathbb{R}^n \setminus \{\mathbf{0}\}$  and the (n-1)-dimensional sphere  $S^{n-1}$  are isomorphic for all n > 1, where  $\mathbf{p} \in S^{n-1}$ .

16. Let X be a topological space, and let  $\alpha: [0,1] \to X$  and  $\beta: [0,1] \to X$  be paths in X. We say that the path  $\beta$  is a *reparameterization* of the path  $\alpha$  if there exists a strictly increasing continuous function  $\sigma: [0,1] \to [0,1]$  such that  $\sigma(0) = 0$ ,  $\sigma(1) = 1$  and  $\beta = \alpha \circ \sigma$ . (Note that if  $\beta$  is a reparameterization of  $\alpha$  then  $\alpha(0) = \beta(0)$ ,  $\alpha(1) = \beta(1)$ , and the paths  $\alpha$  and  $\beta$  have the same image in X.)

(a) Show that there is a well-defined equivalence relation on the set of all paths in X, where a path  $\alpha$  is related to a path  $\beta$  if and only if  $\beta$  is a reparameterization of a path  $\alpha$ . [Hint: use the basic result of analysis which states that a strictly increasing continuous function mapping one interval onto another has a continuous inverse.]

(b) Show that if the path  $\beta$  is a reparameterization of the path  $\alpha$ , then  $\beta \simeq \alpha \operatorname{rel}\{0, 1\}$ .

Given paths  $\gamma_1, \gamma_2, \ldots, \gamma_n$  in a topological space X, where  $\gamma_i(1) = \gamma_{i+1}(0)$  for  $i = 1, 2, \ldots, n-1$ , we define the *concatenation*  $\gamma_1.\gamma_2. \ldots. \gamma_n$  of the paths by the formula  $(\gamma_1.\gamma_2. \ldots. \gamma_n)(t) = \gamma_i(nt - i + 1)$  for all t satisfying  $(i - 1)/n \leq t \leq i/n$ .

(c) Show that the path  $(\gamma_1, \ldots, \gamma_r).(\gamma_{r+1}, \ldots, \gamma_n)$  is a reparameterization of  $\gamma_1.\gamma_2. \cdots .\gamma_n$  for any r between 1 and n-1.

(d) By making repeated applications of (c), or otherwise, show that  $(\gamma_1.\gamma_2).\gamma_3.(\gamma_4.\gamma_5)$  is a reparameterization of  $\gamma_1.(\gamma_2.\gamma_3.\gamma_4).\gamma_5$  for all paths  $\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5$  in X satisfying  $\gamma_i(1) = \gamma_{i+1}(0)$  for i = 1, 2, 3, 4.

17. Let X be a topological space.

(a) Show that, given any path  $\alpha: [0,1] \to X$  in X, there is a welldefined homomorphism  $\Theta_{\alpha}: \pi_1(X, \alpha(1)) \to \pi_1(X, \alpha(0))$  of fundamental groups which sends the homotopy class  $[\gamma]$  of any loop  $\gamma$  based at  $\alpha(1)$ to the homotopy class  $[\alpha.\gamma.\alpha^{-1}]$  of the loop  $\alpha.\gamma.\alpha^{-1}$ , where

$$(\alpha.\gamma.\alpha^{-1})(t) = \begin{cases} \alpha(3t) & \text{if } 0 \le t \le \frac{1}{3}, \\ \gamma(3t-1) & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ \alpha(3-3t) & \text{if } \frac{2}{3} \le t \le 1 \end{cases}$$

(i.e.,  $\alpha . \gamma . \alpha^{-1}$  represents ' $\alpha$  followed by  $\gamma$  followed by  $\alpha$  reversed').

(b) Show that  $\Theta_{\alpha,\beta} = \Theta_{\alpha} \circ \Theta_{\beta}$  for all paths  $\alpha$  and  $\beta$  in X satisfying  $\beta(0) = \alpha(1)$ .

(c) Show that  $\Theta_{\alpha}$  is the identity homomorphism whenever  $\alpha$  is a constant path.

(d) Let  $\alpha$  and  $\hat{\alpha}$  be paths in X satisfying  $\alpha(0) = \hat{\alpha}(0)$  and  $\alpha(1) = \hat{\alpha}(1)$ . Suppose that  $\alpha(0) \simeq \hat{\alpha}(0)$  rel $\{0, 1\}$ . Show that  $\Theta_{\alpha} = \Theta_{\hat{\alpha}}$ .

(e) Explain why the homomorphism  $\Theta_{\alpha}: \pi_1(X, \alpha(1)) \to \pi_1(X, \alpha(0))$  is an isomorphism for all paths  $\alpha$  in X. (This shows that, up to isomorphism, the fundamental group of a path-connected topological space does not depend on the choice of basepoint.)

18. Let X and Y be topological spaces, and let  $x_0$  and  $y_0$  be points of X and Y. Prove that  $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$ . [Hint: you should make use of the result that a function mapping a topological space into  $X \times Y$  is continuous if and only if its components are continuous.]